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Intermediate Pure Mathematics

By J. BLAKEY Ph.D.

Lecturer in Mathematics
Sunderland Technical College

SECOND EDITION

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Preface to Second Edition

The additions now made to this evidently acceptable text have been carefully chosen so as to make the text cover practically all examinations at the General Certificate Advanced Level in Pure Mathematics. They include discussions of Polar Co-ordinates; Logarithmic, Sine and Cosine Series; Coaxal Circles; Determinants; Curvature; Differentiation of Inverse Trigonometric Functions; Integration by Partial Fractions and by Substitution; Mean Values: Integration by Parts; and Differential Equations.

Apart from the worked examples throughout the book there are now over 900 problems practically all of them taken by permission from London University and Northern Matriculation Board examination papers. Examples worked out in the text are marked L.U. when taken from those of the former body. Although in the interests of keeping the book to a moderate compass there is no exposition of formal geometry, a set of 100 problems on plane and solid geometry is provided.

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The author hopes that the book will continue to find favour in Grammar Schools, High Schools, Technical Colleges and Universities both in Britain and in many parts of the Commonwealth, and that his explanations together with the liberal provision of examples worked in detail, will commend it to students working alone.

His grateful thanks are due to the Senate of the London University and to the Northern Universities Joint Matriculation Board for their kind permission to use problems from their examination papers; and he would also like to express his indebtedness to his colleague Mr. P. McKenna, B.Sc., and to his son Michael for their assistance with the proof-reading.

Some minor errors—gratifyingly small in number considering the the complicated nature of the printing—have now been rectified, but author would still be glad of any comment or suggestions which teachers and students care to send to him care of his publishers.

Sunderland,
February 1960

JOSEPH BLAKEY

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CHAPTER I

Quadratic Expressions and Equations in One or More Variables

Numbers. Before proceeding to algebraic methods it is necessary to become conversant with the various types of numbers with which mathematical reasoning is concerned.

Consider a straight line that can be extended indefinitely in either direction. On this line choose a point 0 to represent zero and, using a suitable scale, mark off to the right one, two, three units, etc., to represent the positive numbers 1, 2, 3, etc., respectively. The negative numbers -1, -2, -3, etc., will be similarly marked to the left of 0 along this line.

Any point on this line will represent a number according to the chosen scale, and any such number is known as a *real number.*

Real numbers are subdivided into two groups, (i) rational numbers, (ii) irrational numbers.

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A *rational number* is any number that can be expressed as the quotient of two integers (whole numbers). Thus $\frac{1}{2}$, $\frac{4}{3}$, $-2\frac{1}{5}$, etc., are all rational numbers.

An *irrational number* is a real number that cannot be expressed as the quotient of two integers. Thus $\sqrt{3}$, $\sqrt{7}$, π are examples of irrational numbers.

Generally a real number which forms neither a terminating nor a recurring decimal when put in the decimal form is an irrational number.

Now consider the equation

$$x^2 + 2x + 2 = 0,$$

$$\text{i.e. } (x + 1)^2 + 1 = 0.$$

From this

$$(x + 1)^2 = -1,$$

$$\therefore x + 1 = \pm \sqrt{(-1)},$$

$$\therefore x = -1 \pm \sqrt{(-1)}.$$

The quantity $\sqrt{(-1)}$ found in this result is usually denoted by the letter i, which is the first letter of the word 'imaginary' and numbers involving $\sqrt{(-1)} = i$ are sometimes known as 'imaginary numbers', though this is not a very suitable term; any number of the form $a + ib$, where $i = \sqrt{(-1)}$ and a and b are real numbers is known as a *complex number.*

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Definitions. (1) *A quadratic expression* in x is an expression of the form $ax^2 + bx + c$, where a, b, c are constants.

(2) *A quadratic equation* in a single variable x is an equation of the second degree in x of the type $ax^2 + bx + c = 0$, where a, b, c are constants, or an equation that can be reduced to this form.

In attempting the solution of any quadratic equation in a single variable x it is important to try first the method of factorisation. This entails the initial clearing of fractions by multiplying through the equation by the L.C.M. of the denominators of the fractions.

Next all terms are transferred to the left-hand side (L.H.S.) of the equation and like terms collected together, so that the equation will now appear in the form

$$ax^2 + bx + c = 0.$$

Then the L.H.S. is factorised where possible and each of the linear factors in x is equated to zero, giving a solution (or root) of the given equation. If time permits, the roots obtained should be verified in the original equation as shown in the following example.

Example. Solve the equation

$$\frac{3}{(x-3)} - \frac{4}{(x-4)} + \frac{5}{(x-1)} = 0.$$

Multiplying through the given equation by $(x-1)(x-3)(x-4)$ to clear fractions

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$$3(x-4)(x-1) - 4(x-1)(x-3) + 5(x-3)(x-4) = 0,$$
$$\text{i.e. } 3(x^2-5x+4) - 4(x^2-4x+3) + 5(x^2-7x+12) = 0,$$
$$\text{i.e. } 3x^2 - 15x + 12 - 4x^2 + 16x - 12 + 5x^2 - 35x + 60 = 0,$$
$$\therefore 4x^2 - 34x + 60 = 0,$$
$$\therefore 2x^2 - 17x + 30 = 0,$$
$$\text{i.e. } (2x-5)(x-6) = 0,$$
$$\therefore 2x - 5 = 0 \text{ or } x - 6 = 0,$$
$$\text{i.e. } x = 5/2, \text{ or } 6.$$

Check. $x = 5/2$ L.H.S. $= \frac{3}{-1/2} - \frac{4}{-3/2} + \frac{5}{3/2}$
$= -6 + 8/3 + 10/3$
$= -6 + 18/3 = -6 + 6 = 0,$

$x = 6$ L.H.S. $= \frac{3}{3} - \frac{4}{2} + \frac{5}{5}$
$= 1 - 2 + 1 = 0.$

A second-degree (quadratic) equation in one variable always has two roots, a third-degree equation (cubic) in one variable always has three solutions, a fourth-degree equation (quartic) in one variable has four roots, and so on.

When the roots of an equation involve $i = \sqrt{(-1)}$ they are said to be complex numbers, and otherwise they are real roots. If two roots of an equation are equal they are said to be *coincident*.

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Thus the roots of a quadratic equation in one unknown can be real, coincident, or complex numbers, and it is to be noted that complex number roots can be shown to occur in pairs.

When a quadratic equation, which has been manipulated into the form $ax^2 + bx + c = 0$, cannot be solved by factorisation it will be solved by means of the formula obtained as follows:

$$ax^2 + bx + c = 0 \qquad (1)$$

Dividing through (1) by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\therefore\ x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Completing the square of the L.H.S. by adding the square of half the coefficient of x to each side

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\text{i.e. } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore\ x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)}$$

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$$= \pm\frac{\sqrt{(b^2 - 4ac)}}{2a}$$

$$\therefore\ x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

NOTE. It can be shown, as follows, that the quadratic equation $ax^2 + bx + c = 0$ cannot have more than two roots (a is not zero, i.e. $a \neq 0$).

If possible let the equation $ax^2 + bx + c = 0$ have three roots α, β, γ which are different.

Since each of these roots must satisfy the equation $ax^2 + bx + c = 0$, it follows that

$$a\alpha^2 + b\alpha + c = 0 \qquad (2).$$
$$a\beta^2 + b\beta + c = 0 \qquad (3).$$
$$a\gamma^2 + b\gamma + c = 0 \qquad (4).$$

(2) − (3) gives $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$

i.e. $a(\alpha - \beta)(\alpha + \beta) + b(\alpha - \beta) = 0$

therefore since $\alpha \neq \beta$ by hypothesis

$$a(\alpha + \beta) + b = 0 \qquad (5).$$

Similarly from (3) and (4)

$$a(\beta + \gamma) + b = 0 \qquad (6).$$

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(5) − (6) gives $a(\alpha - \gamma) = 0$, which is impossible since $a \neq 0$ and by hypothesis $\alpha \neq \gamma$.

Hence there cannot be three different roots.

From the previous formula it can be seen that the roots of the given quadratic will be:

(i) real if $b^2 \geqslant 4ac$,
(ii) complex numbers if $b^2 < 4ac$,
(iii) coincident if $b^2 = 4ac$,
(iv) rational if $b^2 - 4ac$ be a complete square.

EXAMPLES: (i) Find to two decimal places the roots of the equation $2x^2 - 7x + 4 = 0$.
(ii) Find the roots of the equation $5x^2 - 4x + 2 = 0$.

The solutions of the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

(i) In this case $a = 2$, $b = -7$, $c = 4$,

$$\therefore x = \frac{-(-7) \pm \sqrt{[(-7)^2 - 4 \times 2 \times 4]}}{2 \times 2} = \frac{7 \pm \sqrt{(49 - 32)}}{4}$$

$$= \frac{7 \pm \sqrt{17}}{4} = \frac{7 \pm 4{\cdot}123}{4} \text{ (using square root tables)}$$

$$= \frac{11{\cdot}123}{4} \text{ or } \frac{2{\cdot}877}{4}$$

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$= 2{\cdot}78$ or $0{\cdot}72$ to two decimal places.

(ii) In this case $a = 5$, $b = -4$, $c = 2$

$$\therefore x = \frac{-(-4) \pm \sqrt{[(-4)^2 - 4 \times 5 \times 2]}}{2 \times 5}$$

$$= \frac{4 \pm \sqrt{[16 - 40]}}{10} = \frac{4 \pm \sqrt{(-24)}}{10}$$

$$= \frac{4 \pm 2\sqrt{6} \times \sqrt{-1}}{10} = \frac{2 \pm \sqrt{6} \times \sqrt{-1}}{5}.$$

NOTE. Since the result in (ii) involves $\sqrt{-1}$ it cannot be worked any further.

Theorem. *If α and β ($\alpha > \beta$) be the roots of the equation $ax^2 + bx + c = 0$, to prove that $\alpha + \beta = -b/a$, and $\alpha\beta = c/a$.*

From the previous formula

$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a}, \quad \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$\therefore \alpha + \beta = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} + \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

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$$\alpha\beta = \left\{\frac{-b + \sqrt{(b^2 - 4ac)}}{2a}\right\}\left\{\frac{-b - \sqrt{(b^2 - 4ac)}}{2a}\right\}$$

$$= \frac{(-b)^2 - [\sqrt{(b^2 - 4ac)}]^2}{4a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Otherwise: the equations

$$ax^2 + bx + c = 0 \quad\ldots\ldots\ldots\ldots(1),$$

and

$$(x - \alpha)(x - \beta) = 0 \quad\ldots\ldots\ldots\ldots(2)$$

will have the same roots and are known as *equivalent equations*, and the coefficients of the various powers of x in the two equations must be proportional.

Expanding equation (2) it becomes,

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad\ldots\ldots\ldots\ldots(3).$$

Comparing equations (3) and (1)

$$\frac{1}{a} = -\frac{(\alpha + \beta)}{b} = \frac{\alpha\beta}{c}$$

$$\left.\begin{aligned}\therefore \alpha + \beta &= -b/a\\ \alpha\beta &= c/a\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots(4)$$

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NOTE. If it be required to form an equation whose roots are α_1 and β_1, where α_1 and β_1 are functions of the above roots α and β, the required equation will be, from (3) above,

$$x^2 - (\alpha_1 + \beta_1)x + \alpha_1\beta_1 = 0 \quad\ldots\ldots\ldots\ldots(5)$$

and the quickest method usually is to find both $(\alpha_1 + \beta_1)$ and $\alpha_1\beta_1$ in terms of $\alpha + \beta$ and $\alpha\beta$, and then use the results (4) to obtain them in terms of a, b, c. The values thus found will be used in equation (5) and the result simplified so that there are no fractions involved.

It is advisable to remember the following:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta.$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = (\alpha + \beta)[(\alpha^2 + 2\alpha\beta + \beta^2) - 3\alpha\beta]$$

$$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta].$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2 \text{ (using value of } \alpha^2 + \beta^2)$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}.$$

EXAMPLE (L.U.). If α and β be the roots of the equation $ax^2 + bx + c = 0$, prove that $a(\alpha + \beta) = -b$, $a\alpha\beta = c$.

Form the equation whose roots are (i) $1/\alpha$, $1/\beta$; (ii) $\alpha + 1/\alpha$, $\beta + 1/\beta$, giving the coefficients in each case in terms of a, b, c.

The first part of the question has already been proved.

Let α_1 and β_1 be the roots in each case. Then the required equation is

$$x^2 - (\alpha_1 + \beta_1)x + \alpha_1\beta_1 = 0.$$

(i) $$\alpha_1 + \beta_1 = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-b/a}{+c/a} = \frac{-b}{c}.$$

$$\alpha_1\beta_1 = \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c},$$

therefore required equation is

$$x^2 - \left(\frac{-b}{c}\right)x + \frac{a}{c} = 0$$

$$\text{i.e. } cx^2 + bx + a = 0.$$

(ii) $$\alpha_1 + \beta_1 = (\alpha + 1/\alpha) + (\beta + 1/\beta) = (\alpha + \beta) + (1/\alpha + 1/\beta)$$
$$= -\frac{b}{a} - \frac{b}{c} \text{ (using part (i))}$$
$$= -b(a + c)/ac.$$
$$\alpha_1\beta_1 = (\alpha + 1/\alpha)(\beta + 1/\beta) = \alpha\beta + \alpha/\beta + \beta/\alpha + 1/(\alpha\beta)$$
$$= \alpha\beta + \frac{\alpha^2 + \beta^2}{\alpha\beta} + \frac{1}{\alpha\beta}$$
$$= \alpha\beta + \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} + \frac{1}{\alpha\beta}$$
$$= \frac{c}{a} + \frac{b^2/a^2 - 2c/a}{c/a} + \frac{a}{c}$$
$$= \frac{a^2 + b^2 + c^2 - 2ac}{ac} \quad \left(\frac{b^2/a^2 - 2c/a}{c/a} = \frac{b^2 - 2ac}{ac}\right)$$

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therefore required equation is

$$x^2 - \left\{-\frac{b(a + c)}{ac}\right\}x + \frac{a^2 + b^2 + c^2 - 2ac}{ac} = 0$$

$$\text{i.e. } acx^2 + b(a + c)x + a^2 + b^2 + c^2 - 2ac = 0.$$

EXAMPLE (L.U.). Show that the roots of the equation

$$x^2 - 2(a - 2)x + 2a - 10 = 0$$

are real if a be real.

Find the possible values of a, when the roots of the equation differ by 6.

The roots of the equation $ax^2 + bx + c = 0$ are real if $b^2 \geqslant 4ac$ therefore the roots of $x^2 - 2(a - 2)x + 2a - 10 = 0$ are real if

$$[2(a - 2)]^2 \geqslant 4(2a - 10)$$
$$\text{i.e. if } 4(a^2 - 4a + 4) \geqslant 4(2a - 10)$$
$$\text{i.e. if } a^2 - 4a + 4 \geqslant 2a - 10$$
$$\text{i.e. if } a^2 - 6a + 14 \geqslant 0$$
$$\text{i.e. if } (a - 3)^2 + 5 \geqslant 0 \text{ (complete square of } a \text{ portion)}$$

i.e. if a be real, since the square of a real quantity is always positive.

Let α and β be the roots of the given equation. Then

$$\alpha + \beta = 2(a - 2) \quad \ldots\ldots (1)$$
$$\alpha\beta = 2a - 10 \quad \ldots\ldots (2)$$
$$\text{and } \alpha - \beta = 6 \quad \ldots\ldots (3)$$

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(1) + (3) gives $2\alpha = 2a + 2 \quad \therefore \alpha = a + 1$
(1) − (3) gives $2\beta = 2a - 10 \quad \therefore \beta = (a - 5)$.

Using these in (2)

$$(a + 1)(a - 5) = 2(a - 5)$$
$$\therefore (a + 1)(a - 5) - 2(a - 5) = 0$$
$$\text{i.e. } (a - 5)(a + 1 - 2) = 0$$
$$\therefore (a - 5)(a - 1) = 0$$
$$\therefore a = 1 \text{ or } 5.$$

Theorem. *To find the condition that two quadratic equations should have a common root.*

Let α be the common root and the two quadratic equations be

$$a_1x^2 + b_1x + c_1 = 0 \quad \ldots\ldots(1),$$
$$a_2x^2 + b_2x + c_2 = 0 \quad \ldots\ldots(2),$$

where α is not zero.

Hence
$$a_1\alpha^2 + b_1\alpha + c_1 = 0 \quad \ldots\ldots(3),$$
$$a_2\alpha^2 + b_2\alpha + c_2 = 0 \quad \ldots\ldots(4).$$

(3) × c_2 gives $a_1c_2\alpha^2 + b_1c_2\alpha + c_1c_2 = 0 \ldots\ldots(5)$.
(4) × c_1 gives $a_2c_1\alpha^2 + b_2c_1\alpha + c_1c_2 = 0 \ldots\ldots(6)$.
(5) − (6) gives $(a_1c_2 - a_2c_1)\alpha^2 + (b_1c_2 - b_2c_1)\alpha = 0$,

$$\text{i.e. } \alpha = -\frac{(b_1c_2 - b_2c_1)}{a_1c_2 - a_2c_1} \qquad (\alpha \neq 0)$$

(3) × a_2 gives $a_1a_2\alpha^2 + a_2b_1\alpha + a_2c_1 = 0 \ldots\ldots(7)$.
(4) × a_1 gives $a_1a_2\alpha^2 + a_1b_2\alpha + a_1c_2 = 0 \ldots\ldots(8)$.
(7) − (8) gives $(a_2b_1 - a_1b_2)\alpha + a_2c_1 - a_1c_2 = 0$

$$\text{i.e. } \alpha = -\frac{(a_2c_1 - a_1c_2)}{a_2b_1 - a_1b_2} = -\frac{(a_1c_2 - a_2c_1)}{a_1b_2 - a_2b_1}$$

Equating these results for α, the required condition is

$$\frac{b_1c_2 - b_2c_1}{a_1c_2 - a_2c_1} = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},$$

i.e. $(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) = (a_1c_2 - a_2c_1)^2$

Theorem. *To find the condition that the quadratic expression $ax^2 + bx + c$ shall be positive, and also the condition that it shall be negative, if $b^2 \geqslant 4ac$.*

There are two cases to be considered, (i) when a is positive, (ii) when a is negative, and in both cases α and β are taken to be the roots of the equation $ax^2 + bx + c = 0$, so that

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

where α is taken as being greater than β.

Case (i). a positive.
When $x > \alpha$, $(x - \alpha)$ is positive and $(x - \beta)$ is positive
$$\therefore a(x - \alpha)(x - \beta) \equiv ax^2 + bx + c \text{ is } \textit{positive}.$$

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When $x < \beta$, $(x - \alpha)$ is negative and $(x - \beta)$ is negative

$$\therefore\ a(x - \alpha)(x - \beta) \equiv ax^2 + bx + c \text{ is } \textit{positive}.$$

When $\alpha > x > \beta$, $(x - \alpha)$ is negative and $(x - \beta)$ is positive

$$\therefore\ a(x - \alpha)(x - \beta) \equiv ax^2 + bx + c \text{ is } \textit{negative}.$$

Thus if a be positive, $ax^2 + bx + c$ is *positive* for $x > \alpha$ or $< \beta$, and *negative* for $\alpha > x > \beta$.

Case (ii). a negative.

As in Case (i), if a be negative, $ax^2 + bx + c$ is *positive* for $\alpha > x > \beta$, and *negative* if $x > \alpha$ or $< \beta$.

This method is best shown in the following tabular form.

Size of x	*Sign of* a	*Sign of* $x - \alpha$	*Sign of* $(x - \beta)$	*Sign of* $ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$
$x > \alpha$	+	+	+	+
$x < \beta$	+	−	−	+
$\alpha > x > \beta$	+	−	+	−
$x > \alpha$	−	+	+	−
$x < \beta$	−	−	−	−
$\alpha > x > \beta$	−	−	+	+

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Theorem. *To find the maximum or minimum value of the quadratic function* $ax^2 + bx + c$.

NOTE. The square of any *real* quantity is always positive, and it is assumed that all quantities dealt with are real unless otherwise stated.

Case (i). a positive.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a[(x + b/2a)^2 + (c/a - b^2/4a^2)].$$

(complete square of x-portion)

Since $(x + b/2a)^2$ is a complete square, it must be positive and it will have its least value when $x = -b/2a$. In this case the value of the quadratic expression will be a minimum and this minimum value will be

$$a\left(\frac{c}{a} - \frac{b^2}{4a^2}\right) = \frac{4ac - b^2}{4a}.$$

Case (ii). a negative.

As in Case (i) it can be seen that the maximum value occurs when $x = -b/2a$ and this value is

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$$\frac{4ac - b^2}{4a}.$$

NOTE. In the case of an equation involving the sum of a number of squares equal to zero, the only *real* possible solutions are given by equating to zero each one of the squares (since the square of a real quantity can only be positive).

Thus, if $x^2 + y^2 + z^2 = 0$, and x, y, z be real, then $x = 0$, $y = 0$, $z = 0$ simultaneously.

Theorem. *To find the limits between which $f(x)/\phi(x)$ must lie, where $f(x)$ and also $\phi(x)$ is either a quadratic in x or linear in x, one at least being a quadratic expression (or function).*

NOTE. In all cases of this description, x is taken to be real, whether this fact is stated in the question or not.

Equate the given fraction to k and clear the fractions thus obtaining a quadratic equation in x. Next state the condition (an inequality) that the roots of quadratic equation in x shall be real, thus obtaining a certain quadratic in k greater than zero. From this inequality the limits for k can be found by using the results of k when $ak^2 + bk + c$ has to be positive, and when it has to be negative.

NOTE. When an inequality is divided through by a negative quantity the sign of inequality must be changed, e.g. $3 > 2$, but $-3 < -2$.

EXAMPLE (L.U.). Show that, for all real values of x, the expression

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$$\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$$

lies between 3 and $\frac{1}{3}$.

Let
$$\frac{x^2 - 2x + 4}{x^2 + 2x + 4} = k$$

$$\therefore x^2 - 2x + 4 = kx^2 + 2kx + 4k$$

i.e. $x^2(1 - k) - 2(1 + k)x + (4 - 4k) = 0.$

Since x is real

$$4(1 + k)^2 \geqslant 4(1 - k)(4 - 4k), \quad \begin{pmatrix} b^2 \geqslant 4ac \text{ in equation} \\ ax^2 + bx + c = 0 \end{pmatrix}$$

i.e. $(1 + k)^2 \geqslant 4(1 - k)^2,$

i.e. $1 + 2k + k^2 \geqslant 4 - 8k + 4k^2,$

i.e. $-3k^2 + 10k - 3 \geqslant 0,$

i.e. $-(k - 3)(3k - 1) \geqslant 0.$

If k lies between 3 and $\frac{1}{3}$ the L.H.S. of the above inequality is positive and for all other values of k the L.H.S. is negative. Hence, for real values of x, the expression $(x^2 - 2x + 4)/(x^2 + 2x + 4)$ must lie between 3 and $\frac{1}{3}$.

EXAMPLE. Show that the function

$$\frac{(2x - 1)(x - 2)}{x - 1}$$

can take all values.

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Let $$\frac{(2x - 1)(x - 2)}{x - 1} = k,$$

$$\therefore\ 2x^2 - 5x + 2 = kx - k,$$
$$\text{i.e. } 2x^2 - (5 + k)x + (2 + k) = 0.$$

Since x is to be real $$(5 + k)^2 \geqslant 8(2 + k),$$
$$\text{i.e. } k^2 + 10k + 25 \geqslant 16 + 8k,$$
$$\therefore\ k^2 + 2k + 9 \geqslant 0,$$
$$\text{i.e. } (k + 1)^2 + 8 \geqslant 0.$$

Since for real values of k, $(k + 1)^2$ is always positive, the L.H.S. of the inequality is always positive for real values of k.

Thus the inequality is satisfied for all real values of k, and the given expression can take *all real values*.

EXAMPLE. Show that there are two values between which

$$\frac{(2x - 1)(2x + 5)}{2x - 3}$$

cannot lie and find them.

Draw a rough graph of the function.

Let $$\frac{(2x - 1)(2x + 5)}{2x - 3} = k$$

$$\therefore\ 4x^2 + 8x - 5 = 2kx - 3k$$
$$\text{i.e. } 4x^2 + (8 - 2k)x + (3k - 5) = 0.$$

Since x is real $$(8 - 2k)^2 \geqslant 16(3k - 5)$$
$$\text{i.e. } (4 - k)^2 \geqslant 4(3k - 5)$$
$$\therefore\ 16 - 8k + k^2 \geqslant 12k - 20$$
$$\text{i.e. } k^2 - 20k + 36 \geqslant 0$$
$$\text{i.e. } (k - 18)(k - 2) \geqslant 0.$$

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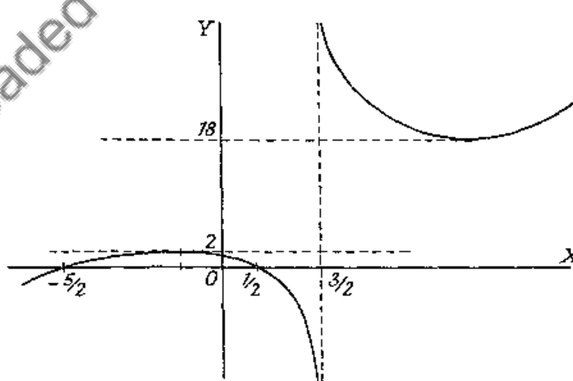
It can be seen from this inequality that k cannot lie between 2 and 18, i.e. $[(2x - 1)(2x + 5)]/(2x - 3)$ cannot lie between 2 and 18.



NOTE. For a rough graph of a given function y the following data is usually required:

(i) Points in which the graph cuts OX and OY (i.e. where $y = 0$ and $x = 0$).

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(ii) Maxima and minima. (Usually obtained by calculus as shown later.)

(iii) The asymptotes (i.e. lines which the curve gradually approaches but never actually meets).

Data for rough graph of $y = (2x - 1)(2x + 5)/(2x - 3)$.

(i) It has been shown that y cannot lie between 2 and 18.

(ii) When $x = 0$, $y = \frac{5}{3}$, and when $y = 0$, $x = \frac{1}{2}$ or $-\frac{5}{2}$.

(iii) When x is slightly $> \frac{3}{2}$, y is large and positive and when x is slightly $< \frac{3}{2}$, y is large and negative, i.e. $x = \frac{3}{2}$ is an *asymptote*.

Special Quadratic Equations. Certain equations which do not appear to be of the quadratic types can often be reduced to this form by means of a substitution as in the following example.

EXAMPLE. Solve the following equations:

(i) $z^4 - 3z^2 + 2 = 0$,

(ii) $y(y + 1) + \dfrac{12}{y(y + 1)} = 8$,

(iii) $x^4 + 2x^3 - x^2 + 2x + 1 = 0$.

(i) Let $z^2 = u$ and the given equation becomes the quadratic equation

$$\begin{aligned} u^2 - 3u + 2 &= 0 \\ \text{i.e. } (u - 1)(u - 2) &= 0, \\ \therefore u &= 1 \text{ or } 2, \text{ i.e. } z^2 = 1 \text{ or } 2, \\ \therefore z &= \pm 1, \pm\sqrt{2}. \end{aligned}$$

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(ii) Here, if $y(y + 1)$ be replaced by u, the equation becomes

$$\begin{aligned} u + 12/u &= 8 \\ \text{i.e. } u^2 + 12 &= 8u, \\ \therefore u^2 - 8u + 12 &= 0, \\ \therefore (u - 2)(u - 6) &= 0, \\ \text{i.e. } u &= 2 \text{ or } 6, \\ \therefore y(y + 1) &= 2 \quad \ldots\ldots (1) \\ \text{or } y(y + 1) &= 6 \quad \ldots\ldots (2) \end{aligned}$$

From (1)

$$\begin{aligned} y^2 + y - 2 &= 0, \\ \text{i.e. } (y + 2)(y - 1) &= 0, \\ \therefore y &= 1 \text{ or } -2. \end{aligned}$$

From (2)

$$\begin{aligned} y^2 + y - 6 &= 0, \\ \text{i.e. } (y + 3)(y - 2) &= 0, \\ \therefore y &= 2 \text{ or } -3. \end{aligned}$$

Hence, the complete solution is $y = 1, 2, -2$, or -3.

(iii) This example comes under the general equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0,$$

which is dealt with by first dividing by x^2, giving

$$x^2 + ax + b + \frac{a}{x} + \frac{1}{x^2} = 0,$$

$$\text{i.e. } (x^2 + 1/x^2) + a(x + 1/x) + b = 0 \quad \ldots\ldots (1)$$

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Next $x + 1/x$ is replaced by u, giving

$$(x + 1/x)^2 = u^2,$$
$$\text{i.e. } x^2 + 2 + 1/x^2 = u^2,$$
$$\text{i.e. } x^2 + 1/x^2 = u^2 - 2$$

and the equation (1) becomes

$$u^2 - 2 + au + b = 0$$
$$\text{i.e. } u^2 + au + (b - 2) = 0,$$

which is a quadratic in u and can be solved in the usual manner.

In the given example $a = 2$ and $b = -1$, therefore the quadratic in u is

$$u^2 + 2u - 3 = 0,$$
$$\text{i.e. } (u + 3)(u - 1) = 0,$$
$$\therefore u = -3 \text{ or } 1$$
$$\therefore x + 1/x = -3 \qquad \ldots\ldots(2),$$
$$\text{or } x + 1/x = 1 \qquad \ldots\ldots(3).$$

From (2)
$$x^2 + 1 = -3x$$
$$\text{i.e. } x^2 + 3x + 1 = 0$$

i.e. using the formula

$$x = \frac{-3 \pm \sqrt{(9 - 4)}}{2} = \frac{-3 \pm \sqrt{5}}{2}.$$

From (3)
$$x^2 + 1 = x$$
$$\text{i.e. } x^2 - x + 1 = 0$$

$$\therefore x = \frac{1 \pm \sqrt{(1 - 4)}}{2} = \frac{1 \pm \sqrt{(-3)}}{2} \text{ (complex roots).}$$

Equations involving Square Roots. When dealing with the equation $x = 2$, if it be squared the result is $x^2 = 4$ from which the solutions are $x = \pm 2$. Thus it can be seen that if a given equation be squared there is a possibility of introducing a value for the variable which does not actually satisfy the original equation and must be discarded as being an *extraneous root* introduced by squaring.

Hence, when dealing with equations which are solved by means of squaring, the roots obtained *must* always be tested in the *original* equation and any value that does not satisfy the original equation must be discarded.

When an equation involves square roots of functions of the variable (usually three in number), the square root sign always indicates the *positive* square root, and the method of solution to be adopted is to have two of the quantities on one side of the equation and the third square root on the other side.

Next square the two sides of the equation and transfer all terms not involving a square root to one side and have the square root term on the other side. After squaring this new equation an equation will be obtained from which the values of the variable will be determined, and these must be tested, as stated previously, in the original equation.

The following examples will illustrate this method of procedure.

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EXAMPLE (L.U.). Solve the equation

$$2\sqrt{(x+1)} - 3\sqrt{(2x-5)} = \sqrt{(x-2)}.$$

Squaring the given equation

$$4(x+1) - 12\sqrt{[(x+1)(2x-5)]} + 9(2x-5) = x-2.$$

i.e. $4x + 4 - 12\sqrt{(2x^2 - 3x - 5)} + 18x - 45 = x - 2,$

$$\therefore\ 21x - 39 = 12\sqrt{(2x^2 - 3x - 5)},$$

$$\text{i.e. } 7x - 13 = 4\sqrt{(2x^2 - 3x - 5)}.$$

Squaring again

$$49x^2 - 182x + 169 = 16(2x^2 - 3x - 5)$$
$$= 32x^2 - 48x - 80,$$
$$\therefore\ 17x^2 - 134x + 249 = 0,$$
$$\text{i.e. } (17x - 83)(x - 3) = 0,$$
$$\therefore\ x = 83/17, \text{ or } 3.$$

Using $x = 83/17$ in the original equation

$$\text{L.H.S.} = 2\sqrt{(83/17 + 1)} - 3\sqrt{(166/17 - 5)}$$
$$= 2\sqrt{100/17} - 3\sqrt{81/17} = 20/\sqrt{17} - 27/\sqrt{17} = -7/\sqrt{17};$$
$$\text{R.H.S.} = \sqrt{(83/17 - 2)} = \sqrt{(49/17)} = 7/\sqrt{17};$$

therefore $x = 83/17$ does not satisfy the original equation and must be discarded as an extraneous root.

Using $x = 3$

$$\text{L.H.S.} = 2\sqrt{4} - 3\sqrt{1} = 4 - 3 = 1;$$
$$\text{R.H.S.} = \sqrt{1} = 1;$$

therefore the solution is $x = 3$.

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EXAMPLE. Solve the equation $\sqrt{(x+2)} + \sqrt{(x+9)} = 7.$

NOTE. When there are only two square roots in the equation, as in the given example, it is advisable to have one square root on one side of the equation and the other square root on the other side.

The given equation is thus written $\sqrt{(x+2)} = 7 - \sqrt{(x+9)}.$

Squaring $\quad x + 2 = 49 - 14\sqrt{(x+9)} + (x+9),$

$$\text{i.e. } -56 = -14\sqrt{(x+9)},$$
$$\therefore\ 4 = \sqrt{(x+9)}.$$

Squaring again $\quad 16 = x + 9 \quad \therefore\ x = 7.$

This value for x will be found to check in the original equation and hence the required solution is $x = 7$.

EXAMPLE. Solve the equation $\sqrt{(x+2)} - \sqrt{(2x-3)} = 3\sqrt{(3x-5)}.$

Squaring the given equation

$$x + 2 - 2\sqrt{[(2x-3)(x+2)]} + (2x-3) = 9(3x-5)$$
$$= 27x - 45,$$
$$\text{i.e. } -2\sqrt{[(2x-3)(x+2)]} = 24x - 44,$$
$$\therefore\ \sqrt{(2x^2 + x - 6)} = 22 - 12x.$$

Squaring again

$$2x^2 + x - 6 = 144x^2 - 528x + 484,$$
$$\therefore\ 142x^2 - 529x + 490 = 0,$$
$$\text{i.e. } (x-2)(142x - 245) = 0,$$
$$\therefore\ x = 2 \text{ or } 245/142.$$

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Using $x = 2$ in the original equation

$$\text{L.H.S.} = -\sqrt{1} + \sqrt{4} = -1 + 2 = 1;$$
$$\text{R.H.S.} = 3\sqrt{1} = 3;$$
$$\therefore x = 2 \text{ is an extraneous value.}$$

Using $x = 245/142$

$$\text{L.H.S.} = -\sqrt{\left(\frac{490}{142} - 3\right)} + \sqrt{\left(\frac{245}{142} + 2\right)} = -\sqrt{\frac{64}{142}} + \sqrt{\frac{529}{142}}$$
$$= -\frac{8}{\sqrt{142}} + \frac{23}{\sqrt{142}} = \frac{15}{\sqrt{142}};$$
$$\text{R.H.S.} = 3\sqrt{\left(\frac{735}{142} - 5\right)} = 3\sqrt{\frac{25}{142}} = \frac{15}{\sqrt{142}}.$$

Hence, $x = 245/142$ is the required solution.

NOTE. It is possible that all the values obtained, in a question of this nature, are true roots of the equation, although this does not appear to be the case in the worked-out examples.

NOTE. In certain problems the working is made easier by means of a substitution as in the following example.

EXAMPLE. Solve completely the following equations,

(i) $\sqrt{(x^2 - 3x + 6)} - \sqrt{(x^2 - 3x + 3)} = 1,$

(ii) $\sqrt{\left(\frac{x-1}{3x+2}\right)} + 2\sqrt{\left(\frac{3x+2}{x-1}\right)} = 3.$

(i) Here it is to be noted that $x^2 - 3x$ occurs in both of the quantities under the square root signs, and by replacing this by p the equation can be written

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$$\sqrt{(p+6)} - \sqrt{(p+3)} = 1,$$
$$\text{i.e. } \sqrt{(p+6)} = 1 + \sqrt{(p+3)}.$$

Squaring this equation,

$$p + 6 = 1 + 2\sqrt{(p+3)} + p + 3,$$
$$\text{i.e. } 2 = 2\sqrt{(p+3)}, \quad \therefore \quad 1 = \sqrt{(p+3)}.$$

Squaring again,

$$1 = p + 3,$$
$$\therefore p = -2,$$
$$\text{i.e. } x^2 - 3x = -2,$$
$$\therefore x^2 - 3x + 2 = 0,$$
$$\therefore (x-1)(x-2) = 0,$$
$$\therefore x = 1 \text{ or } 2.$$

Check. $x = 1$. L.H.S. $= \sqrt{4} - \sqrt{1} = 2 - 1 = 1$
$x = 2$. L.H.S. $= \sqrt{4} - \sqrt{1} = 2 - 1 = 1$

(ii) In this case it can be seen that

$$\sqrt{\left(\frac{3x+2}{x-1}\right)}$$

is the inverse of

$$\sqrt{\left(\frac{x-1}{3x+2}\right)}$$

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and thus, using $$p = \sqrt{\left(\frac{x-1}{3x+2}\right)}$$

the equation becomes
$$\begin{aligned} p + 2/p &= 3, \\ \text{i.e. } p^2 + 2 &= 3p, \\ \text{i.e. } p^2 - 3p + 2 &= 0, \\ \therefore (p-2)(p-1) &= 0, \\ \therefore p &= 2, \text{ or } 1, \\ \therefore p^2 &= 4, \text{ or } 1. \end{aligned}$$

Using $p^2 = 4$,
$$\begin{aligned} (x-1)/(3x+2) &= 4, \\ \therefore x - 1 &= 12x + 8, \\ \text{i.e. } -9 &= 11x \\ \therefore x &= -9/11. \end{aligned}$$

Using $p^2 = 1$,
$$\begin{aligned} (x-1)/(3x+2) &= 1, \\ \therefore x - 1 &= 3x + 2, \\ \therefore -3 &= 2x \\ \therefore x &= -3/2. \end{aligned}$$

Thus $x = -9/11$ or $-3/2$.

The results can be checked in the *original* equation in the usual manner.

Simultaneous Linear Equations. A linear equation in two or more variables is an equation involving these variables in the first degree only. Thus, a linear equation in the three variables x, y, z will be of the general form $ax + by + cz = d$, where a, b, c, d are constants.

A series of n equations in n variables is said to be *independent* if no single equation of the series can be formed from a combination of the remaining $(n - 1)$ equations.

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Two independent linear equations in x, y, will be required in order to find unique values of x and y; three independent linear equations in x, y, z will be required to find unique values of x, y, and z, and in general n independent linear equations in n variables will be required to find unique values of these variables.

NOTE. The two equations $x - 3y + 2 = 0$, $2x - 6y + 4 = 0$, are not independent as the second equation can be obtained by multiplying the first equation by 2.

The method of solution of n independent equations in n unknowns, as shown in the examples, is to eliminate the variables one by one until an equation in only one variable is obtained.

EXAMPLE (L.U.). Solve the equations

$$x + 4y + 4z = 7 \quad \ldots\ldots (1)$$
$$3x + 2y + 2z = 6 \quad \ldots\ldots (2)$$
$$9x + 6y + 2z = 14 \quad \ldots\ldots (3)$$

It is clear from the equations that it is easiest to eliminate the variable z first.

(3) − (2) gives $$6x + 4y = 8 \quad \ldots\ldots (4).$$
(2) × 2 gives $$6x + 4y + 4z = 12 \quad \ldots\ldots (5).$$

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(5) − (1) gives $5x = 5 \therefore x = 1.$

(5) − (4) gives $4z = 4 \therefore z = 1$

Using $x = 1$ in (4) $6 + 4y = 8 \therefore 4y = 2 \therefore y = \frac{1}{2}.$

Hence the solution is $\left.\begin{aligned} x &= 1 \\ y &= \tfrac{1}{2} \\ z &= 1 \end{aligned}\right\}$

It will be found that these results will check in the original equations (1), (2), and (3).

EXAMPLE. If the equation $y = a + bx + cx^2 + dx^3$ be satisfied by $x = 1$, $y = 2$; $x = 0$, $y = 4$; $x = -1$, $y = 3$; $x = 2$, $y = 0$, find the values of a, b, c, d and the value of y when $x = -2$, and the other values of x when $y = 4$.

Using the *data given* in the question

$$2 = a + b + c + d \quad \ldots\ldots (1)$$
$$4 = a \quad \ldots\ldots (2)$$
$$3 = a - b + c - d \quad \ldots\ldots (3)$$
$$0 = a + 2b + 4c + 8d \quad \ldots\ldots (4)$$

Using $a = 4$ in (1), (2), (3) and (4) these become

$$b + c + d = -2 \quad \ldots\ldots (5),$$
$$-b + c - d = -1 \quad \ldots\ldots (6),$$
$$2b + 4c + 8d = -4$$
$$\text{i.e. } b + 2c + 4d = -2 \quad \ldots\ldots (7).$$

(5) + (6) gives $2c = -3 \therefore c = -3/2.$

Using this value for c in (5) and (7)

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$$b + d = -\tfrac{1}{2} \quad \ldots\ldots (8)$$
$$b + 4d = 1 \quad \ldots\ldots (9)$$

(9) − (8) gives $3d = 3/2 \therefore d = \frac{1}{2}.$

Substituting $d = \frac{1}{2}$ in (8)

$$b + \tfrac{1}{2} = -\tfrac{1}{2} \therefore b = -1.$$

Hence $\left.\begin{aligned} a &= 4 \\ b &= -1 \\ c &= -3/2 \\ d &= \tfrac{1}{2} \end{aligned}\right\}$

Thus $$y = 4 - x - \frac{3x^2}{2} + \frac{x^3}{2}.$$

When $x = -2$, $y = 4 + 2 - 6 - 4 = -4.$

When $y = 4$, $$4 = 4 - x - \frac{3x^2}{2} + \frac{x^3}{2}$$

$$\therefore \tfrac{1}{2}x^3 - \frac{3}{2}x^2 - x = 0$$

$$\text{i.e. } \tfrac{1}{2}x(x^2 - 3x - 2) = 0$$

$$\therefore x = 0 \text{ or } x^2 - 3x - 2 = 0.$$

Excluding the value $x = 0$ the values of x are given by the formula for the solution of a quadratic equation and are

$$x = \frac{3 \pm \sqrt{(9 + 8)}}{2} = \frac{3 \pm \sqrt{17}}{2}.$$

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Simultaneous Quadratic Equations. When there are equations containing n variables it will be necessary to have n of these equations (independent) to find unique values of the variables.

NOTE. A *homogeneous* expression of the second degree in x and y is one in which each term is of the second degree in x and y and will be of the form $ax^2 + bxy + cy^2$, where a, b, c are constants. (A homogeneous expression of the third degree in x and y will be of the form

$$ax^3 + bx^2y + cxy^2 + dy^3$$

and so on.)

There are three different types of simultaneous quadratic equations in x and y to be considered, and also other cases that reduce to simultaneous quadratic equations.

The three types are:

(i) One linear and one quadratic equation.

(ii) Two homogeneous quadratic expressions equal to a constant to form two equations.

(iii) Any other types of quadratic equations.

Type (i). This type is solved as follows:

From the linear equation find the value of one of the variables in terms of the other and substitute this in the quadratic equation, thus obtaining a quadratic equation in a single variable from which two values of that variable can be obtained. The corresponding values of the other variable will now be obtained by using these values in the *linear* equation.

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EXAMPLE (L.U.). Solve the equations

$$3x^2 + 4xy - y^2 = 1 \qquad (1)$$

$$2x + 3y = 1 \qquad (2)$$

From equation (2) $\quad x = (1 - 3y)/2 \qquad (3)$

Substituting for x from (3) in (1).

$$3\frac{(1 - 3y)^2}{4} + 2(1 - 3y)y - y^2 = 1.$$

$$\therefore 3(1 - 6y + 9y^2) + 8y(1 - 3y) - 4y^2 = 4$$

$$\therefore 3 - 18y + 27y^2 + 8y - 24y^2 - 4y^2 = 4$$

$$\therefore y^2 + 10y + 1 = 0$$

$$\therefore y = \frac{-10 \pm \sqrt{(100 - 4)}}{2} = \frac{-10 \pm \sqrt{96}}{2}$$

$$= \frac{-10 \pm 4\sqrt{6}}{2} = -5 \pm 2\sqrt{6}.$$

Substituting these values of y in (3)

$$x = \frac{1 - 3(-5 \pm \sqrt{6})}{2} = \frac{1 + 15 \mp 6\sqrt{6}}{2}$$

$= 8 \mp 3\sqrt{6}$ (NOTE. Signs reversed to those in value of y

$$\therefore \begin{matrix} x = & 8 - 3\sqrt{6} \\ y = & -5 + 2\sqrt{6} \end{matrix} \Bigg\} \quad \text{or} \quad \begin{matrix} 8 + 3\sqrt{6} \\ -5 - 2\sqrt{6} \end{matrix} \Bigg\}$$

Type (ii)

Let the two homogeneous expressions in x and y be

$$a_1x^2 + b_1xy + c_1y^2 \text{ and } a_2x^2 + b_2xy + c_2y^2$$

and the given equations be

$$a_1x^2 + b_1xy + c_1y^2 = d_1 \quad \ldots\ldots(1),$$
$$a_2x^2 + b_2xy + c_2y^2 = d_2 \quad \ldots\ldots(2).$$

Multiply equation (1) by d_2 and equation (2) by d_1 and subtract one result from the other thus obtaining an equation of the form

$$Ax^2 + Bxy + Cy^2 = 0$$

from which two values of one variable can be determined in terms of the other variable.

These values are now substituted in one of the equations (1) or (2) each giving two equations in a single variable, from which, in all, four values of that variable can be found.

To find the corresponding values of the other variable these values are substituted in the *linear equations* which were used to obtain the specific value.

EXAMPLE (L.U.). Solve the equations

(i) $x^2 - 2xy + 8y^2 = 8$, $3xy - 2y^2 = 4$. (ii) $x^2 + y^2 - 3 = 3xy$, $2x^2 + y^2 = 6$.

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(i)

$$x^2 - 2xy + 8y^2 = 8 \quad \ldots\ldots (1)$$
$$3xy - 2y^2 = 4 \quad \ldots\ldots (2)$$

(2) × 2 gives $\quad 6xy - 4y^2 = 8 \quad \ldots\ldots (3)$

(1) − (3) gives $\quad x^2 - 8xy + 12y^2 = 0$

i.e. $(x - 6y)(x - 2y) = 0$

$$\therefore x = 6y \quad \ldots\ldots (4)$$
$$\text{or } x = 2y \quad \ldots\ldots (5)$$

Using (4) in (2) $\quad 18y^2 - 2y^2 = 4 \; \therefore 16y^2 = 4$

$\therefore y^2 = \frac{1}{4}$, and $y = \pm\frac{1}{2}$.

Using $y = \pm\frac{1}{2}$ in (4) $\quad x = \pm 3$.

Using (5) in (2) $6y^2 - 2y^2 = 4$, $\therefore 4y^2 = 4$, $\therefore y^2 = 1$, $\therefore y = \pm 1$.

Substituting $y = \pm 1$ in (5) $\quad x = \pm 2$.

Hence, the complete solution is

$$\left.\begin{array}{l} x = \pm 3 \\ y = \pm\frac{1}{2} \end{array}\right\} \text{ or } \left.\begin{array}{l} \pm 2 \\ \pm 1 \end{array}\right\}$$

These results could be stated more fully in the tabular form

x	$+3$	-3	$+2$	-2
y	$+\frac{1}{2}$	$-\frac{1}{2}$	$+1$	-1

NOTE. The results, when simple, should be checked in the original equations.

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(ii)

$$x^2 - 3xy + y^2 = 3 \quad \dots\dots (1)$$
$$2x^2 + y^2 = 6 \quad \dots\dots (2)$$

(1) × 2 gives $$2x^2 - 6xy + 2y^2 = 6 \quad \dots\dots (3)$$

(3) − (2) gives $$-6xy + y^2 = 0$$
$$\therefore y(y - 6x) = 0$$
$$\therefore y = 0 \quad \dots\dots (4)$$
$$\text{or } y = 6x \quad \dots\dots (5)$$

Using (4) in (1) $\quad x^2 = 3 \;\therefore\; x = \pm\sqrt{3}$.

Using (5) in (1) $\quad x^2 - 18x^2 + 36x^2 = 3$

$$\therefore 19x^2 = 3, \text{ and } x = \pm\sqrt{(3/19)}.$$

Substituting $x = \pm\sqrt{(3/19)}$ in (5) $\quad y = \pm 6\sqrt{(3/19)}$.

Hence, the complete solution is

$$\left.\begin{array}{l} x = \pm\sqrt{3} \\ y = 0 \end{array}\right\}, \text{ or } \left.\begin{array}{l} \pm\sqrt{(3/19)} \\ \pm 6\sqrt{(3/19)} \end{array}\right\}$$

Type (iii)

For this type there are no special methods of approach and the method for two problems will be given.

EXAMPLE. Solve the equations

$$(x - 2)(y - 1) = 3 \quad \dots\dots (1)$$
$$(x + 2)(2y - 5) = 15 \quad \dots\dots (2)$$

From equation (1) $\quad x - 2 = 3/(y - 1)$

$$\therefore x + 2 = \frac{3}{y - 1} + 4 = \frac{3 + 4y - 4}{y - 1} = \frac{4y - 1}{y - 1}.$$

Using this in equation (2) it becomes

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$$\frac{(4y - 1)}{y - 1}(2y - 5) = 15$$
$$\text{i.e. } (4y - 1)(2y - 5) = 15(y - 1)$$
$$\therefore 8y^2 - 22y + 5 = 15y - 15$$
$$\therefore 8y^2 - 37y + 20 = 0$$
$$\text{i.e. } (y - 4)(8y - 5) = 0$$
$$\therefore y = 4 \text{ or } \tfrac{5}{8}.$$

Using these values for y in equation (1),

when $y = 4 \quad (x - 2) \times 3 = 3, \;\therefore\; x - 2 = 1, \;\therefore\; x = 3,$

when $y = \frac{5}{8} \quad (x - 2)(-\frac{3}{8}) = 3, \;\therefore\; x - 2 = -8, \;\therefore\; x = -6.$

Thus the solutions are $\left.\begin{array}{l} x = 3 \\ y = 4 \end{array}\right\}, \text{ or } \left.\begin{array}{l} -6 \\ \frac{5}{8} \end{array}\right\}$

EXAMPLE. Solve the equations

$$2x^2 - xy - y^2 = 8 \quad \dots\dots (1)$$
$$xy = 6 \quad \dots\dots (2)$$

From equation (2) $y = 6/x$, and using this in (1)

$$2x^2 - 6 - \frac{36}{x^2} = 8$$
$$\text{i.e. } 2x^4 - 6x^2 - 36 = 8x^2$$
$$\therefore 2x^4 - 14x^2 - 36 = 0$$
$$\therefore x^4 - 7x^2 - 18 = 0$$
$$\text{i.e. } (x^2 - 9)(x^2 + 2) = 0$$
$$\therefore x^2 = 9 \text{ or } -2$$
$$\text{and } x = \pm 3 \text{ or } \pm\sqrt{-2} \text{ (complex).}$$

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Using these results in equation (2)

$$y = \pm 2 \text{ when } x = \pm 3$$
$$y = \pm \frac{6}{\sqrt{-2}} \text{ when } x = \pm\sqrt{-2}.$$

Hence the complete solution is

$$\left.\begin{aligned} x &= \pm 3 \\ y &= \pm 2 \end{aligned}\right\}, \text{ or } \left.\begin{aligned} &\pm\sqrt{-2} \\ &\pm 6/\sqrt{-2} \end{aligned}\right\}$$

NOTE. These solutions could have also been obtained by using the method of Type (ii) under which category these equations fall.

An example on equations which can be reduced to at least one quadratic equation will demonstrate the method of dealing with simultaneous equations of *higher degree than the second* in two unknowns.

EXAMPLE. Solve the equations

$$x^3 + y^3 = 35 \ldots\ldots\ldots\ldots(1),$$
$$x^2y + xy^2 = 30 \ldots\ldots\ldots\ldots(2),$$

giving only real roots.

Factorising, the equations become

$$(x + y)(x^2 - xy + y^2) = 35 \ldots\ldots\ldots\ldots (3)$$
$$xy(x + y) = 30 \ldots\ldots\ldots\ldots (4)$$

(3) ÷ (4) gives $\dfrac{x^2 - xy + y^2}{xy} = \dfrac{7}{6}$

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$$6x^2 - 6xy + 6y^2 = 7xy,$$
$$\text{i.e. } 6x^2 - 13xy + 6y^2 = 0,$$
$$\therefore (3x - 2y)(2x - 3y) = 0.$$

From this $y = \dfrac{3x}{2} \ldots\ldots\ldots\ldots(5),$

or $y = \dfrac{2x}{3} \ldots\ldots\ldots\ldots (6)$

Using (5) in (1)

$$x^3 + 27x^3/8 = 35,$$
$$\therefore 35x^3/8 = 35, \text{ i.e. } x^3 = 8,$$
$$\therefore x = 2 \text{ (neglecting complex roots).}$$

When $x = 2$ in (5) $y = 3.$

Using (6) in (1) $x^3 + 8x^3/27 = 35$

$$\therefore 35x^3/27 = 35, \text{ and } x^3 = 27.$$
$$\therefore x = 3 \text{ (real root).}$$

When $x = 3$ in (6) $y = 2.$

Hence, the solutions are $\left.\begin{aligned} x &= 3 \\ y &= 2 \end{aligned}\right\}$ or $\left.\begin{aligned} 2 \\ 3 \end{aligned}\right\}$

Equations involving at least *one non-linear equation in three or more unknowns* can only be solved in special cases, and two examples are given to demonstrate the method in particular cases.

EXAMPLE. Solve the equations

$$(y - 2)(z - 1) = 4 \ldots\ldots\ldots\ldots (1)$$

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$$(z - 1)(x + 1) = 20 \qquad (2)$$
$$(x + 1)(y - 2) = 5 \qquad (3)$$

In this example it is to be noted that $(x + 1)$, $(y - 2)$, $(z - 1)$ occur throughout the equations.

Multiplying (1), (2) and (3)

$$(x + 1)^2(y - 2)^2(z - 1)^2 = 400$$
$$\therefore (x + 1)(y - 2)(z - 1) = \pm 20 \qquad (4)$$

(4) ÷ (1) gives $x + 1 = \pm 5, \quad \therefore x = 4 \text{ or } -6$

(4) ÷ (2) gives $y - 2 = \pm 1, \quad \therefore y = 3 \text{ or } 1.$

(4) ÷ (3) gives $z - 1 = \pm 4, \quad \therefore z = 5 \text{ or } -3.$

Hence, the solutions are
$$\left.\begin{array}{l} x = 4 \\ y = 3 \\ z = 5 \end{array}\right\}, \text{ or } \left.\begin{array}{r} -6 \\ 1 \\ -3 \end{array}\right\}$$

EXAMPLE (L.U.). Solve the simultaneous equations

$$xy + x + y = 23 \qquad (1)$$
$$xz + x + z = 41 \qquad (2)$$
$$yz + y + z = 27 \qquad (3)$$

From (1) $\quad y(x + 1) = 23 - x$, i.e. $y = \dfrac{23 - x}{x + 1} \qquad (4)$

From (2) $\quad z(x + 1) = 41 - x$, i.e. $z = \dfrac{41 - x}{x + 1} \qquad (5)$

Substituting results (4) and (5) in equation (3)

$$\frac{(23 - x)(41 - x)}{(x + 1)^2} + \frac{23 - x}{x + 1} + \frac{41 - x}{x + 1} = 27$$

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i.e. $(23 - x)(41 - x) + (23 - x + 41 - x)(x + 1) = 27(x + 1)^2$

$$\text{i.e. } 943 - 64x + x^2 + 64 + 62x - 2x^2 = 27x^2 + 54x + 27$$
$$\therefore 28x^2 + 56x - 980 = 0$$
$$\text{i.e. } x^2 + 2x - 35 = 0$$
$$\therefore (x - 5)(x + 7) = 0$$
$$\therefore x = 5 \text{ or } -7.$$

Using $x = 5$ in (4) and (5) $\quad y = 3$, and $z = 6$.

Using $x = -7$ in (4) and (5) $y = -5$, and $z = -8$.

Thus the complete solution is
$$\left.\begin{array}{l} x = 5 \\ y = 3 \\ z = 6 \end{array}\right\}, \text{ or } \left.\begin{array}{r} -7 \\ -5 \\ -8 \end{array}\right\}$$

A much neater solution of this example is contained in the following.

Adding unity to each side of the equations (1), (2), (3)

$$xy + x + y + 1 = 24, \text{ i.e. } (x + 1)(y + 1) = 24 \qquad (4)$$

and similarly
$$(z + 1)(x + 1) = 42 \qquad (5)$$
$$(y + 1)(z + 1) = 28 \qquad (6)$$

(4) × (5) × (6) gives

$$(x + 1)^2(y + 1)^2(z + 1)^2 = 24 \times 42 \times 28$$
$$\therefore (x + 1)(y + 1)(z + 1) = \pm 168 \qquad (7)$$

(7) ÷ (4) gives $z + 1 = \pm 7$, i.e. $z = 6$ or -8

(7) ÷ (5) gives $y + 1 = \pm 4$, i.e. $y = 3$ or -5

(7) ÷ (6) gives $x + 1 = \pm 6$, i.e. $x = 5$ or -7

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EXAMPLES I

1. If the roots of the equation $x^2 - bx + c = 0$ are α and β, and the roots of the equation $x^2 + \lambda bx + \lambda^2 c = 0$ are γ and δ, show that the equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$ is $x^2 - \lambda b^2 x + 2\lambda^2 c(b^2 - 2c) = 0$.

Show that the roots of this equation are always real.

2. State and prove the condition that the roots of $ax^2 + 2bx + c = 0$ shall be equal.

If the equation $a^2x^2 + 6abx + ac + 8b^2 = 0$ has equal roots, prove that the roots of the equation $ac(x + 1)^2 = 4b^2x$ are also equal.

3. The roots of the equation $x^2 + 2ax + b^2 = 0$ are α_1 and β_1, and those of $x^2 + 2cx + d^2 = 0$ are α_2, β_2.

Prove that (i) if $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, then $a^2 + d^2 = b^2 + c^2$;

(ii) if $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$, then $b^2d^2 = a^2d^2 + b^2c^2$.

4. Prove that the roots of the equation $(a - b - c)x^2 + ax + b + c = 0$ are real if a, b, c are real.

Prove also that, if one root be double the other, then $b + c = a/3$ or $2a/3$.

5. If x be real, show that the expression $y = (x^2 + x + 1)/(x + 1)$ can have no real value between -3 and $+1$.

6. (i) Obtain the conditions satisfied by a, b, c if the expression $ax^2 + bx + c$ is positive for all real values of x. (ii) Show that, for real values of x, the function $(x^2 - \alpha\beta)/(2x - \alpha - \beta)$ cannot assume any values between α and β.

7. What is the condition that the roots of the equation $ax^2 + bx + c = 0$ may be real?

Show that, if the roots of this equation be real, the roots of the equation $(a + c - b)x^2 - 2(a - c)x + (a + c + b) = 0$ are also real.

Prove that, if α, β are the roots of the first equation, the product of the roots of the second equation is $(1 - \alpha)(1 - \beta)/(1 + \alpha)(1 + \beta)$.

8. What do you know about the roots of the quadratic equation

$$x^2 + ax + b = 0$$

if (*a*) $a^2 - 4b > 0$, (*b*) b is negative?

If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are α^4 and β^4.

9. Find expressions for (i) the sum, (ii) the product, of the roots of the equation $ax^2 + bx + c = 0$.

If the difference of the roots of the equation in x

$$1/x + 1/(a - x) = 1/b$$

is equal to c, find a in terms of b and c.

10. (i) Determine the limits to the value of λ so that the equation

$$x^2 - 6x - 1 + \lambda(2x + 1) = 0$$

may have real roots.

(ii) If α, β are the roots of the equation $ax^2 + bx + c = 0$ form the equation whose roots are $1/(\alpha - 4\beta)$, $1/(\beta - 4\alpha)$.

11. Show that the roots of $x^2 - (b + c)x + bc - a^2 = 0$ are real if a, b, c are real. If this equation has coincident roots, what deductions can be made as to the values of a, b, c?

If the roots of $2x^2 + 10bx + c = 0$ are the roots of $x^2 + bx + c = 0$ each increased by 4, find the values of b and c.

12. Obtain the condition for $x^2 + px + q$ to be positive for all real values of x.

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Prove that $5/(2x^2 + 3x + 3)$ is positive for all real values of x and find its greatest value.

Draw a rough graph of the function for values of x between -6 and $+3$.

13. (i) Find the equation whose roots are the squares of the roots of the equation $2x^2 + 3x - 1 = 0$. (ii) Show that, if x be real, the function $(4x - 3)/(x^2 + 1)$ can only take values in a certain range and find this range.

14. Prove that, if one root of the equation $ax^2 + bx + c = 0$ is equal to the square of the other root, then $b^3 = ac(3b - a - c)$.

Find the value of y, if one root of the equation in x,

$$27x^2 + 6x - (y + 2) = 0,$$

is the square of the other root.

15. Obtain expressions for the sum and product of the roots of the quadratic equation $ax^2 + bx + c = 0$ in terms of the coefficients.

If α, β are the roots of $x^2 + px + q = 0$ and α_1, β_1 are the roots of $x^2 + p_1x + q_1 = 0$, express $[(\alpha - \alpha_1)(\alpha - \beta_1)] + [(\beta - \alpha_1)(\beta - \beta_1)]$ in terms of p, q, p_1, q_1.

16. If α, β are the roots of the equation $ax^2 + 2bx + c = 0$, show that $\alpha + \beta = -2b/a$ and $\alpha\beta = c/a$. Find the equation whose roots are $(2\alpha + 3)/(\alpha + 1)$, and $(2\beta + 3)/(\beta + 1)$.

17. If α and β are the roots of $ax^2 + 2bx + c = 0$, show that

$$\alpha + \beta = -2b/a, \ \alpha\beta = c/a.$$

If the above equation has real roots, and if m, n are real numbers such that $m^2 > n > 0$, show that the equation $ax^2 + 2mbx + nc = 0$ also has equal roots.

18. If α and β are the roots of $px^2 + qx + r = 0$, prove that

$$\alpha + \beta = -q/p, \ \alpha\beta = r/p.$$

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If γ and δ are the roots of the equation $qx^2 + rx + p = 0$, prove that

(i) $(\alpha - \gamma)(\alpha - \delta) = (q\alpha^2 + r\alpha + p)/q$,

(ii) $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = (p^3 + q^3 + r^3 - 3pqr)/pq^2$.

Hence, or otherwise, deduce the condition that the equations have a common root.

19. If $ax^2 + 2bx + c = 0$ and $y = x + 1/x$, prove that

$$acy^2 + 2b(c + a)y + (a - c)^2 + 4b^2 = 0.$$

If α and β are the roots of $ax^2 + 2bx + c = 0$, prove that

$$(\alpha + 1/\alpha)^2 + (\beta + 1/\beta)^2 = [4b^2(a^2 + c^2) - 2ac(a - c)^2]/a^2c^2.$$

20. What are the conditions to be satisfied by a, b, c in order that both roots of the equation $ax^2 + bx + c = 0$ may be real and positive?

If this equation have real roots, show that the roots of the equation

$$a^2x^2 + (2ac - b^2)x + c^2 = 0$$

are real and positive.

21. Prove that, if a quadratic equation has real coefficients, the sum of the squares of its roots is always real. Prove the same of the cubes of the roots.

Express in a form with rational coefficients the quadratic equation whose roots are $\frac{3}{4} \pm \frac{1}{2}\sqrt{5}$.

22. Obtain the condition that the roots of the equation $ax^2 + bx + c = 0$ should be real and unequal.

If a, b, n, d are four numbers such that $a + b = ab + n$ and $a - b = d$, express a and b in terms of n and d.

23. Show that the expression $y = ax^2 + 2bx + c$ has always the same sign as a if $ac > b^2$.

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Show that, if $a > 0$, y has one minimum value, and if $a < 0$, y has one maximum value. Determine this value (calculus must not be used).

24. Prove that the expression

$$x^2 - (a + b + c)x + a^2 + b^2 + c^2 + 2bc - ca - ab$$

can never be negative if x, a, b, c are real.

25. The roots of the equation $ax^2 + bx + c = 0$ are α and β, and the roots of the equation $px^2 + qx + r = 0$ are γ and δ. Form an equation whose roots are $\alpha\gamma + \beta\delta$ and $\alpha\delta + \beta\gamma$.

If the roots of this equation are real, prove that $b^2 - 4ac$ and $q^2 - 4pr$ must be both positive or both negative.

26. If $y = (ax^2 + bx + c)/(cx^2 + bx + a)$, where x is real, show that y can have real values between $-\infty$ and $+\infty$, if $b^2 > (a + c)^2$.

27. If α, β are the roots of the equation $x^2 - 7x + 2 = 0$, find, without solving the equation, the value of

$$\text{(i) } \alpha^2 + \beta^2, \quad \text{(ii) } \frac{2 - \alpha}{3 + \beta} + \frac{2 - \beta}{3 + \alpha}.$$

28. Obtain the roots of the equation

$$bcx^2 + [b^2 + c^2 - a(b + c)]x + (a - b)(a - c) = 0.$$

Find the quadratic equation whose roots are the reciprocals of the roots of this equation.

29. α, β are the roots of the quadratic equation $ax^2 + 2bx + c = 0$, and γ, δ are the roots of $a_1x^2 + 2b_1x + c_1 = 0$. Show that if $(\alpha - \gamma)(\beta - \delta) + (\alpha - \delta)(\beta - \gamma) = 0$, then $ac_1 + a_1c = 2bb_1$.

Show also that if $k = (\alpha - \gamma)(\beta - \delta)/(\alpha - \delta)(\beta - \gamma)$, then

$$\frac{k - 1}{k + 1} = \pm 2\frac{\sqrt{[(b^2 - ac)(b_1^2 - a_1c_1)]}}{ac_1 + a_1c - 2bb_1}$$

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30. If β, γ are the roots of the equation $x^2 + p_1x + q_1 = 0$ and γ, α are the roots of $x^2 + p_2x + q_2 = 0$, prove that α, β are the roots of the equation $(q_1 - q_2)^2x^2 + (p_1 - p_2)(q_1^2 - q_2^2)x + q_1q_2(p_1 - p_2)^2 = 0$.

31. If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$ and α_1, β_1 are the roots of $a_1x^2 + b_1x + c_1 = 0$, prove that

$$a^2a_1^2(\alpha - \alpha_1)(\alpha - \beta_1)(\beta - \alpha_1)(\beta - \beta_1) = (ca_1 - c_1a)^2 - (ab_1 - a_1b)(bc_1 - b_1c).$$

Deduce the condition that the two equations may have a common root.

32. Explain how one can, by inspection of the coefficients, without solving the equation $ax^2 + bx + c = 0$, determine (i) whether the roots are real or not; (ii) if real, whether the roots have opposite signs; (iii) the sign of the roots if both have the same sign; and (iv) the sign of the numerically greater root if the signs are different.

Apply these tests to determine the nature of the roots of the equations:

(i) $2x^2 + 4x + 3 = 0$, (ii) $2x^2 + 4x - 3 = 0$,
(iii) $2x^2 - 4x - 3 = 0$, (iv) $x^2 + 6x + 3 = 0$.

33. Prove that, if the difference between the roots of the equation

$$ax^2 - bx + c = 0$$

is the same as the difference between the roots of $bx^2 - cx + a = 0$, then $b^4 - a^2c^2 = 4ab(bc - a^2)$.

Find the condition that the two equations should have a common root.

34. (i) If α, β are the roots of the equation

$$(a + b + c)x^2 + (b + 2c)x + c = 0,$$

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find the equation whose roots are $\alpha/(\alpha + 1)$, $\beta/(\beta + 1)$, expressing the coefficients in terms of a, b, c.

(ii) Find the range of values of y for which both the roots of the equation in x $x^2 - 2(y - 1)x - (y - 3) = 0$ are real and positive.

35. (i) If α and β are the roots of $ax^2 + bx + c = 0$, prove that the equation whose roots are $k\alpha$ and $k\beta$ is $ax^2 + bkx + ck^2 = 0$.

(ii) If the roots of the equation $ax^2 + 8(b - a)x + 4(4a - 8b + c) = 0$ are $(4 - 2\alpha)$ and $(4 - 2\beta)$, find the equation whose roots are α and β.

36. (i) Show that $2x^2 + 3x + 7$ is always positive for real values of x and find its minimum value. (ii) Find the equation whose roots are the squares of the roots of $2x^2 + 3x + 7 = 0$, without solving the equation.

37. Find, without differentiation, the least value of $x^2 - x$.

For what values of x is the expression $x(x^2 - x - 2)(x^2 - x + 1)$ positive?

38. (i) Prove that (*a*) If a and b are rational numbers, the roots of the equation $(a + 2b)x^2 + 2(a - b)x + (a - 4b) = 0$ are rational, and (*b*) If the roots of this equation are equal, b must be zero.

(ii) The roots of the equation $2x^2 + x + 5 = 0$ are α and β, and those of $2x^2 - 3x + 2k = 0$ are $(\alpha + 1)$ and $(\beta + 1)$. Find the value of k.

39. Show that if the equations

$$ax^2 + 2bx + c = 0, \quad a_1x^2 + 2b_1x + c_1 = 0,$$

have a common root, then $4(bc_1 - b_1c)(ab_1 - a_1b) = (ca_1 - c_1a)^2$.

If the equations $kx^2 + 2x + 1 = 0$, $x^2 + 2x + k = 0$ have a common root, find the possible values of k and the value of the common root in each case.

40. Solve the equations (i) $\sqrt{(x - 9)} - \sqrt{(x - 16)} = \sqrt{(x - 24)}$.

(ii) $x/3 - y/2 = -2 = xy + y^2$.

41. Solve the equations (i) $2x - y = 5$, $2x + xy = 2$.

(ii) $\sqrt{(3 - x)} - \sqrt{(7 + x)} = \sqrt{(16 + 2x)}$.

42. If $2^{x^2} = 16^{x-1}$ find x.

43. Solve (i) $5x + 4y - 2z = 2$, $x - 7y + 3z = 3$, $12y + z = -4$. (ii) $x - 2y = 3$, $x^2 + y^2 = 26$

44. Find the *real* roots of the equations

$$\text{(i)}\quad x^2 + \frac{9}{x^2} - 4\left(x + \frac{3}{x}\right) - 6 = 0;$$

$$\text{(ii)}\quad x^2 - 2x + y^2 - 2y = 14, \quad xy = 5.$$

45. If $y = ax(x - 1)(x - 2) + b(x + 1)(x - 1)(x - 2) + c(x + 1)x(x - 2) + d(x + 1)x(x - 1)$,

and if the corresponding values of x and y are given by the table

x	-1	0	1	2
y	36	8	-4	15,

find the values of a, b, c, d, and hence find the value of y when $x = \frac{1}{2}$.

46. If $x + y = u$, $xy = v$, express $x^4 + y^4$ in terms of u and v. (Hint: consider $(x + y)^4$.)

Find all the real solutions of the equations $x^4 + y^4 = 641$, $x + y = 7$.

47. Solve the equations

$$\text{(i)}\quad \frac{x - 1}{x + 1} + \frac{x + 2}{x - 2} = 2 + \frac{2}{x + 4}$$

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$$\left(\text{Hint: } \frac{x-1}{x+1} = 1 - \frac{2}{x+1}, \quad \frac{x+2}{x-2} = 1 + \frac{4}{x-2}\right)$$

(ii) $2x^2 + 3xy + 4y^2 = 1, \quad 4x^2 + 3xy + 2y^2 = 2.$

48. Solve the simultaneous equations

$$yz = 2(y + z - 6),$$
$$zx = 2(z + x - 4),$$
$$xy = 2(x + y - 1).$$

49. Solve the equations

(i) $$\frac{x+7}{x+5} + \frac{x+9}{x+7} = \frac{x+6}{x+4} + \frac{x+10}{x+8}.$$

(Simplify as in example 47(i).)

(ii) $x(3y - 5) = 4, \quad y(2x + 7) = 27.$

50. (i) Solve the equation $\sqrt{(x-1)} + \sqrt{(x+4)} = \sqrt{(3x+10)}$, where each term is real.

(ii) Solve the equations $xy = 1$, $yz = 9$, $zx = 16$, and deduce the solution of the equations

$$(y+z)(z+x) = 1, \quad (z+x)(x+y) = 9, \quad (x+y)(y+z) = 16.$$

51. (i) Simplify $$\frac{2+\sqrt{3}}{x-1+\sqrt{3}} + \frac{2-\sqrt{3}}{x-1-\sqrt{3}} + \frac{10}{x^2-2x-2}.$$

(ii) Solve the equation $$6\sqrt{\left(\frac{2x}{x-1}\right)} + 5\sqrt{\left(\frac{x-1}{2x}\right)} = 13.$$

52. Solve the equations

(i) $\sqrt{(4x-2)} - 2\sqrt{(x-3)} = 2.$

(ii) $x^2 + 3xy - 23x + 8y = 0, \quad x - y + 1 = 0.$

53. Solve for x, y, z the equations

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$$y(z-x) = 3; \quad x(y+z) = 32; \quad x + y + z = 12.$$

54. Solve the equations

(i) $x^2 - xy + 3y^2 = 15, \quad 3x^2 - 2y^2 = -5.$

(ii) $$x(x+1) + \frac{12}{x(x+1)} = 8.$$

55. Find all values of x and y which satisfy the simultaneous equations

$$x^2 - 2xy - y^2 = 14, \quad 2x^2 + 3xy + y^2 = -2.$$

56. Solve the equations

(i) $\sqrt{(8x+1)} = \sqrt{(2x+3)} - \sqrt{(x+1)}.$

(ii) $x + y = 1, \quad 2(x^2 + y^2) = 17.$

57. Solve the equations

(i) $xy - 3x - 3y + 12 = 0, \quad 2xy + 4x + 4y = 56.$

(ii) $2^x + 3^y = 44, \quad 2^{x+2} + 3^{y+2} = 221.$

Giving the answers as correctly as the tables will allow.

58. (i) Solve the equation $\sqrt{(1 + x + x^2)} + \sqrt{(1 - x + x^2)} = 3.$

(ii) If $x + y + z = 0$, $ax + by + cz = 0$, and

$$x^3 + y^3 + z^3 = 3(b-c)(c-a)(a-b),$$

prove that $$\frac{x}{b-c} = \frac{y}{c-a} = \frac{z}{a-b} = 1.$$

59. (i) Solve the equation $6(x^2 + 1/x^2) + 5(x + 1/x) - 38 = 0.$

(ii) If $z = x - y$, obtain from the equations $xy - x + y = 13$, $x^2 + y^2 = 25$, a quadratic in z, and thence find the values of x and y satisfying the equations.

60. (i) If $yz = a^2$, $zx = b^2$, $xy = c^2$, show that $a^2x = b^2y = c^2z = \pm abc.$

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(ii) Solve the equations $x(y+z) = 33$, $y(z+x) = 35$, $z(x+y) = 14$.

61. (i) Solve the equation

$$\sqrt{\left(\frac{a+x}{b+x}\right)} - \sqrt{\left(\frac{b+x}{a+x}\right)} = \frac{3}{2}.$$

(ii) If $y/z + z/y = a$, $z/x + x/z = b$, $x/y + y/x = c$, prove that $a^2 + b^2 + c^2 = abc + 4$.

62. (i) Solve the equation

$$\sqrt{(2^{x+2} + 5)} + \sqrt{(5 \cdot 2^x - 9)} = \sqrt{(3 \cdot 2^{x+2} + 21)}.$$

(ii) Solve the simultaneous equations

$$x(y-c) = p, \quad (y-c)(z+c) = q, \quad (z+c)x = r.$$

63. Solve for x and y the equations

$$\left\{\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c}\right\}\left\{\frac{x}{b-c} + \frac{y}{c-a} - \frac{1}{a-b}\right\}$$

$$= \left\{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right\}(x+y-1) = \frac{x}{a} + \frac{y}{b} - \frac{1}{c}.$$

64. Solve the equations

(i) $\sqrt{(3x-5)} - \sqrt{(2x-5)} = 1$.
(ii) $x^2 + y^2 = 13$, $3x^2 + xy + y^2 = 15$.

65. Solve the equations

(i) $x^2 + xy + y^2 = 7$, $x^2 - xy + y^2 = 13$.
(ii) $\sqrt{(x^2 - 5x)} - \sqrt{(x^2 - 5x - 20)} = 2$.

66. Solve the simultaneous equations $x(14 - y) = 2$, $2x + y = 11$. Prove that there are two pairs of values of x and y that satisfy the equations

$$(a-y)x + h = 0, \quad hx + b - y = 0,$$

and that these values are real if a, b and h are real.

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67. (i) Solve the simultaneous equations $x(y+3) = 4$, $3y(x-4) = 5$.

(ii) If p and q are real and not zero, find the condition that the roots of the equation $2p^2x^2 + 2pqx + q^2 - 3p^2 = 0$ are real.

If the roots of this equation are α and β, prove that $\alpha^2 + \beta^2$ is independent of p and q.

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CHAPTER II

Theory of Indices, Square Roots of Surds, Logarithms, etc.

The Theory of Indices. Earlier work will have confirmed the following laws of indices (m and n being positive integers):

Law I. $$a^m \times a^n = a^{m+n}.$$

Extension $$a^m \times a^n \times a^p \times \ldots = a^{m+n+p+\ldots}.$$

Law II. $$a^m \div a^n = a^{m-n}. \qquad (m > n)$$

Law III. $$(a^m)^n = a^{mn}.$$

Extension $$\left(\frac{a^m b^n \times \ldots}{c^p \times \ldots}\right)^q = \frac{a^{mq} b^{nq} \times \ldots}{c^{pq} \times \ldots}.$$

It is now assumed that these results are valid for *all* values of m, n, etc., and on this assumption it will be necessary to find meanings for a^0, a^{-m}, $a^{m/n}$ (where m, n are positive integers or negative integers).

Theorem I. *To find a meaning for* a^0.

By the second law of indices $a^m/a^m = a^{m-m} = a^0$.

But $$a^m/a^m = 1$$
$$\therefore\ a^0 = 1.$$

Theorem II. *To find a meaning for* a^{-m} *where* m *is a positive rational number.*

By the first law of indices

$$a^m \times a^{-m} = a^{m-m} = a^0.$$

But $$a^0 = 1,$$
$$\therefore\ a^m \times a^{-m} = 1$$
$$\therefore\ a^{-m} = 1/a^m.$$

Thus $$2^{-2} = \frac{1}{2^2} = \tfrac{1}{4};$$

$$(\tfrac{1}{2})^{-3} = \frac{1}{(\frac{1}{2})^3} = \frac{1}{\frac{1}{8}} = 8;$$

$$(-5)^{-3} = \frac{1}{(-5)^3} = \frac{1}{-125} = -\frac{1}{125};$$

$$7^{-\frac{1}{2}} = 1/7^{\frac{1}{2}}.$$

Theorem III. *To find a meaning for* $a^{m/n}$, *where* m *and* n *are any integers (positive or negative).*

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By the third law of indices $(a^{m/n})^n = a^m$

$$\therefore\ a^{m/n} = \sqrt[n]{a^m}.$$

When $m = 1$, the result becomes

$$a^{1/n} = \sqrt[n]{a}.$$

Thus
$$4^{\frac{1}{2}} = \sqrt{4} = 2;$$

$$\left(\frac{27}{8}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{27}{8}} = \frac{3}{2};$$

$$16^{-\frac{1}{4}} = \frac{1}{16^{\frac{1}{4}}} = \frac{1}{\sqrt[4]{16}} = \quad .$$

NOTE. When dealing with the multiplication or division of expressions involving fractional or negative indices the methods used are the same as when dealing with expressions involving positive integral indices.

EXAMPLE. Simplify and express with positive indices

$$\sqrt{(x^{2/3}y^{1/2})} \times x^{-2/3}y^{3/2} \times \sqrt[3]{(x^{3/4}y^{-1/2})}.$$

$$\begin{aligned}
\text{The given expression} &= (x^{2/3}y^{1/2})^{1/2} \times x^{-2/3}y^{3/2} \times (x^{3/4}y^{-1/2})^{1/3} \\
&= x^{1/3}y^{1/4} \times x^{-2/3}y^{3/2} \times x^{1/4}y^{-1/6} \\
&= x^{1/3-2/3+1/4}y^{1/4+3/2-1/6} \\
&= x^{-1/12}y^{19/12} = y^{19/12}/x^{1/12}.
\end{aligned}$$

EXAMPLE. Obtain the value of

$$(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})(a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}} + c^{\frac{1}{2}})(b^{\frac{1}{2}} + c^{\frac{1}{2}} - a^{\frac{1}{2}})$$

$$\begin{aligned}
\text{The given expression} &= [(a^{\frac{1}{2}} + b^{\frac{1}{2}}) + c^{\frac{1}{2}}][(a^{\frac{1}{2}} + b^{\frac{1}{2}}) - c^{\frac{1}{2}}] \\
&\qquad \times [c^{\frac{1}{2}} + (a^{\frac{1}{2}} - b^{\frac{1}{2}})][c^{\frac{1}{2}} - (a^{\frac{1}{2}} - b^{\frac{1}{2}})] \\
&= [(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - c][c - (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2] \\
&\qquad \text{(using the difference of two squares)} \\
&= [(a + b - c) + 2a^{\frac{1}{2}}b^{\frac{1}{2}}][2a^{\frac{1}{2}}b^{\frac{1}{2}} - (a + b - c)] \\
&= (2a^{\frac{1}{2}}b^{\frac{1}{2}})^2 - (a + b - c)^2 \\
&= 4ab - (a^2 + b^2 + c^2 + 2ab - 2bc - 2ca) \\
&= 2bc + 2ca + 2ab - a^2 - b^2 - c^2.
\end{aligned}$$

EXAMPLE. Divide $a^{5/2} - 5a^2b^{1/3} + 10a^{3/2}b^{2/3} - 10ab + 5a^{1/2}b^{4/3} - b^{5/3}$ by $a^{1/2} - b^{1/3}$.

$$a^2 - 4a^{3/2}b^{1/3} + 6ab^{2/3} - 4a^{1/2}b + b^{4/3}$$

$$a^{1/2} - b^{1/3}\ \big)\ a^{5/2} - 5a^2b^{1/3} + 10a^{3/2}b^{2/3} - 10ab + 5a^{1/2}b^{4/3} - b^{5/3}$$

$$a^{5/2} - a^2b^{1/3}$$

$$- 4a^2b^{1/3} + 10a^{3/2}b^{2/3}$$

$$- 4a^2b^{1/3} + 4a^{3/2}b^{2/3}$$

$$6a^{3/2}b^{2/3} - 10ab$$

$$6a^{3/2}b^{2/3} - 6ab$$

$$- 4ab + 5a^{1/2}b^{4/3}$$

$$- 4ab + 4a^{1/2}b^{4/3}$$

$$+ a^{1/2}b^{4/3} - b^{5/3}$$

$$a^{1/2}b^{4/3} - b^{5/3}$$

$$\cdot \qquad \cdot$$

Quotient is
$$a^2 - 4a^{3/2}b^{1/3} + 6ab^{2/3} - 4a^{1/2}b + b^{4/3}.$$

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Surds and Square Roots of Surds. As stated previously, a rational number is any number that can be expressed in the form p/q $[q \neq 0]$, where p and q are integers, whilst any number, such as π, $\sqrt{2}$, $\sqrt[3]{5}$, etc., that is a real number which cannot be expressed in this form, is known as an irrational number.

Real numbers can also be considered as numbers that lie on the number scale, which is a straight line that can be extended to infinity in both directions with some convenient point on it chosen as the zero point, i.e. the point representing zero, and a suitable length is chosen for unity. *Infinity* (symbol ∞) is a number beyond the bounds of computation.

Surd is usually used to denote an expression involving $\sqrt[n]{a}$, where a is not a perfect nth power of a rational number. Thus $2 + \sqrt{3}$, $\sqrt{2} + \sqrt[3]{4}$, $\sqrt{5} - 2\sqrt{7} + 3\sqrt{2}$ are surds, but $\sqrt[3]{8} + \sqrt{9}$ is not a surd as it can be simplified to $2 + 3 = 5$.

When calculating a number such as $4/\sqrt{3}$ using the value of $\sqrt{3}$ from tables, it is advisable to multiply both numerator and denominator by $\sqrt{3}$ giving $4\sqrt{3}/3$

$$= 4 \times 1{\cdot}7321/3 \text{ (using tables)}$$
$$= 2{\cdot}309 \text{ to three decimal places.}$$

This method is known as *rationalising* the denominator.

If $\sqrt{a} + \sqrt{b}$ represent a surdic quantity, then $\sqrt{a} - \sqrt{b}$ is known as its *conjugate surd* (and vice versa), and the product of the two is $(a - b)$ which is a rational number providing a and b are rational.

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When simplifying a fraction involving a surdic denominator, it is necessary to multiply numerator and denominator by the conjugate surd of the denominator, thus obtaining a rational denominator.

Example. Simplify (i) $3/(\sqrt{3} + \sqrt{2})$, (ii) $\dfrac{\sqrt{5} - 2}{\sqrt{5} + 2} + \dfrac{\sqrt{5} + 2}{\sqrt{5} - 2}$.

(i) $$\frac{3}{\sqrt{3} + \sqrt{2}} = \frac{3(\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = \frac{3(\sqrt{3} - \sqrt{2})}{3 - 2} = 3(\sqrt{3} - \sqrt{2}).$$

(ii) $$\frac{\sqrt{5} - 2}{\sqrt{5} + 2} + \frac{\sqrt{5} + 2}{\sqrt{5} - 2} = \frac{(\sqrt{5} - 2)^2}{(\sqrt{5} + 2)(\sqrt{5} - 2)} + \frac{(\sqrt{5} + 2)^2}{(\sqrt{5} - 2)(\sqrt{5} + 2)}$$
$$= \frac{5 - 4\sqrt{5} + 4}{5 - 4} + \frac{5 + 4\sqrt{5} + 4}{5 - 4} = 18.$$

Note. In general when dealing with surdic quantities only square roots will be encountered, but it is possible for cube and higher roots to occur. In the case of cube roots, use is made of the results

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2),$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Replacing x by $a^{\frac{1}{3}}$, y by $b^{\frac{1}{3}}$ these become

$$a + b = (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}),$$
$$a - b = (a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}).$$

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Thus if a and b be rational the conjugate surd of $a^{\frac{1}{3}} + b^{\frac{1}{3}}$ is $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$, and that of $a^{\frac{1}{3}} - b^{\frac{1}{3}}$ is $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

EXAMPLE. Simplify $\dfrac{2 - \sqrt[3]{5}}{2 + \sqrt[3]{5}}$.

$$\begin{aligned}\frac{2 - \sqrt[3]{5}}{2 + \sqrt[3]{5}} &= \frac{(2 - \sqrt[3]{5})(4 - 2\sqrt[3]{5} + \sqrt[3]{25})}{(2 + \sqrt[3]{5})(4 - 2\sqrt[3]{5} + \sqrt[3]{25})} \\ &= \frac{8 - 4\sqrt[3]{5} + 2\sqrt[3]{25} - 4\sqrt[3]{5} + 2\sqrt[3]{25} - 5}{2^3 + (\sqrt[3]{5})^3} \\ &= \frac{3 - 8\sqrt[3]{5} + 4\sqrt[3]{25}}{8 + 5} = \frac{3 - 8\sqrt[3]{5} + 4\sqrt[3]{25}}{13}.\end{aligned}$$

NOTE. An example will show the method of procedure when the denominator consists of more than two terms.

EXAMPLE. Simplify $\dfrac{2 - 2\sqrt{2} + \sqrt{5}}{2 - \sqrt{2} - \sqrt{5}}$.

$$\begin{aligned}&\frac{2 - 2\sqrt{2} + \sqrt{5}}{2 - \sqrt{2} - \sqrt{5}} \\ &= \frac{(2 - 2\sqrt{2} + \sqrt{5})(2 + \sqrt{2} + \sqrt{5})}{[2 - (\sqrt{2} + \sqrt{5})][2 + (\sqrt{2} + \sqrt{5})]} \\ &= \frac{4 - 4\sqrt{2} + 2\sqrt{5} + 2\sqrt{2} - 4 + \sqrt{10} + 2\sqrt{5} - 2\sqrt{10} + 5}{4 - (2 + 5 + 2\sqrt{10})} \\ &= \frac{5 - 2\sqrt{2} + 4\sqrt{5} - \sqrt{10}}{-(3 + 2\sqrt{10})} \\ &= \frac{(5 - 2\sqrt{2} + 4\sqrt{5} - \sqrt{10})(3 - 2\sqrt{10})}{-(3 + 2\sqrt{10})(3 - 2\sqrt{10})} \\ &= \frac{15 - 6\sqrt{2} + 12\sqrt{5} - 3\sqrt{10} - 10\sqrt{10} + 4\sqrt{20} - 8\sqrt{50} + 20}{-(9 - 40)} \\ &= \frac{15 - 6\sqrt{2} + 12\sqrt{5} - 3\sqrt{10} - 10\sqrt{10} + 8\sqrt{5} - 40\sqrt{2} + 20}{31} \\ &= \frac{35 - 46\sqrt{2} + 20\sqrt{5} - 13\sqrt{10}}{31}.\end{aligned}$$

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Theorem. *If the two surdic quantities* $(\sqrt{a} + b)$ *and* $(\sqrt{c} + d)$ *are equal, then* $a = c$ *and* $b = d$, *where* a, b, c, d *are rational and* a *and* c *are not perfect squares.*

The product of zero and *any* finite number is zero. Thus it is clear, from the nature of rational and irrational numbers, that a rational number can only equal an irrational number if each be zero, which is a neutral number (i.e. can be considered as being either rational or irrational).

Since
$$\begin{aligned}\sqrt{a} + b &= \sqrt{c} + d \\ b - d &= \sqrt{c} - \sqrt{a}\end{aligned}$$

i.e. a rational number = an irrational number, which is impossible unless each be zero.

$$\therefore\ b - d = 0 \quad \text{and} \quad \sqrt{c} - \sqrt{a} = 0$$
$$\text{i.e. } b = d \quad \text{and} \quad \sqrt{c} = \sqrt{a}$$
$$\therefore\ b = d \quad \text{and} \quad c = a.$$

Theorem. *To find the square roots of the surdic quantities* (i) $a + \sqrt{b}$ *and* (ii) $a - \sqrt{b}$, *where a and b are rational and b is not a perfect square.*

Let $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$ (x, y rational).

$$\text{Squaring } a + \sqrt{b} = x + y + 2\sqrt{xy}.$$

From the previous theorem

$$a = x + y \quad \ldots\ldots\ldots\ldots (1)$$
$$b = 4xy \quad \ldots\ldots\ldots\ldots (2)$$

The equations (1) and (2) can be solved for x and y, in any particular case, by inspection or by ordinary algebraic means.

(ii). Let $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}$.

$$\text{Squaring } a - \sqrt{b} = x + y - 2\sqrt{xy},$$

and as before $a = x + y$, $b = 4xy$, from which x and y can be found as in (i).

EXAMPLE. Find the square roots of (i) $17 + 12\sqrt{2}$, (ii) $22 - 12\sqrt{2}$.

(i) Let $\sqrt{(17 + 12\sqrt{2})} = \sqrt{x} + \sqrt{y}$, where x, y are rational.

Squaring $17 + 12\sqrt{2} = x + y + 2\sqrt{xy}$

Hence, $17 = x + y \quad \ldots\ldots\ldots\ldots (1)$

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$6\sqrt{2} = \sqrt{xy} \quad \therefore\ 72 = xy \quad \ldots\ldots\ldots\ldots (2)$

From equations (1) and (2), two numbers are required whose sum is 17 and product 72 and by inspection these are seen to be 9 and 8.

$$\therefore\ \sqrt{(17 + 12\sqrt{2})} = \sqrt{9} + \sqrt{8} = 3 + 2\sqrt{2}.$$

(ii) Let $\sqrt{(22 - 12\sqrt{2})} = \sqrt{x} - \sqrt{y}$. $(x > y)$

Squaring $22 - 12\sqrt{2} = x + y - 2\sqrt{xy}$.

$$\therefore\ 22 = x + y \quad \ldots\ldots\ldots\ldots (1)$$
$$6\sqrt{2} = \sqrt{xy}, \text{ i.e. } 72 = xy \quad \ldots\ldots\ldots\ldots (2)$$

From (2), $y = 72/x$.
Using this in equation (1)

$$22 = x + 72/x$$
$$\text{i.e. } x^2 - 22x + 72 = 0,$$
$$\therefore\ (x - 18)(x - 4) = 0,$$
$$\therefore\ x = 4 \text{ or } 18.$$

From equation (2) using these values $y = 18$ or 4.
But $x > y$ $\therefore\ x = 18$ and $y = 4$.
Hence, $\sqrt{(22 - 12\sqrt{2})} = \sqrt{18} - \sqrt{4} = 3\sqrt{2} - 2$.

Logarithms

Definition. If $y = a^x$, then x is said to be the *logarithm* of y to the base a.

This can be written; if $y = a^x$, then $x = \log_a y$.

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Thus, since $8 = 2^3$, $\log_2 8 = 3$;

since $81 = 3^4$, $\log_3 81 = 4$;

since $\frac{1}{16} = \frac{1}{4^2} = 4^{-2}$, $\log_4 \frac{1}{16} = -2$.

This definition can also be stated in the following form:

The logarithm of y to the base a is the power to which a must be raised in order to give the value y.

When the base is 10, the logarithms are known as *common* logarithms, and when the base is e ($= 2{\cdot}71828\ldots$), the logarithms are known as *Natural or Napierian logarithms.*

From the definition of a logarithm it can be seen that $a^{\log_a y} = y$. (Let $N = a^{\log_a y}$, then $\log_a N = \log_a y$, therefore $N = y$.) Thus

$$5^{\log_5 3} = 3, \quad 7^{\log_7 11} = 11.$$

Using the above second definition of a logarithm, it follows that

$$\log_2 16 = \log_2 2^4 = 4,$$
$$\log_3 1/9 = \log_3 3^{-2} = -2,$$
$$\log_5 0{\cdot}04 = \log_5 1/25 = \log_5 5^{-2} = -2.$$

Theorem. *To prove that* $\log_a (M \times N) = \log_a M + \log_a N$.

Let $M = a^x$ .... (1)

$\therefore \log_a M = x.$

Let $N = a^y$ .... (2)

$\therefore \log_a N = y.$

From (1) $\times$ (2) $M \times N = a^x \times a^y = a^{x+y}$ (first law of indices)

$\therefore \log_a M \times N = x + y = \log_a M + \log_a N.$

Theorem. *To prove that* $\log_a M/N = \log_a M - \log_a N$.

Using the same notation as in the previous theorem

$M/N = a^x/a^y = a^{x-y}$ (second law of indices)

$\therefore \log_a M/N = x - y = \log_a M - \log_a N.$

Theorem. *To prove that* $\log_a M^p = p \log_a M$.

Let $M = a^x$ .... (1)

$\therefore \log_a M = x.$

From (1) $M^p = (a^x)^p = a^{xp}$ (third law of indices)

$\therefore \log_a M^p = px$

$= p \log_a M.$

NOTE. These three theorems can be combined so as to give the logarithm of an expression including products, quotients, and powers of quantities.

Theorem. *To prove that* $\log_b M = \log_a M/\log_a b$.

Let $M = b^x$ .... (1)

$\therefore \log_b M = x.$

Taking logarithms to the base a in (1)

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$$\log_a M = \log_a b^x = x \log_a b.$$
$$\therefore x = \log_a M / \log_a b$$
$$\text{i.e. } \log_b M = \log_a M / \log_a b.$$

Using $M = a$ in this result

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}.$$

This last result (i.e. $\log_b M = \log_a M / \log_a b$) can be used for converting the logarithm to any base b to a common logarithm (quotient) by putting $a = 10$, in order to find its value using tables of common logarithms.

EXAMPLE. Find, using tables, the value of $\log_2 5$.

Using the previous theorem with $b = 2$, $m = 5$, $a = 10$

$$\log_2 5 = \log_{10} 5 / \log_{10} 2$$
$$= \frac{0{\cdot}69897}{0{\cdot}30103} = 2{\cdot}322 \text{ to 4 significant figures.}$$

NOTE. When dealing with five-figure tables the *final* result can only be taken to *four* significant figures.

EXAMPLE. Find, without using tables, the values of (i) $5^{-3 \log_5 2}$, (ii) $a^{4 \log_a \frac{1}{3}}$.

(i) Using the theory of logarithms

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$$5^{-3 \log_5 2} = 5^{\log_5 2^{-3}} = 5^{\log_5 1/8} = 1/8.$$

(ii) $$a^{4 \log_a 1/3} = a^{\log_a (1/3)^4} = a^{\log_a 1/81} = 1/81.$$

EXAMPLE. Solve the equations

(i) $$\log_a (5x - 6) + \log_a (2x + 3) = \log_a (10x^2 - 3x - 5)$$
(ii) $$2 \log_a (x - 2) = \log_a (2x - 5)$$

(i) Using the theory of logarithms the equation can be written

$$\log_a (5x - 6)(2x + 3) = \log_a (10x^2 - 3x - 5),$$
$$\therefore (5x - 6)(2x + 3) = 10x^2 - 3x - 5,$$
$$\text{i.e. } 10x^2 + 3x - 18 = 10x^2 - 3x - 5,$$
$$\therefore 6x = 13, \text{ i.e. } x = 13/6.$$

(ii) The equation can be written

$$\log_a (x - 2)^2 = \log_a (2x - 5)$$
$$\therefore (x - 2)^2 = 2x - 5$$
$$\therefore x^2 - 4x + 4 = 2x - 5$$
$$\text{i.e. } x^2 - 6x + 9 = 0$$
$$\therefore (x - 3)^2 = 0, \quad \therefore x = 3 \text{ (twice).}$$

Indicial Equations. Any equation involving the variable (or variables) in the indices of the numbers present is known as an *indicial equation.*

There are two main types of indicial equations to be considered as follows:

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(i) the straightforward cases which can be solved by taking logarithms to the base 10.

(ii) those which can be reduced to a quadratic in y, where $y = a^{f(x)}$ and $f(x)$ means any expression involving x (*function* of x).

NOTE. No base is shown in the case of common logarithms.

EXAMPLES (on type (i)).

(*a*) Solve the equation $2^x \times 5^{x-1} = 8^{2x+1}$.

Taking logarithms to the base 10 for the equation

$$\log 2^x + \log 5^{x-1} = \log 8^{2x+1}$$
$$\text{i.e. } x \log 2 + (x-1)\log 5 = (2x+1)\log 8$$
$$\text{i.e. } x\log 2 + x\log 5 - 2x\log 8 = \log 8 + \log 5$$
$$\therefore\ x[\log 2 + \log 5 - 2\log 8] = \log 8 + \log 5$$
$$\text{i.e. } x \log (2 \times 5)/8^2 = \log (8 \times 5)$$
$$\therefore\ x = (\log 40)/\log \tfrac{5}{32}$$
$$= (\log 40)/(\log 5 - \log 32)$$
$$= 1{\cdot}60206/(0{\cdot}69897 - 1{\cdot}50515)$$
$$= -1{\cdot}60206/0{\cdot}80618$$
$$\therefore\ x = -1{\cdot}987 \text{ to 4 significant figures.}$$

(*b*) The pulls T_1 and T_2 at the ends of a taut rope which passes round a circular post are related by $T_2 = T_1 e^{\mu\theta}$, where $e = 2{\cdot}718$, $\mu = 0{\cdot}22$ and θ radians is the angle through which the rope turns.

Find (i) θ given that $T_2 = 3T_1$,

(ii) T_1 and T_2 given that $\theta = 3{\cdot}1$ and $T_2 - T_1 = 745$ lb. wt.

(i)
$$T_2 = 3T_1 \quad \therefore\ 3T_1 = T_1 \times 2{\cdot}718^{0\cdot 22\theta}$$
$$\text{i.e. } 2{\cdot}718^{0\cdot 22\theta} = 3.$$

Taking logarithms to the base 10, and using four-figure tables, since e is only given to four significant figures

$$0{\cdot}22\theta \log 2{\cdot}718 = \log 3$$
$$\text{i.e. } \theta = \frac{\log 3}{0{\cdot}22 \log 2{\cdot}718} = \frac{0{\cdot}4771}{0{\cdot}22 \times 0{\cdot}4343} = \frac{0{\cdot}4771}{0{\cdot}09555}$$
$$= 5{\cdot}00 \text{ to 3 significant figures.}$$

NOTE. Three significant figures only are used in the result since four-figure tables are used.

(ii)
$$T_2 - T_1 = 745 \quad \ldots\ldots\ldots\ldots (1)$$
$$T_2 = T_1 e^{0\cdot 22 \times 3\cdot 1} = T_1 e^{0\cdot 682} \quad \ldots\ldots\ldots\ldots (2)$$

Using equation (2) in equation (1) $T_1 e^{0\cdot 682} - T_1 = 745$.

Let
$$x = e^{0\cdot 682} = 2{\cdot}718^{0\cdot 682}$$
$$\therefore\ \log x = 0{\cdot}682 \log 2{\cdot}718$$
$$= 0{\cdot}682 \times 0{\cdot}4343 = 0{\cdot}2962$$
$$\therefore\ x = 1{\cdot}978.$$

Hence,
$$1{\cdot}978T_1 - T_1 = 745$$
$$\text{i.e. } 0{\cdot}978T_1 = 745$$
$$\therefore\ T_1 = 745/0{\cdot}978 = 762 \text{ lb. wt. to 3 significant figures.}$$

Using this result in equation (1)

$$T_2 = T_1 + 745 = 762 + 745 = 1507 \text{ lb. wt.}$$

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(c) (L.U.). If $y = a + bx^n$ is satisfied by the values $x = 2$, $y = 10$; $x = 4$, $y = 15$; $x = 8$, $y = 27$, show that $n = \log 2{\cdot}4 / \log 2$ and deduce the values of a and b.

Using the given data in $y = a + bx^n$,

$$10 = a + b \cdot 2^n \quad \text{.......(1)},$$
$$15 = a + b \cdot 4^n, \quad \text{i.e. } 15 = a + b \cdot 2^{2n} \quad \text{.......(2)},$$
$$27 = a + b \cdot 8^n, \quad \text{i.e. } 27 = a + b \cdot 2^{3n} \quad \text{.......(3)}.$$

(2) − (1) gives $5 = b(2^{2n} - 2^n)$, i.e. $5 = b \cdot 2^n(2^n - 1)$ ....(4)

(3) − (2) gives $12 = b(2^{3n} - 2^{2n})$, i.e. $12 = b \cdot 2^{2n}(2^n - 1)$ ....(5)

(5) ÷ (4) gives $\dfrac{2^{2n}}{2^n} = \dfrac{12}{5}$, i.e. $2^n = 2{\cdot}4$ ....(6)

Taking logarithms to the base 10 (could use any base),

$$n \log 2 = \log 2{\cdot}4$$
$$\therefore n = (\log 2{\cdot}4) / \log 2.$$

Using equation (6) in equation (4)

$$5 = b \times 2{\cdot}4 \times 1{\cdot}4 = 3{\cdot}36b$$
$$\therefore b = 5/3{\cdot}36 = 125/84 = 1{\cdot}488 \text{ (to 4 significant figures).}$$

Using equation (6) in equation (1) and this value for b

$$10 = a + \frac{125}{84} \times 2{\cdot}4 = a + \frac{25}{7}$$
$$\therefore a = 10 - 25/7 = 45/7 = 6{\cdot}429 \text{ (to 4 significant figures).}$$

EXAMPLE (type (ii)). Solve the equations

$$(a) \quad 4^x - 6 \times 2^x - 7 = 0,$$
$$(b) \quad 3^{2x} - 3^{x+1} + 2 = 0.$$

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(a) If 2^x be replaced by y the equation becomes

$$y^2 - 6y - 7 = 0, \qquad (\text{since } 4^x = 2^{2x} = y^2)$$
$$\therefore (y - 7)(y + 1) = 0,$$
$$\therefore y = 7 \text{ or } -1.$$

But if x be real then 2^x must be positive

$$\therefore y = 2^x = 7.$$

Taking logarithms to the base 10

$$x \log 2 = \log 7$$
$$\therefore x = (\log 7)/\log 2 = 0{\cdot}84510/0{\cdot}30103$$
$$= 2{\cdot}807 \text{ correct to 4 significant figures.}$$

(b) Now $3^{x+1} = 3 \times 3^x$, and $3^{2x} = (3^x)^2$, therefore replacing 3^x by y the equation becomes

$$y^2 - 3y + 2 = 0$$
$$\text{i.e. } (y - 1)(y - 2) = 0$$
$$\therefore y = 1 \text{ or } 2$$
$$\text{i.e. } 3^x = 1 \quad \text{.......(1)},$$
$$\text{or } 3^x = 2 \quad \text{.......(2)}$$

From (1), $x = 0$.

From (2), taking logarithms to the base 10

$$x \log 3 = \log 2$$
$$\therefore x = (\log 2)/\log 3 = 0{\cdot}30103/0{\cdot}47712 = 0{\cdot}6309 \text{ to 4 significant figures}$$
$$\therefore x = 0 \text{ or } 0{\cdot}6309.$$

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EXAMPLE (L.U.). Without the use of tables show that approximately

$$5 \log 2 + 2 \log 3 = 2 \log 17,$$
$$1 + 2 \log 2 + 2 \log 3 = 2 \log 19.$$

By eliminating log 3 between these equations, deduce that 38/17 is a close approximation to $\sqrt{5}$.

Find from the tables the percentage error in this approximation.

Now, $5 \log 2 + 2 \log 3 = \log 2^5 + \log 3^2$
$= \log 32 + \log 9 = \log 32 \times 9$
$= \log 288.$

Also $2 \log 17 = \log 17^2 = \log 289.$

Hence, approximately $5 \log 2 + 2 \log 3 = 2 \log 17$ .... (1)

Also, $1 + 2 \log 2 + 2 \log 3 = \log 10 + \log 2^2 + \log 3^2$
$= \log (10 \times 2^2 \times 3^2) = \log 360.$

But, $2 \log 19 = \log 19^2 = \log 361$

$\therefore$ approximately $1 + 2 \log 2 + 2 \log 3 = 2 \log 19$ ...... (2)

(2) − (1) gives $1 - 3 \log 2 \simeq 2 \log 19 - 2 \log 17$
i.e. $\log 10 - \log 2 \simeq 2 \log 19 - 2 \log 17 + 2 \log 2$
$\simeq 2 \log (19 \times 2/17)$
i.e. $\log (10/2) \simeq 2 \log 38/17$
i.e. $\frac{1}{2} \log 5 \simeq \log 38/17$
$\therefore \log 5^{\frac{1}{2}} \simeq \log 38/17$
i.e. $\sqrt{5} = 38/17$ approximately.

NOTE. The sign $\simeq$ means 'approximately equal' and the signs $\doteqdot$ and $\triangleq$ have also been used for this purpose.

The error in the approximation $= \sqrt{5} - 38/17$
$= 2{\cdot}2361 - 2{\cdot}2353 = 0{\cdot}0008$

$$\therefore \text{ percentage error} = \frac{0{\cdot}0008 \times 100}{2{\cdot}2361} = \frac{0{\cdot}08}{2{\cdot}2361}$$
$= 0{\cdot}036$ to 3 decimal places.

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EXAMPLES II

1. (i) Simplify $(x^4yz^{-3})^2 \times \sqrt{(x^{-5}y^2z)} \div (xz)^{7/2}$.

(ii) If $x = \sqrt[3]{(p+q)} + \sqrt[3]{(p-q)}$ and $p^2 - q^2 = r^3$ prove that
$$x^3 - 3rx - 2p = 0.$$

2. If a, b, c, d are rational numbers so that $a + \sqrt{b} = c + \sqrt{d}$, and neither b nor d is a complete square, prove that $a = c$, $b = d$.

Express the square root of $18 - 12\sqrt{2}$ in the form $\sqrt{x} - \sqrt{y}$, where x and y are rational.

3. State the Index Laws and explain how meanings are obtained for $x^{\frac{1}{3}}$, x^0, and $x^{-\frac{1}{3}}$.

Find the quotient when $x^{4/3} - y^{-8/5}$ is divided by $x^{\frac{2}{3}} + y^{-2/5}$ and evaluate it when $x = 2{\cdot}731$, $y = 3{\cdot}142$.

4. (i) Find the square root of $19 - 4\sqrt{21}$ in surd form. (ii) Find the square root of $121x^6 + 44x^5 - 18x^4 + 18x^3 + 5x^2 - 2x + 1$.

5. (i) If $x = 3 + \sqrt{5}$, express as fractions with rational denominators $x - x^{-1}$ and $x^2 - x^{-2}$. (ii) If a, b, c, d are rational numbers, and if neither

b nor d is a perfect square, prove that the product of $(a + \sqrt{b})(c + \sqrt{d})$ can only be a rational number if

$$\frac{c}{a} = -\frac{\sqrt{d}}{\sqrt{b}}.$$

6. Prove, as shortly as possible, that

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4.$$

Hence write down the factors by which $p + \sqrt{q} + \sqrt{r}$ must be multiplied in order to form a rational product.

Reduce $1/(\sqrt{3} + \sqrt{2} + 1)$ to a form in which the denominator is rational.

7. (i) Prove that $(\log_a b)(\log_b a) = 1$ and check your result by using $a = b$.
(ii) Find y, if $x^y x^3 = 10^{5z}$, where $z = \log_{10} x$.

8. (i) With the aid of tables calculate approximately the value of x that satisfies $5^{2x} = 3(2^x)$.

(ii) Prove that $2 \log (a + b) = 2 \log a + \log \left(1 + \frac{2b}{a} + \frac{b^2}{a^2}\right)$.

(iii) Calculate $\log_{10} 5$ and $\log_{10} 0{\cdot}125$ given that $\log_{10} 2 = 0{\cdot}3010299957$.

9. (i) Given that $\log_{10} 3 = 0{\cdot}47712$ find the number of digits in the integral part of $(\sqrt{3})^{89}$. (ii) If x and y are positive numbers each greater than unity and if $x^{x+y} = y^{12}$, $y^{x+y} = x^3$, prove that $\log x = 2 \log y$ and hence find x and y.

10. (i) If $\log_x 10{\cdot}24 = 2$, find x. (ii) If $a = \log_b c$, $b = \log_c a$, $c = \log_a b$, prove that $abc = 1$.

11. (i) Prove that $\log_c a^n = n \log_c a$.

(ii) Solve the equations (a) $\log_{10} (x^2 + 2x) = 0{\cdot}90309$, (b) $(2{\cdot}4)^x = 0{\cdot}59$.

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12. Evaluate $\log_{10} \frac{(1728)^{\frac{1}{3}}}{(5^2 + 12^2)^{\frac{1}{2}}}$, $(0{\cdot}001529)^{2/9}$, $\log_{10} 53{\cdot}9$.

13. Prove that if x, y, p are any real numbers, (i) $\log_a xy = \log_a x + \log_a y$; (ii) $\log_a x^p = p \log_a x$.

Use logarithms to calculate the value of $[x^4y^2 - \sqrt{(xz^3)}]^{\frac{1}{3}}$ given that $x = 4{\cdot}7163$, $y = 0{\cdot}16542$, $z = 0{\cdot}18639$.

14. Prove that $\log_a x = -\log_{1/a} x$. Draw the graph of 6^x for values of x from 0 to 1. Hence find $\log_6 4$ and deduce $\log_{1/6} 4$ and $\log_4 6$.

15. Prove that $\log_a xy = \log_a x + \log_a y$ and that $\log_a x = \log_b x \times \log_a b$. Given that, to some base a, $\log 2744 = 5{\cdot}15757$, $\log 4732 = 5{\cdot}51256$, $\log 4459 = 5{\cdot}47384$, find $\log_a 28$ without using tables.

16. Solve the equations, (i) $(2{\cdot}93)^x = (0{\cdot}56)^{3-x}$; (ii) $4^{x+2y} = 5$, $2^{x-3y} = 8$.

17. Prove that $\log_b a \times \log_a b = 1$. Given that $\log_{10} 2 = 0{\cdot}30103$, find, *without the use of tables*, (i) the value of $\log_8 5$ correct to four decimal places; (ii) the value of x when $2 \cdot 10^{1+x} - 2 \cdot 10^{-x} = 3$.

18. If $a = \log (10/9)$, $b = \log (25/24)$, $c = \log (81/80)$, prove that $\log 2 = 7a - 2b + 3c$, and express $\log 3$ and $\log 5$ in like manner in terms of a, b, c.

19. (i) Without using tables, show that

$$\frac{\log \sqrt{27} + \log \sqrt{8} - \log \sqrt{125}}{\log 6 - \log 5} = \frac{3}{2}.$$

(ii) Solve the equation $5^{2x} - 5^{x+1} + 4 = 0$.

20. Define the logarithm of a number to a given base, and from your definition prove that, (i) $\log_b a \times \log_a b = 1$; (ii) $\log_b a = -\log_{1/b} a$.

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With the aid of tables, find the values of

$$\text{(i) } (16{\cdot}81)^{\frac{1}{4}} \times (3{\cdot}5^2)^{\frac{1}{3}}, \quad \text{(ii) } (0{\cdot}092)^{\frac{1}{4}} - (3178/6245)^{\frac{1}{2}}$$

correct to four significant figures.

21. Establish the ordinary rule for writing down by inspection the integral part of the logarithm to the base 10 of any number exceeding unity.

For what integral values of n is 2^n a number of exactly 50 digits?

Solve the equation $\log_x 3{\cdot}052 = 5$.

22. Prove that if a, b, c are any positive numbers $\log_a b \times \log_b c = \log_a c$.

If $x^2 + y^2 = 7xy$, prove that $\log(x + y) = \log 3 + \frac{1}{2}\log x + \frac{1}{2}\log y$.

23. Show how to find $\log_a x$ when $\log_{10} x$ and $\log_{10} a$ are known.

Find as accurately as your tables permit the value of $x^{2/5}y^{5/2}/(x^{2/5} + y^{5/2})$, when $x = 0{\cdot}243$, $y = 0{\cdot}784$.

24. Prove from the definition of a logarithm that if m, n, x are any positive numbers,

$$\log_{mn} x = \frac{\log_n x}{1 + \log_n m}.$$

25. Two quantities x and y are known to be such that y varies as a power of x. When $x = 4$, $y = 10{\cdot}8$, and when $x = 9$, $y = 36{\cdot}45$. Determine the relation between x and y, and find x when $y = 7$.

26. Determine by the aid of logarithms the value of the expression

$$E = (x^2 - y^2)^{2/3}(x + y)^{1/4}/(x - y)^{3/5},$$

when $x = 763$, $y = 249$, correct to four significant figures. Find also the logarithm of E to the base 7.

27. Define $\log_y x$, where x and y are any positive numbers.

If $\log_b a = 4$, $\log_c b = 3$ and $a = 32c^2$, find a, b, c.

28. Justify the statement that $x^{-n} = 1/x^n$, where n is a positive integer.

Solve the simultaneous equations $3^x - 2^{y+2} = 10$, $2^y - 3^{x-2} = 2$.

29. Define a logarithm and use your definition to find the integral part of the logarithm of $0{\cdot}03$ to the base $4/5$.

Find the values of x that satisfy the equation $2 \cdot 27^x - 5 \cdot 9^x + 3^{x+1} = 3^x$.

30. Prove that (i) $\log_c a + \log_c b = \log_c ab$;

(ii) $\log_a b \times \log_b c \times \log_c a = 1$;

(iii) $\dfrac{1}{\log_a (abc)} + \dfrac{1}{\log_b (abc)} + \dfrac{1}{\log_c (abc)} = 1$.

31. If $a = \log_e (1 + \frac{1}{15})$, $b = \log_e (1 + \frac{1}{24})$, $c = \log_e (1 + \frac{1}{80})$, prove that

$$\begin{aligned} \log_e 2 &= 7a + 5b + 3c, \\ \log_e 3 &= 11a + 8b + 5c, \\ \log_e 5 &= 16a + 12b + 7c. \end{aligned}$$

Given that $\log_e (1 + x)$ is approximately equal to x, when x is small, find from the above relations alone the approximate value of $\log_{10} 2$ to three decimal places.

32. (i) Prove that $\log_b a^n = n \log_b a$, and without using tables show that $\log_{10} 3$ is approximately equal to $\frac{1}{4} + \frac{3}{4}\log_{10} 2$. (ii) Solve the equation $2 \times 3^{2x+3} - 7 \times 3^{x+1} - 68 = 0$.

33. (i) Use tables to solve the equation $x^{0{\cdot}4142} = 0{\cdot}6335$. (ii) If $y = a^{1/(1-\log_a x)}$ and $z = a^{1/(1-\log_a y)}$, show that $x = a^{1/(1-\log_a z)}$.

34. (i) Prove that, whatever be the base used, $\log ab = \log a + \log b$, $\log a^b = b \log a$. If $p = \log_{10} 2$, $q = \log_{10} 3$, express in terms of p and q, $\log_{10} 108$, $\log_{10} 10{\cdot}8$, $\log \sqrt{3}/5$. (ii) Solve the equation $3^{2x} - 3^{x+1} + 2 = 0$.

35. (i) Solve the simultaneous equations $\log(x-1) + 2\log y = 2\log 3$, $\log x + \log y = \log 6$. (ii) Find the smallest integral value of n for which $(0{\cdot}861)^n$ is less than 0·01.

36. (i) Evaluate the logarithm of 3 to the base 2. (ii) Solve the simultaneous equations $4^x = 6^y$, $2(2^x) = 3(2^y)$, giving your results to three decimal places.

37. (i) Solve the equation $4^x - 3 \cdot 2^{x+1} - 2 = 0$. (ii) The relation $y = a + bx^n$ is satisfied by the pairs of values

x	2	4	8
y	10	15	27

Show that $n = \dfrac{\log 2{\cdot}4}{\log 2}$.

38. (i) Solve the equation $5^{2x} - 5^{x+1} + 6 = 0$. (ii) Given that $\log_{10} 2 = 0{\cdot}3010300$ and $\log_{10} 3 = 0{\cdot}4771213$, find the values of $\log_{10} 96$ and $\log_{10} 0{\cdot}0375$ to three decimal places.

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CHAPTER III

Remainder and Factor Theorems, Identities, Proportion, Graphs

Definitions. *A conditional equation* is an equation that is satisfied for only a finite number of values of the variable contained in it.

Thus $\frac{1}{3}x + \frac{1}{2}x = \frac{1}{4}$ is a conditional equation since it is satisfied by the value $x = 3/10$ only, and $x^2 - 3x + 2 = 0$ is a conditional equation since it is only satisfied by the two values $x = 1$ and $x = 2$.

An *identity* is an equation that is satisfied for *all* values of the variable (or variables) present in it (or, when both sides are simplified they are identical).

Thus $x(x + 1) = x^2 + x$ is true for all values of x and is an identity. It is usually written $x(x + 1) \equiv x^2 + x$. Also $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ for all values of x and y and can be written in the form $(x + y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3$ being an identity.

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From the definition of an identity it can be seen that the equation will be valid for any values of the variable or variables contained in it. Also the two sides of the equation must be identical when simplified and the coefficients of like terms on the two sides of the equation can be equated, giving a second method of dealing with identities.

A *function* of x is any expression involving x, and is usually denoted by any one of the symbols $f(x)$, $F(x)$, $\varphi(x)$, $\psi(x)$, etc.

A *rational, integral, algebraical function* of x is an expression in x in which the powers of x present are positive integers and the coefficients of terms in x are all rational. The general expression (polynomial) of this type in x is therefore

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n,$$

where n is a positive integer and a_0, a_1, a_2, etc., are rational constants. This is an expression of the nth degree in x, since the highest power of x present is the nth.

Given any function $f(x)$ involving x, the quantity $f(y)$ means the result of replacing x by y in the expression $f(x)$.

Thus, if

$$f(x) = 2x^2 - 3x + 5,$$
$$f(0) = 2 \times 0 - 3 \times 0 + 5 = 5,$$

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$$f(1) = 2 \times 1^2 - 3 \times 1 + 5 = 2 - 3 + 5 = 4,$$
$$f(-2) = 2(-2)^2 - 3(-2) + 5 = 8 + 6 + 5 = 19,$$
$$f(y) = 2y^2 - 3y + 5,$$
$$f(x^2) = 2(x^2)^2 - 3(x^2) + 5 = 2x^4 - 3x^2 + 5.$$

The Remainder Theorem. This states that, if $f(x)$ be a rational, integral, algebraical function of x, the remainder on dividing $f(x)$ by $ax + b$ is $f(-b/a)$, where a, b are constants and $ax + b \neq 0$.

When $f(x)$ is divided by $(ax + b)$ let the quotient be $\varphi(x)$ and the remainder R. Then it is known from the ordinary rules of division in algebra that R will be a constant and will not contain x.

Hence $f(x) \equiv (ax + b)\varphi(x) + R$ (this is an identity being true for *all* values of x).

Using $x = -b/a$ in this identity

$$f(-b/a) = 0 \times \varphi(-b/a) + R$$
$$\therefore\ R = f(-b/a).$$

When $a = 1$ and b is replaced by $(-\alpha)$ the result becomes $R = f(\alpha)$, i.e. when $f(x)$ is divided by $(x - \alpha)$ the remainder has the value $f(\alpha)$.

EXAMPLE. Find the remainder on dividing $(3x^3 - 2x + 3)$ by $(2x + 1)$.

Let $f(x) \equiv 3x^3 - 2x + 3$.

Therefore required remainder $= f(-\frac{1}{2})$
$$= 3(-\tfrac{1}{2})^3 - 2(-\tfrac{1}{2}) + 3$$
$$= -\tfrac{3}{8} + 1 + 3 = 3\tfrac{5}{8}.$$

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The Factor Theorem. This states that, if $(ax + b)$ be a factor of $f(x)$, then $f(-b/a) = 0$, and conversely, if $f(-b/a) = 0$, then $(ax + b)$ is a factor of $f(x)$.

If $f(x)$ has $(ax + b)$ as a factor, then the remainder R obtained by dividing $f(x)$ by $(ax + b)$ must be zero, and it has been shown that

$$R = f(-b/a)$$
$$\therefore\ f(-b/a) = 0$$

if $(ax + b)$ be a factor of $f(x)$.

Also if $f(-b/a) = 0$ then $R = 0$ and $(ax + b)$ *must be a factor of* $f(x)$.

Similarly, if $(x - \alpha)$ be a factor of $f(x)$, then $f(\alpha) = 0$, and, if $f(\alpha) = 0$, then $(x - \alpha)$ is a factor of $f(x)$.

NOTE. If the equation $f(x) = 0$ cannot be solved by previous methods, it is usual to use the factor theorem in the following manner:

Find, by trial and error, values of x that make $f(x) = 0$, then these values will be some or all of the roots of the given equation.

In the case of a cubic equation $f(x) = 0$, if one root $x = \alpha$ be found by this method, then $f(x)$ can be divided by $(x - \alpha)$ and the remaining two roots will be found by equating the quadratic quotient obtained to zero.

Similar methods can be applied to a quartic (fourth-degree equation), after finding two roots by trial and error, and so on for fifth- and sixth-degree equations, etc.

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Example. For what value of p is $(x^3 + px - 3)$ divisible by $(x - 3)$?

Let $$f(x) \equiv x^3 + px - 3.$$
$$\therefore f(3) = 27 + 3p - 3 = 24 + 3p.$$

Since $f(x)$ is divisible by $(x - 3)$, it follows that $f(3) = 24 + 3p = 0$, from which $p = -8$.

Example. Find the factors of (i) $x^3 - x^2 - 14x + 24$,
(ii) $x^4 - 3x^3 + 3x^2 - 3x + 2$.

(Note. This means factorise as *fully as possible*.)

(i) Considering the constant term 24, the only possible factors are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, ± 24, and hence the only possible factors of the given function $f(x)$ are $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 4)$, $(x \pm 6)$, $(x \pm 8)$, $(x \pm 12)$, $(x \pm 24)$. (Lowest values used first.)

Let $$f(x) \equiv x^3 - x^2 - 14x + 24$$
$$\therefore f(1) = 1 - 1 - 14 + 24 \neq 0 \qquad (\neq \text{ denotes not equal to})$$
$$\therefore (x - 1) \text{ is } \textit{not} \text{ a factor of } f(x)$$
$$f(2) = 8 - 4 - 28 + 24 = 0$$
$$\therefore (x - 2) \text{ is a factor of } f(x).$$

$$\begin{array}{r|l}
 & x^2 + x - 12 \\
\hline
x - 2 & x^3 - x^2 - 14x + 24 \\
 & x^3 - 2x^2 \\
\hline
 & \quad x^2 - 14x \\
 & \quad x^2 - 2x \\
\hline
 & \qquad -12x + 24 \\
 & \qquad -12x + 24 \\
\hline
 & \qquad \cdot \quad \cdot
\end{array}$$

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$$\therefore f(x) = (x - 2)(x^2 + x - 12)$$
$$= (x - 2)(x - 3)(x + 4).$$

(ii) Let $$f(x) \equiv x^4 - 3x^3 + 3x^2 - 3x + 2.$$

The only factors of 2 are ± 1, ± 2.
$$f(1) = 1 - 3 + 3 - 3 + 2 = 0,$$
$$\therefore (x - 1) \text{ is a factor of } f(x).$$
$$f(-1) = 1 + 3 + 3 + 3 + 2 \neq 0,$$
$$\therefore (x + 1) \text{ is not a factor of } f(x).$$
$$f(2) = 16 - 24 + 12 - 6 + 2 = 0,$$
$$\therefore (x - 2) \text{ is a factor of } f(x).$$

Dividing $f(x)$ by $(x - 1)(x - 2) = x^2 - 3x + 2$,

$$\begin{array}{r|l}
 & x^2 + 1 \\
\hline
x^2 - 3x + 2 & x^4 - 3x^3 + 3x^2 - 3x + 2 \\
 & x^4 - 3x^3 + 2x^2 \\
\hline
 & \qquad\qquad x^2 - 3x + 2 \\
 & \qquad\qquad x^2 - 3x + 2 \\
\hline
 & \qquad\qquad \cdot \quad \cdot \quad \cdot
\end{array}$$

$$\therefore \text{ factors of } f(x) \text{ are } (x - 1)(x - 2)(x^2 + 1).$$

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NOTE. $x^2 + 1$ cannot be expressed as the product of real factors and is known as a *quadratic factor*.

EXAMPLE. Find the roots of the equation $2x^3 + 3x^2 - 3x - 2 = 0$.

Let $\quad f(x) \equiv 2x^3 + 3x^2 - 3x - 2.$

Now $\quad f(1) = 2 + 3 - 3 - 2 = 0.$

$\therefore (x - 1)$ is a factor of $f(x)$.

$$2x^3 + 3x^2 - 3x - 2 = (2x^3 - 2x^2) + (5x^2 - 3x - 2)$$

(adding and subtracting $2x^2$)

$$= 2x^2(x - 1) + (x - 1)(5x + 2)$$
$$= (x - 1)(2x^2 + 5x + 2)$$
$$= (x - 1)(2x + 1)(x + 2).$$

NOTE. This method, which replaces the division method, makes use of the fact that $(x - 1)$ is a factor of $f(x)$, therefore the given equation can be written

$$(x - 1)(2x + 1)(x + 2) = 0$$
$$\therefore x = 1, -\tfrac{1}{2}, \text{ or } -2.$$

EXAMPLE (L.U.). n being a positive integer and

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$$

vanishing when x is put equal to α, prove that $(x - \alpha)$ is a factor of the expression.

Use this theorem to determine, by trial, the integral root of

$$16x^3 + 31{\cdot}6x^2 - 6{\cdot}8x - 12 = 0,$$

and find the values of the remaining roots.

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The first part of the question is the factor theorem and has already been proved.

Let $\quad f(x) \equiv 16x^3 + 31{\cdot}6x^2 - 6{\cdot}8x - 12.$

It will be found on trial that $f(1) \neq 0,\ f(-1) \neq 0,\ f(2) \neq 0.$

$$f(-2) = -128 + 126{\cdot}4 + 13{\cdot}6 - 12 = 0$$

$\therefore x + 2$ is a factor of $f(x)$.

$$\begin{array}{r|l}
 & 16x^2 - 0{\cdot}4x - 6 \\
\hline
x + 2 & 16x^3 + 31{\cdot}6x^2 - 6{\cdot}8x - 12 \\
 & 16x^3 + 32x^2 \\
\hline
 & -0{\cdot}4x^2 - 6{\cdot}8x \\
 & -0{\cdot}4x^2 - 0{\cdot}8x \\
\hline
 & -6x - 12 \\
 & -6x - 12 \\
\hline
 & \cdot \quad \cdot
\end{array}$$

$$\begin{aligned}
\therefore f(x) &\equiv (x + 2)(16x^2 - 0{\cdot}4x - 6) \\
&\equiv 0{\cdot}2(x + 2)(80x^2 - 2x - 30) \equiv 0{\cdot}4(x + 2)(40x^2 - x - 15) \\
&\equiv 0{\cdot}4(x + 2)(5x + 3)(8x - 5)
\end{aligned}$$

therefore the given equation can be written

$$0{\cdot}4(x + 2)(5x + 3)(8x - 5) = 0,$$

from which $\quad x = -2,\ -\tfrac{3}{5}, \text{ or } \tfrac{5}{8}.$

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Application of the Factor Theorem to Symmetrical Expressions. An expression in a, b, c is said to be *symmetrical* if the value of the expression is unchanged when a is replaced by b, b by c, and c by a simultaneously (using three letters only).

Hence, it follows that, if $(a - b)$ be a factor of a symmetrical expression in a, b, c then $(b - c)$ and $(c - a)$ must also be factors of the expression. Similarly, if $(a + b)$ be a factor, then $(b + c)$ and $(c + a)$ must also be factors.

NOTE. It must first be established that the function is a symmetrical one before these results can be used.

The symmetrical expressions of the first and second degrees in a, b, and c are $k_0(a + b + c)$, and $k_1(a^2 + b^2 + c^2) + k_2(bc + ca + ab)$ respectively, where k_0, k_1, k_2 are constants whose values must be determined in specific examples (when used) by using the properties of identities.

If, for example, a fifth-degree *homogeneous* expression in a, b, c be symmetrical and $(a - b)$ be a factor of the expression, the remaining factors must be $(b - c)$, $(c - a)$, and

$$k_1(a^2 + b^2 + c^2) + k_2(bc + ca + ab)$$

to make up the fifth degree, where k_1 and k_2 have to be determined by the use of the properties of identities. ('Homogeneous' means each term of same degree.)

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EXAMPLE. Factorise the following:
(i) $bc(b - c) + ca(c - a) + ab(a - b)$;
(ii) $x^3(y^2 - z^2) + y^3(z^2 - x^2) + z^3(x^2 - y^2)$;
(iii) $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$.

(i) It can be readily shown that the given expression is of third degree and is symmetrical in a, b, c.

Let
$$f(a) \equiv bc(b - c) + ca(c - a) + ab(a - b).$$
$$\therefore f(b) = bc(b - c) + cb(c - b) + b^2 \times 0$$
$$= 0$$

therefore $(a - b)$ is a factor of $f(a)$, and hence it follows that $(c - a)$ and $(b - c)$ are factors of $f(a)$. Hence, there can only be a constant factor in addition to these factors.

Let $bc(b - c) + ca(c - a) + ab(a - b) \equiv k(b - c)(c - a)(a - b)$, where k is a constant.

Equating coefficients of a^2b in this identity

$$1 = -k \ \therefore\ k = -1.$$
$$\therefore f(a) \equiv -(b - c)(c - a)(a - b).$$

(ii) The given expression is a fifth-degree symmetrical expression in x, y, and z.

Let
$$f(x) \equiv x^3(y^2 - z^2) + y^3(z^2 - x^2) + z^3(x^2 - y^2)$$
$$\therefore f(y) = y^3(y^2 - z^2) + y^3(z^2 - y^2) + z^3 \times 0$$
$$= 0.$$

Hence, by the factor theorem, $(x - y)$ is a factor of $f(x)$, therefore

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$(y - z)$ and $(z - x)$ are also factors of $f(x)$, and the remaining factor will be $k_1(x^2 + y^2 + z^2) + k_2(yz + zx + xy)$, where k_1 and k_2 are constants.

Thus, $x^3(y^2 - z^2) + y^3(z^2 - x^2) + z^3(x^2 - y^2)$

$$\equiv (y - z)(z - x)(x - y)[k_1(x^2 + y^2 + z^2) + k_2(yz + zx + xy)].$$

Equating coefficients of x^3y^2 and x^4y in this identity

$$1 = k_1 - k_2$$
$$0 = -k_1$$
$$\therefore\ k_1 = 0 \text{ and } k_2 = -1.$$

Hence, $f(x) \equiv -(y - z)(z - x)(x - y)(yz + zx + xy)$.

(iii) This can be seen to be a symmetrical expression of the fourth degree in a, b, c.

Let
$$f(a) \equiv a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$$
$$\therefore f(b) = b(b^3 - c^3) + b(c^3 - b^3) + c(b^3 - b^3)$$
$$= 0.$$

Thus, by the factor theorem, $(a - b)$ is a factor of $f(a)$, and it follows that $(b - c)$ and $(c - a)$ are also factors of $f(a)$ and the remaining factor which must be linear in a, b, c will be $k(a + b + c)$, where k is a constant.

Therefore $a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$

$$\equiv k(b - c)(c - a)(a - b)(a + b + c).$$

Equating coefficients of a^3b in this identity

$$-1 = -k \quad \therefore \quad k = 1$$
$$\therefore f(a) \equiv (b - c)(c - a)(a - b)(a + b + c).$$

Theorem. *To prove that $(a+b+c)$ is a factor of $a^3 + b^3 + c^3 - 3abc$, and to find the other factor.*

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Let $\quad f(a) \equiv a^3 + b^3 + c^3 - 3abc$

$$\therefore f(-[b + c]) = (-b - c)^3 + b^3 + c^3 - 3(-b - c)bc$$
$$= -b^3 - 3b^2c - 3bc^2 - c^3 + b^3 + c^3 + 3b^2c + 3bc^2 = 0$$

therefore *$(a + b + c)$ is a factor of $f(a)$.*

By division, the other factor is found to be

$$a^2 + b^2 + c^2 - bc - ca - ab.$$

The result

$$a^3 + b^3 + c^3 - 3abc \equiv (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$$

is well worth memorising.

Partial Fractions. When a given fraction $f(x)/\varphi(x)$, where $f(x)$ and $\varphi(x)$ are rational, integral, algebraical functions of x (polynomials), is expressed as the sum of two or more simpler fractions according to *certain definite rules*, it is said to be expressed in (or resolved into) *partial fractions*.

Thus,
$$\frac{2}{3 - x} + \frac{1}{2 + x} = \frac{7 + x}{(3 - x)(2 + x)},$$
$$\therefore \frac{7 + x}{(3 - x)(2 + x)} = \frac{2}{3 - x} + \frac{1}{2 + x},$$

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and $2/(3 - x) + 1/(2 + x)$ are the partial fractions in this special case corresponding to the complex fraction $(7 + x)/(3 - x)(2 + x)$, as they will be found to fit in with the rules laid down.

The method of procedure in the case of the general fraction $f(x)/\varphi(x)$ is as follows, where the degree of $f(x)$ is less than the degree of $\varphi(x)$.

(i) The given fraction will be equated identically to the sum of simpler fractions involving constants A, B, C, etc. (whose values have to be determined later) according to the following rules.

(a) To every linear factor $(a_1x + b_1)$ of $\varphi(x)$ there will be a corresponding partial fraction $A/(a_1x + b_1)$.

(b) To every repeated factor $(a_2x + b_2)^2$ of $\varphi(x)$ there will be two corresponding partial fractions $B/(a_2x + b_2) + C/(a_2x + b_2)^2$.

(c) To every factor of the form $(a_3x + b_3)^3$ of $\varphi(x)$ there will be three corresponding partial fractions

$$D/(a_3x + b_3) + E/(a_3x + b_3)^2 + F/(a_3x + b_3)^3,$$

and so on.

NOTE. It will be assumed that the problems that are encountered will only involve linear factors of $\varphi(x)$, i.e. all the factors of $\varphi(x)$ will be linear in x. Also it will be taken that the degree of $f(x)$ in x is always lower than that of $\varphi(x)$.

(ii) To determine the constants A, B, C, etc., the whole identity written down is multiplied throughout by $\varphi(x)$ giving rise to a second identity, from which the values of the constants are obtained by giving values to x that make *each linear factor of* $\varphi(x)$ *vanish* (this simplifies the working), and by equating coefficients of like terms if all the required constants have not then been determined.

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EXAMPLE (L.U.). Find the partial fractions equal to

$$(3x^2 - 7)/(x^3 + 2x^2 - 8x).$$

Now $x^3 + 2x^2 - 8x \equiv x(x^2 + 2x - 8) \equiv x(x - 2)(x + 4)$.

Let $$\frac{3x^2 - 7}{x(x - 2)(x + 4)} \equiv \frac{A}{x} + \frac{B}{x + 4} + \frac{C}{x - 2}$$

therefore, multiplying through by $x(x - 2)(x + 4)$

$$3x^2 - 7 \equiv A(x - 2)(x + 4) + Bx(x - 2) + Cx(x + 4).$$

Since this is an identity it is true for *all* values of x.

Using the values of x that make each linear factor of the denominator vanish,

$$x = 0 \qquad -7 = -8A \quad \therefore \quad A = 7/8.$$
$$x = -4 \qquad 41 = 24B \quad \therefore \quad B = 41/24.$$
$$x = 2 \qquad 5 = 12C \quad \therefore \quad C = 5/12.$$

Hence, $$\frac{3x^2 - 7}{x^3 + 2x^2 - 8x} = \frac{7}{8x} + \frac{41}{24(x + 4)} + \frac{5}{12(x - 2)}.$$

Example. Express in partial fractions

$$\text{(i)}\ \frac{9}{(x-1)(x+2)^2}, \qquad \text{(ii)}\ \frac{4+3x+2x^2}{(1-2x)(1-x^2)}.$$

(i) Let $$\frac{9}{(x-1)(x+2)^2} \equiv \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

therefore multiplying through by $(x-1)(x+2)^2$,

$$9 \equiv A(x+2)^2 + B(x-1)(x+2) + C(x-1).$$

In this identity, using:

$$\begin{aligned} x &= 1 & 9 &= 9A & \therefore\ A &= 1 \\ x &= -2 & 9 &= -3C & \therefore\ C &= -3 \\ x &= 0 & 9 &= 4A - 2B - C = 4 - 2B + 3 \end{aligned}$$

$$\therefore\ 2B = -2, \text{ i.e. } B = -1.$$

Hence, $$\frac{9}{(x-1)(x+2)^2} \equiv \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}.$$

(ii) $(1-x^2) = (1-x)(1+x)$.

Let $$\frac{4+3x+2x^2}{(1-2x)(1-x)(1+x)} \equiv \frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{1+x},$$

therefore $4 + 3x + 2x^2$

$$\equiv A(1-x)(1+x) + B(1-2x)(1+x) + C(1-x)(1-2x)$$

(multiplying through by $(1-2x)(1-x)(1+x)$).

In this identity, using the values,

$$\begin{aligned} x &= \tfrac{1}{2} & 4 + \tfrac{3}{2} + \tfrac{1}{2} &= \tfrac{3}{4}A, & \therefore\ \tfrac{3}{4}A &= 6, & \therefore\ A &= 8. \\ x &= 1 & 9 &= -2B, & \therefore\ B &= -9/2. \\ x &= -1 & 3 &= 6C, & \therefore\ C &= \tfrac{1}{2}. \end{aligned}$$

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$$\therefore\ \frac{4+3x+2x^2}{(1-2x)(1-x^2)} \equiv \frac{8}{1-2x} - \frac{9}{2(1-x)} + \frac{1}{2(1+x)}.$$

The following are examples on the use of identities in factor questions.

Example (L.U.). (i) If $x^2 + ax + b$ and $x^2 + cx - b$ have a common linear factor and b is not zero, prove that $4b = a^2 - c^2$, and the factor is $x + \frac{1}{2}(a + c)$.

(ii) Determine the conditions that

$$x^3 + px^2 + qx + r \text{ and } x^3 + rx^2 + qx + p$$

may have a common factor of the second degree.

(i) Let $(x - \alpha)$ be the common factor. (Coefficient of x^2 is unity in each case.)

Then by the factor theorem

$$\alpha^2 + a\alpha + b = 0 \ldots\ldots\ldots\ldots (1)$$

$$\alpha^2 + c\alpha - b = 0 \ldots\ldots\ldots\ldots (2)$$

(1) − (2) gives $\alpha(a - c) + 2b = 0,$

$$\therefore\ \alpha = -2b/(a-c) \ldots\ldots\ldots\ldots (3)$$

(1) + (2) gives $2\alpha^2 + \alpha(a + c) = 0,$

$$\therefore\ \alpha[2\alpha + (a + c)] = 0$$

$$\therefore\ \alpha = -\tfrac{1}{2}(a + c) \ldots\ldots\ldots\ldots (4)$$

(since $\alpha \neq 0$.)

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From (3) and (4)

$$-2b/(a - c) = -(a + c)/2,$$

$$\text{i.e. } 4b = (a + c)(a - c) = a^2 - c^2.$$

Using the result (4), $x - \alpha = x + \frac{1}{2}(a + c)$.

Hence the common factor is $x + \frac{1}{2}(a + c)$.

(ii) Let $(x - \alpha)$ and $(x - \beta)$ be the remaining linear factors of

$$(x^3 + px^2 + qx + r) \text{ and } (x^3 + rx^2 + qx + p)$$

respectively.

Since they have a common quadratic factor

$$\frac{x^3 + px^2 + qx + r}{x - \alpha} \equiv \frac{x^3 + rx^2 + qx + p}{x - \beta}$$

$$\therefore (x - \beta)(x^3 + px^2 + qx + r) \equiv (x - \alpha)(x^3 + rx^2 + qx + p).$$

In this identity, equating coefficients of powers of x, which can be done by multiplying out mentally,

x^3 $\qquad -\beta + p = -\alpha + r$ .... (1),

x^2 $\qquad q - p\beta = q - \alpha r$, i.e. $p\beta = \alpha r$ .... (2),

x $\qquad r - q\beta = p - q\alpha$ .... (3),

unity $\qquad r\beta = \alpha p$ .... (4).

(2) ÷ (4) gives $\qquad \dfrac{p}{r} = \dfrac{r}{p}$, i.e. $p^2 = r^2$

$$\therefore p = -r \quad \text{........ (5)}$$

since $p = +r$ gives identical expressions.

Using (5) in (4), $\qquad \beta = -\alpha$ .... (6).

Using (5) and (6) in (1), $2\alpha = 2r$ $\therefore \alpha = r$, $\beta = -r$ www.dbraulibrary.org.in

therefore from (3), $\qquad r + qr = -r - rq,$

$$\text{i.e. } r + rq = 0, \text{ i.e. } r(1 + q) = 0$$

$$\therefore r = 0 \text{ or } q = -1.$$

Clearly $r \neq 0$ $\therefore q = -1$.

Hence required conditions are $p = -r$ and $q = -1$.

NOTE on the factorisation of homogeneous quadratic functions in x, y, and z.

Consider the homogeneous quadratic function

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

(each term is of the second degree in x, y, z and the expression is therefore homogeneous), which it is required to factorise where possible. (In certain cases z is taken as unity.)

The given expression is first considered with respect to the terms containing only x and y, viz. the portion $ax^2 + 2hxy + by^2$, and this is factorised in the usual way into $(a_1x + b_1y)(a_2x + b_2y)$ where possible.

The factors of the given expression (if it can be factorised) will therefore be $(a_1x + b_1y + c_1z)$ and $(a_2x + b_2y + c_2z)$, where c_1 and c_2 will be determined by means of the properties of identities; i.e. equate the given expression identically to $(a_1x + b_1y + c_1z)(a_2x + b_2y + c_2z)$, where the a's and b's are known, and equate coefficients of yz and zx to find c_1 and c_2.

EXAMPLE (L.U.). Factorise (i) $2x^2 - y^2 + 2z^2 + xy + 4zx + yz$;
(ii) $2x^2 - 2y^2 + 3xy - x + 8y - 6$.

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(i) The portion not containing z is $2x^2 + xy - y^2 = (2x - y)(x + y)$. Hence, the factors of the original expression will be $(2x - y + c_1z)$ and $(x + y + c_2z)$. Therefore

$$2x^2 - y^2 + 2z^2 + xy + 4zx + yz \equiv (2x - y + c_1z)(x + y + c_2z).$$

Equating coefficients in this identity,

z^2 $\qquad c_1c_2 = 2$ ....(1),

zx $\qquad 2c_2 + c_1 = 4$ ....(2),

yz $\qquad c_1 - c_2 = 1$ ....(3)

From (2) and (3), $c_2 = 1$ and $c_1 = 2$, which check in (1).

Therefore expression $\equiv (2x - y + 2z)(x + y + z)$.

(ii) Factorising the second-degree portion in x and y

$$2x^2 + 3xy - 2y^2 = (2x - y)(x + 2y).$$

$\therefore\ 2x^2 - 2y^2 + 3xy - x + 8y - 6 \equiv (2x - y + c_1)(x + 2y + c_2)$, where c_1 and c_2 are constants.

Equating coefficients in this identity,

x $\qquad 2c_2 + c_1 = -1$ ....(1),

y $\qquad -c_2 + 2c_1 = 8$ ....(2),

unity $\qquad c_1c_2 = -6$ ....(3)

From (1) and (2), $c_1 = 3$, $c_2 = -2$, which satisfy equation (3).

Therefore expression $\equiv (2x - y + 3)(x + 2y - 2)$.

Continued Proportion. If $a/b = c/d = e/f = \ldots$, then the quantities $a, b, c, d, e, f, \ldots$ are said to be in *continued proportion.*

Theorem. *If $a/b = c/d = e/f = \ldots$, to prove that*

$$\frac{la + mc + ne + \ldots}{lb + md + nf + \ldots} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

where $l, m, n, \ldots$ are any quantities.

Let each of the fractions a/b, c/d, e/f, etc., be equal to k.

Then $a = bk$, $c = dk$, $e = fk$, ...

Using these results,

$$\frac{la + mc + ne + \ldots}{lb + md + nf + \ldots} = \frac{lbk + mdk + nfk + \ldots}{lb + md + nf + \ldots}$$

$$= \frac{k(lb + md + nf + \ldots)}{lb + md + nf + \ldots}$$

$$= k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots$$

EXAMPLE (L.U.).

(i) If $y = \dfrac{\sqrt{(px + q)} + \sqrt{(px - q)}}{\sqrt{(px + q)} - \sqrt{(px - q)}}$, prove that $y + \dfrac{1}{y} = \dfrac{2px}{q}$.

(ii) If $\qquad \dfrac{x + y - z}{p} = \dfrac{x - y + z}{q} = \dfrac{-x + y + z}{r}$,

prove that $\qquad \dfrac{(x + y + z)^3}{8xyz} = \dfrac{(p + q + r)^3}{(q + r)(r + p)(p + q)}$.

(i) $y + \dfrac{1}{y} = \dfrac{\sqrt{(px + q)} + \sqrt{(px - q)}}{\sqrt{(px + q)} - \sqrt{(px - q)}} + \dfrac{\sqrt{(px + q)} - \sqrt{(px - q)}}{\sqrt{(px + q)} + \sqrt{(px - q)}}$

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$$= \frac{[\sqrt{(px+q)} + \sqrt{(px-q)}]^2 + [\sqrt{(px+q)} - \sqrt{(px-q)}]^2}{[\sqrt{px+q)} - \sqrt{(px-q)}]\,[\sqrt{(px+q)} + \sqrt{(px-q)}]}$$

$$= \frac{(px+q) + 2\sqrt{[(px+q)(px-q)]} + (px-q) + (px+q) - 2\sqrt{[(px+q)(px-q)]} + (px-q)}{(px+q) - (px-q)}$$

$$= \frac{4px}{2q} = \frac{2px}{q}.$$

(ii) Let $a = x + y - z,\ b = x - y + z,\ c = -x + y + z,$

$$\therefore \frac{a}{p} = \frac{b}{q} = \frac{c}{r}, \text{ and } a + b + c = x + y + z.$$

By the previous theorem each fraction

$$= \frac{al + bm + cn}{pl + qm + rn},$$

where l, m, n are any numbers.

(*a*) Choose $l = m = n = 1$, then each fraction

$$= \frac{a+b+c}{p+q+r} = \frac{x+y+z}{p+q+r}.$$

(*b*) Choose $l = m = 1,\ n = 0,$ then each fraction

$$= \frac{a+b}{n+q} = \frac{2x}{p+q}.$$

Similarly if $l = n = 1,\ m = 0,$ each fraction $= \frac{2y}{r+p},$

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and if $m = n = 1,\ l = 0,$ each fraction $= \frac{2z}{q+r}.$

From the results in (*a*) and (*b*) each fraction cubed

$$= \frac{(x+y+z)^3}{(p+q+r)^3} = \frac{2x}{p+q} \times \frac{2y}{r+p} \times \frac{2z}{q+r} = \frac{8xyz}{(p+q)(r+p)(q+r)}.$$

$$\therefore \frac{(x+y+z)^3}{8xyz} = \frac{(p+q+r)^3}{(q+r)(r+p)(p+q)}.$$

Graphical Work. Under this heading it is only intended to give a brief summary on graphs as the main bulk of the work will have been done previously.

In drawing a graph of the single equation $y = f(x)$, it is essential that the following rules be observed.

(i) After ascertaining the range of values of x, the corresponding values of y are calculated in tabular form on the ordinary sheet of paper, and these should generally be correct to two decimal places.

(ii) The axes $X'OX$, $Y'OY$ are now placed on the graph paper so as to use up as much of the graph paper as possible, their positions being determined by the ranges of values of x and y.

(iii) All the points on the graph paper corresponding to the x (abscissa) and y (ordinate) values in the calculated table will be

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shown by either a dot in a small circle, or by a small cross.

(iv) Only multiples of 1, 2, 4, 5 small squares of the graph paper (which lend themselves to use with decimals) should be used per unit for the scale for either axis, and the scales need not necessarily be the same for the two axes.

(v) The curve itself should first be drawn very faintly, so as to obtain some idea of its general shape, and then a heavier line will be used and unessential points erased.

There should be a sufficiency of points, to act as guides for a smooth accurate curve; otherwise, it will be necessary to calculate and use one or two more intermediate points.

(vi) The point of intersection of the two axes need not be the point where $x = 0$ and $y = 0$. The point is so chosen that as little as possible of the graph paper is wasted.

(vii) A line parallel to the long side of the graph paper can be used for the x-axis, if this be more suitable than the short side.

NOTE. When points are required from the graph that lie between the calculated values, the process is known as *interpolation*. When, in order to find the required point, the graph has to be extended the process is known as *extrapolation*.

Intersection of Curves and Solution of Equations. To find graphically the approximate solutions of the equation $f(x) = 0$, the curve $y = f(x)$ is drawn and the points where the curve cuts the x-axis (i.e. at $y = 0$) will give the required solutions.

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For the solution of $f(x) = a$, where a is a constant, the x-values where the curve cuts the line $y = a$ (i.e. a line parallel to OX at a distance a units from it) will give the required solutions.

In certain cases it is necessary to adopt the *two-graph method* of finding the solutions of a given equation, this being particularly the case when the equation $f(x) = 0$ can be expressed in the form $f_1(x) = f_2(x)$, where $f_1(x)$ is an algebraical function, and $f_2(x)$ is a non-algebraical (*transcendental*) function.

In this case, and also when the curves given by $f_1(x)$ and $f_2(x)$ are well known, the curves $y = f_1(x)$ and $y = f_2(x)$ are drawn on the same sheet of graph paper using the same axes and the same scales of representation for both graphs.

The tables for both graphs must be formed first, and the positions of the axes and the scales of representation will be dependent upon the joint tables. It is advisable to use a dot with a small circle round for points on one curve, and the small diagonal cross for points on the other curve.

At any point of intersection of the two curves the two values of y will be the same, and therefore at these points $f_1(x) = f_2(x)$, i.e. the approximate solutions are the values of x at the points of intersection of the two curves.

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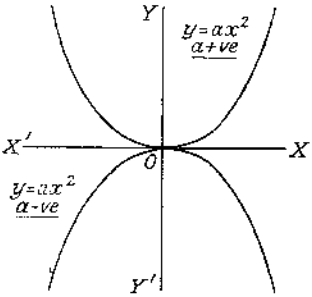
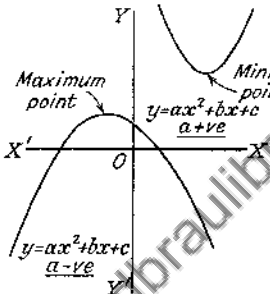
Rough Graphs of Standard Curves. It is advisable to have some idea of the shape of the curve represented by the following equations.

(i) *A linear equation* (first degree in x and y of the form $ax + by + c = 0$, where a and b are constants) represents a straight line, and hence there is only need to calculate *two* values of y corresponding to values of x to draw the line with a ruler, but a *third* point should be calculated as a checking point.

(ii) The general equation $y = ax^2 + bx + c$ represents a *parabola*, for all cases of which a rough diagram is shown.





(iii) The equation $y = ax^3 + bx^2 + cx + d$ represents a *cubic curve*, rough graphs being given below for the various cases.

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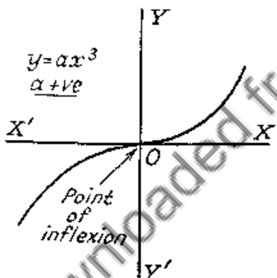

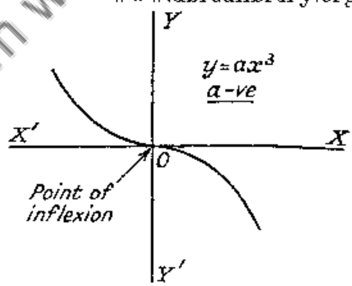

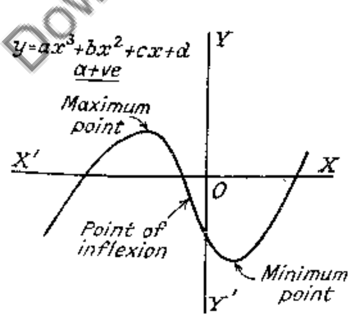

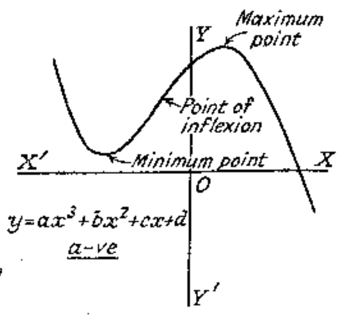

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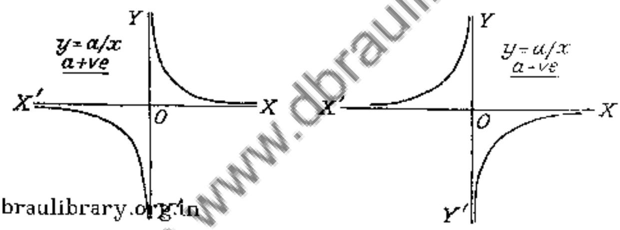
Note. Certain types of points known as maximum and minimum points and points of inflexion are pointed out in the diagrams and will be dealt with more fully in a later chapter.

(iv) The equation $y = a/x$, or $xy = a$ constant, represents the curve known as a *rectangular hyperbola*. It is to be noted that, if a be positive and x small and positive, then y is large and positive; when a is positive and x small and negative, then y is large and negative; with opposite results if a be negative.

The equation can also be written $x = a/y$, and as above it follows that, if a is positive and y is small and positive, x is large and positive, whilst, if a is positive and y is small and negative, x is large and negative, and so on.

The lines $x = 0$ and $y = 0$ which the curve gradually approaches but never meets are known as the *asymptotes* to the curve, and are shown by the axes of x and y in the following rough graphs of the hyperbola.



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The *rectangular* hyperbola is so named because its asymptotes are at *right angles*.

Experimental Data for Curves. It is to be understood that experimental results are liable to human error. Hence, there is always the possibility that the corresponding points plotted on the graph paper will not be strictly accurate. Thus, the curve drawn should be traced so that the points due to the experimental results are for the most part placed on either side of a smooth curve, if they are to follow some mathematical law, so that the errors on either side cancel out.

Normally the variables are chosen as a function of the given variable so that the graph itself is a straight line, and, by observing the slope m of this line and its intercept c on the y-axis, the law of the experiment can be written $y = mx + c$ (see co-ordinate geometry), where x and y are the variables used.

For instance, if it be suspected that the x and y values given satisfy the law $y = ax^2 + c$, it would be advisable to plot the values of y against those of a new variable $X = x^2$, and if a straight line graph be obtained, it would then be known that x and y satisfied the law $y = ax^2 + c$ and a and c would be given by the slope and intercept

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on the y-axis respectively of the straight line drawn *between* the points.

Similarly if it were thought that x and y satisfied the law $y = b \times a^x$, then, taking logarithms to the base 10,

$$\log y = x \log a + \log b.$$

Thus, if $Y = \log y$ be plotted against x, and the graph between the points be a straight line, then it can be assumed that the suspected law holds true, and the slope of the line $= \log a$, and the intercept on the y-axis is $\log b$, from which results the values of a and b can be derived.

EXAMPLE (L.U.). Draw the graph of $y = (x - 2)/(x - 3)$ for values of x between -2 and $+4$.

Draw also with the same axes and on the same scale the graph of $4y = x^2$.

From your graphs show that the equation $x^3 - 3x^2 - 4x + 8 = 0$ has three roots and read off their approximate values.

x	-2	-1	0	1	2	3	4	*Slightly* > 3	*Slightly* < 3
$y = \dfrac{(x - 2)}{(x - 3)}$	0·8	0·75	0·67	0·5	0	$\pm\infty$	2	$+\infty$	$-\infty$
$y = \frac{1}{4}x^2$	1	0·25	0	0·25	1	2·25	4		

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In the case of the curve $y = (x - 2)/(x - 3)$ the following intermediate points are required: $x = 2\frac{1}{2}$, $y = -1$, $x = 2\frac{3}{4}$, $y = -3$, $x = 3\frac{1}{4}$, $y = 5$, $x = 3\frac{1}{2}$, $y = 3$.

The graph is shown on next page.

The equation $x^3 - 3x^2 - 4x + 8 = 0$ can be written

$$x^3 - 3x^2 = 4(x - 2)$$
$$\text{i.e. } x^2(x - 3) = 4(x - 2)$$
$$\text{i.e. } \frac{x^2}{4} = \frac{x - 2}{x - 3}.$$

Hence the solutions of this equation are given by the points of intersection of the two graphs, which are three in number and are (from the graph) approximately $-1·75$, $1·25$, $3·5$.

EXAMPLE (L.U.). Taking $e = 2·718$, draw the graph of $\log_e (x + 3)$ between $x = -2·5$ and $x = 3$.

With the same axes and scales draw the graph of $5y = x^3$, and use these graphs to solve approximately the equation $(x + 3)^5 = e^{x^3}$.

NOTE. It is advisable (to save time) to use the Napierian log tables (logarithms to the base e).

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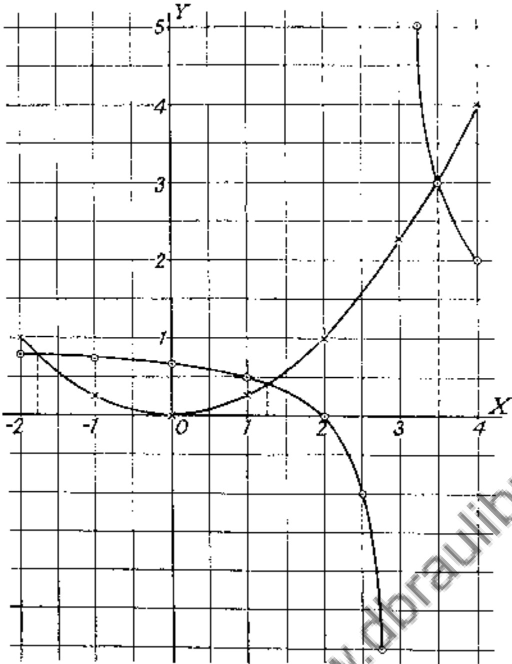

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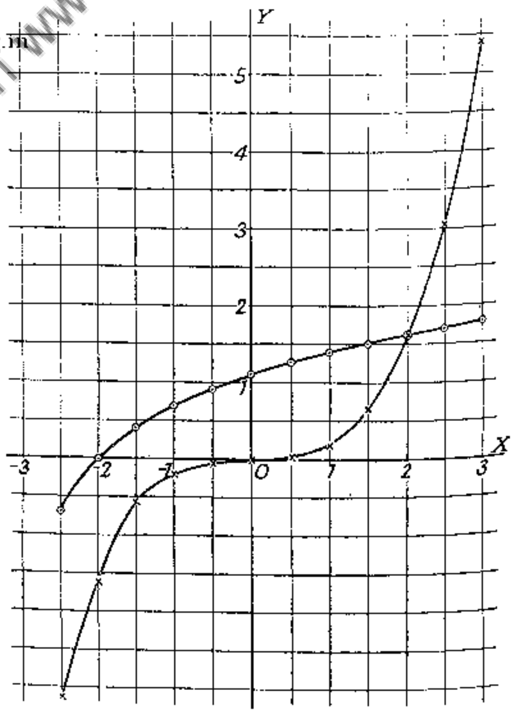

x	$-2{\cdot}5$	-2	$-1{\cdot}5$	-1	$-0{\cdot}5$	0
$y = \log_e(x+3)$	$-0{\cdot}69$	0	0·41	0·69	0·92	1·10
$y = \frac{1}{5}x^3$	$-3{\cdot}125$	$-1{\cdot}6$	$-0{\cdot}675$	$-0{\cdot}2$	$-0{\cdot}025$	0

x	0·5	1	1·5	2	2·5	3
$y = \log_e(x+3)$	1·25	1·39	1·50	1·61	1·70	1·79
$y = \frac{1}{5}x^3$	0·025	0·2	0·675	1·6	3·125	5·4

The graphs are shown opposite.

Where the curves intersect the values of y for the two curves are equal.

$$\therefore \log_e(x+3) = \tfrac{1}{5}x^3 \text{ at the points of intersection}$$
$$\text{i.e. } 5\log_e(x+3) = x^3$$
$$\text{i.e. } \log_e(x+3)^5 = x^3$$
$$\therefore (x+3)^5 = e^{x^3} \quad \dots\dots\dots (1)$$

at the points of intersection of the two graphs.

From the graph an approximation to the root of the equation is $x = 2$.

EXAMPLE (L.U.). Plot the points given in the annexed table of values:

x	2	4·9	7	10·1	16	18	20
y	2·1	2·21	2·32	2·45	2·62	2·72	2·8

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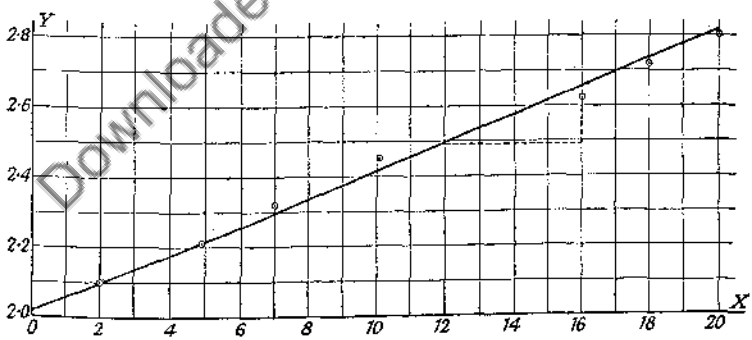
Draw the straight line lying most evenly among the points, and find its equation.

What would probably be the value of y when $x = 21{\cdot}5$ and when $x = 0$?



The graph is shown above and if the equation of the line be $y = mx + c$, it can be seen from the graph that c (intercept on y-axis) is 2·02 and the slope m is

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$$\frac{2{\cdot}65 - 2{\cdot}49}{16 - 12} \quad \text{(best to take whole number base)}$$

$$= \frac{0{\cdot}16}{4} = 0{\cdot}04$$

therefore equation of line is $y = 0{\cdot}04x + 2{\cdot}02$.

When $x = 21{\cdot}5$ probable value of $y = 0{\cdot}04 \times 21{\cdot}5 + 2{\cdot}02 = 2{\cdot}88$.

When $x = 0$ probable value of y is $2{\cdot}02$.

EXAMPLES III

1. (i) Prove that the remainder when a polynomial $f(x)$ is divided by $(x - a)$ is $f(a)$. (ii) Prove the identity

$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2).$$

2. Explain the difference between a conditional equation and an identity, and prove that, if $ax^2 + bx + c$ and $a_1x^2 + b_1x + c_1$ are equal for more than two values of x, the coefficients of like powers of x are equal.

Assuming the numbers A, B, C, D can be found so that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = A + Bn + Cn^2 + Dn^3$$

for all values of n, find the values of these numbers.

3. If $l^2 + m^2 + n^2 = \alpha^2 + \beta^2 + \gamma^2 = 1$, $l\alpha + m\beta + n\gamma = x$, prove that $(l\beta - m\alpha)^2 + (m\gamma - n\beta)^2 + (n\alpha - l\gamma)^2 + x^2 = 1$.

Hence, or otherwise, prove that if $x = \pm 1$, then $l/\alpha = m/\beta = n/\gamma$.

4. Find the H.C.F. and L.C.M. of the expressions $x^3 + 5x^2 + 8x + 4$, $x^2 - x - 6$, and $x^3 + 4x^2 + x - 6$.

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5. (i) Simplify $\dfrac{x + ay + a^2z}{(a - b)(a - c)} + \dfrac{x + by + b^2z}{(b - c)(b - a)} + \dfrac{x + cy + c^2z}{(c - a)(c - b)}$.

(ii) Find the factors of $2x^2 - 2y^2 - 3z^2 + 5yz + 5zx - 3xy$.

6. Simplify $\dfrac{x^2 - 8x + 12}{3x^2 - 17x - 6} - \dfrac{2x^2 + 5x + 2}{6x^2 + x - 1}$,

and find its value when $3x = \sqrt{2} - 1$.

7. Show that $x + y + z$ is a factor of $x^3 + y^3 + z^3 - 3xyz$ and find the other factor.

Eliminate x, y, z from the equations $x + y + z = a$, $x^2 + y^2 + z^2 = b^2$, $x^3 + y^3 + z^3 = c^3$, $xyz = d^3$.

8. If $f(x)$, a polynomial in x, be divided by $(ax + b)$, prove that the remainder is $f(-b/a)$.

Find the polynomial in x of the third degree that vanishes when $x = 1$ and $x = -2$, has the value -6 when $x = 0$, and leaves the remainder $-100/27$ when divided by $3x + 2$.

9. State the Remainder Theorem and use it to prove that $(x + y + z)$ is a factor of $(x^3 + y^3 + z^3 - 3xyz)$, and obtain the other factor.

If $x + y + z = 4$, $x^2 + y^2 + z^2 = 10$, $x^3 + y^3 + z^3 = 16$, find the value of xyz.

10. Substitute $(py + q)/(y + 1)$ for x in the function

$$5(2x + 1)/(3x^2 - 2x + 17)$$

and find the values that p and q must have in order that the resulting fraction may be of the form $(Ay^2 + B)/(Cy^2 + D)$, where A, B, C, D are numbers whose values are to be found.

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11. Find the highest common factor and, in factors, the least common multiple of the following expressions:

$a^2 + ab - 6b^2$; $(a + b)^3 - 4b^2(3a + 7b)$; $a(a + 4b)(a - 3b) + b^2(7a + 3b)$.

12. (i) If $\dfrac{3x^2 - 5}{(x + 2)^3} = \dfrac{A}{x + 2} + \dfrac{B}{(x + 2)^2} + \dfrac{C}{(x + 2)^3}$,

where A, B, C are constants, find A, B, C.

(ii) Show that $x^2 + 6x - 10$ can be expressed in the form

$$(x - \alpha)(x - \beta) + 2(x - \alpha) + 3x$$

in two different ways and find the values of α and β in each case.

13. If the fractions p_1/q_1, p_2/q_2, p_3/q_3, . . . are all equal to one another, prove that each fraction is equal to the fraction

$$\frac{ap_1 + bp_2 + cp_3 + \dots}{aq_1 + bq_2 + cq_3 + \dots}.$$

If $$\frac{x}{11x - 6y + 2z} = \frac{y}{-6x + 10y - 4z} = \frac{z}{2x - 4y + 6z},$$

and if $x + 2y + 2z$ be not equal to zero, prove that each fraction is equal to $\frac{1}{3}$. Hence, or otherwise, find the ratio $x : y : z$.

14. If $a_1 : b_1 = a_2 : b_2 = a_3 : b_3$, prove that each fraction is equal to $(pa_1 + qa_2 + ra_3)/(pb_1 + qb_2 + rb_3)$, where p, q, r are any quantities.

If $(ny + mz)/a = (lz + nx)/b = (mx + ly)/c$, prove that

$$\frac{x}{l(-al + bm + cn)} = \frac{y}{m(al - bm + cn)} = \frac{z}{n(al + bm - cn)}.$$

15. Prove that there are two values of λ for which the expression

$$x^2 - 2y^2 - 3z^2 + \lambda yz + 2zx + xy$$

may be expressed as the product of two linear factors and find the factors in each case.

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16. If $f(x) \equiv a_0x^n + a_1x^{n-1} + \dots + a_n$, prove that $f(c)$ is the remainder when $f(x)$ is divided by $x - c$.

Factorise the expression $(bc + ca + ab)^3 - b^3c^3 - c^3a^3 - a^3b^3$.

17. If $ax + by + cz = 0$, and $a^2x + b^2y + c^2z = 0$, find the ratio $x : y : z$.

Hence, show that, if $ab + bc + ca = 0$, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 0$.

18. If the expression $(x - a)^2 + (y - b)^2 + (a^2 + b^2 - 1)(x^2 + y^2 - 1)$ be denoted by E, prove that E is the sum of two squares of which $(bx - ay)^2$ is one.

If a and b are real quantities, find real values of x and y for which $E = 0$.

19. If $f(x) \equiv a_0x^n + a_1x^{n-1} + \dots + a_n$, where $a_0, a_1, \dots, a_n$ are rational, prove that if $x - \sqrt{p}$ be a factor of $f(x)$, $x + \sqrt{p}$ is also a factor, $\sqrt{p}$ being a surd.

Find an equation with rational coefficients which is satisfied by

$$x = 1 + \sqrt{2} + \sqrt{3}.$$

State the other roots of the equation obtained.

20. If $f(x) \equiv a_0x^n + a_1x^{n-1} + \dots + a_n$, prove that if $f(x)$ be divided by $(x - k)$, the remainder is $f(k)$.

If $(x - k)^2$ be a factor of $x^3 + 3px + q$, prove that $4p^3 + q^2 = 0$, and find the remaining factor.

21. Find a pair of factors of the expression $x^4 - 12x^2 + 4$ each of which is of the form $x^2 + ax + b$, where a is rational and not zero. Hence or

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otherwise, solve the equation $x^4 - 12x^2 + 4 = 0$. (Any square roots involved in the answers need not be evaluated.)

22. Find the three constants A, B, C such that

$$\frac{2x+1}{(x+1)^2(2x-5)} \equiv \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{(2x-5)}.$$

23. (i) Prove that

$$b - c)^3 + (c - a)^3 + (a - b)^3$$
$$= -3[(b + c)^2(b - c) + (c + a)^2(c - a) + (a + b)^2(a - b)].$$

(ii) Find the factors of $(b + c)^3(b - c) + (c + a)^3(c - a) + (a + b)^3(a - b)$.

24. If $(mz + ny)/a = (nx + lz)/b = (ly + mx)/c$, show that $x : y : z = l(bm + cn - al) : m(cn + al - bm) : n(al + bm - cn)$.

If $l = a$, $m = b$, $n = c$, and a, b, c are the sides of a triangle ABC, show that $x : y : z = \cos A : \cos B : \cos C$.

25. Prove that if $f(x)$ denotes a polynomial in x, the remainder obtained when $f(x)$ is divided by $(x - k)$ is $f(k)$.

When $f(x) \equiv ax^3 + bx + c$ is divided by $(x - 2)$, $(x + 1)$, $(x + 3)$ the remainders are 1, 2, and -4 respectively. Find the numerical values of a, b, c and determine the values of x for which $f(x) = 0$.

26. Find what values p must have in order that $(x - p)$ may be a factor of $4x^3 - (3p + 2)x^2 - (p^2 - 1)x + 3$.

Write down the remaining factor of the expression corresponding to each value of p.

27. Prove that $(x + y + z)$ is a factor of $x^3 + y^3 + z^3 - 3xyz$, and show that the other factor can be expressed as half the sum of the squares of the differences of x, y, z.

If $x = k(b + c - a)$, $y = k(c + a - b)$, $z = k(a + b - c)$, prove that $x^3 + y^3 + z^3 - 3xyz = 4k^3(a^3 + b^3 + c^3 - 3abc)$.

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28. If $A = by + \gamma/z$, $B = cz + \alpha/x$, $C = ax + \beta/y$, verify that $ABC - a\alpha A - b\beta B - c\gamma C = abcxyz + \alpha\beta\gamma/xyz$.

Solve the equations $y + 2/z = 15/2$, $z + 2/x = 4/3$, $x + 2/y = 10/3$.

29. Draw the graph of $y = \log_2 x$ for values of x between 0 and 8, taking one inch as the unit for both x and y. Also, with the same axes and unit of length, draw the graph of $y = x^2 - 3x$.

Hence, find the approximate solutions of the equation $2^{x^2-3x} = x$.

30. On the same axes, and with the same scales, draw graphs of

(i) $y = \log_{10} x$; (ii) $y = x^2 - 4x + 3$,

and hence find solutions of the equation $10^{x^2-4x+3} = x$, explaining your method.

Give reasons, based on your graphs, why you would expect the equation $10^{(x-a)(x-b)} = x$ always to have two real roots if either a or b is greater than unity.

31. Draw the graph of $y = x^3 + x^2$ for values of x from $-3{\cdot}5$ to $+3$. Take 1 inch as unit for x and 0·1 inch as unit for y. On the same axes draw the graph of $y + 6 = k(2x + 1)$ for several different values of k.

Explain a method of solving graphically the equation $x^3 + x^2 - 2kx = k - 6$ for any given value of k. Find for what value of k the equation has one root equal to 2, and, for this value of k, find the other two roots.

32. With the same axes and to the same scales draw the graphs of $y = \log_{10} x$, $y = \log_{10} 10x$, and $y = \log_{10}(x + 1)$ for values of x between 0 and 10.

By using your graphs, solve the equation $1 + \log_{10} x = \log_{10}(x + 1)$.

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33. (i) If $2^x = 3^y = 12^z$, prove that $xy = z(x + 2y)$. (ii) Variables x and y are related by a law of the form $y = kx^n$. Approximate values of y for various values of x are given by the table:

x	3	$4\frac{1}{4}$	$5\frac{1}{3}$	8	10	11	12
y	22	26	30	36	40	42	44

From the graph of $\log y$ against $\log x$, deduce the values of the constants k and n.

34. (i) Show that the equation $kx(1 - x) = 1$ has no real roots if $0 < k < 4$. (ii) Draw the graph of $y = 1/x(1 - x)$ for values of x between -3 and $+4$.

From your graph obtain approximate values of the roots of the equation (i) when $k = -1$.

35. Trace the curve $y = 6/(x + 2)$. Find the co-ordinates of the points in which it is met by the straight line $y = m(x + 2)$, and show that the line joining these points is bisected by the axis of x.

36. Trace the curve $y = x - 4/x$ between $x = 1$ and $x = 4$. Find the equation of the line joining the end points of this part of the curve, and the point where this line crosses the axis of x.

37. Find the range of values of k for which the equation

$$k(2x - 5) = x^2 - 3x + 2$$

has real roots.

Illustrate your result by drawing the graphs of $y = x^2 - 3x + 2$, and of $y = k(2x - 5)$.

Determine the range of values of x for which $4x - 10$ is greater than $x^2 - 3x + 2$.

38. Prove that the points of intersection of the circle $(x - 1)^2 + y^2 = 9$ and the curve $y = \frac{1}{2}x^2$ are roots of the equation $x^4 + 4x^2 - 8x - 32 = 0$.

Show from a sketch that the positive root lies between 2 and 3, and find a more exact value from a large-scale graph between the values of x.

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39. Draw in the same figure the graphs of $[8(2x - 1)(x - 1)]$ and $\log_{10} x$ from $x = 0{\cdot}2$ to $x = 1{\cdot}2$, using the same axes and scales.

State from your figure the *number* of real roots of the equation

$$8(2x - 1)(x - 1) = \log_{10} x.$$

Show that one root lies between 0·5 and 0·6 and by inspection of your tables, or otherwise, determine the value of this root correct to two decimal places.

40. By plotting the curves $y = 10 - x^2$ and $y = 1/x$ from $x = -4$ to $x = 4$, find approximate values of the roots of the equation $x^3 - 10x + 1 = 0$.

Find the smaller of the positive roots more accurately by plotting the curves again from $x = 0$ to $x = 1$, using a conveniently larger scale along the axis of x.

41. With the same origin and axes, draw the graphs of the functions $(2 + x)/1 + x)$ and $\frac{1}{4}x^2$ for positive values of x, and hence obtain the positive root of the equation $x^3 + x^2 - 4x - 8 = 0$.

42. With the same axes draw the graphs of $y = x(2x + 1)$ and $y = 4/x^2$ from $x = -2$ to $x = +2$. Deduce that the equation $2x^4 + x^3 - 4 = 0$ has two real roots and find them approximately.

43. Draw the graph of $y = 2x/(x + 1)$ for values of x from -4 to $+4$.

Find the value of m for which the quadratic equation $m(x + 1) = 2x/(x + 1)$ has equal roots, and verify this value from your graph.

44. Sketch the curve $y = x^3 - x - 1$, and calculate, correct to two decimal places, the value of x for which y is zero.

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45. Prove that the roots of the equation $kx(x+3) = (x+1)(x-2)$ are real if k is real. Sketch the curve represented by

$$y = \frac{(x+1)(x-2)}{x(x+3)}.$$

46. If $y = x/a + b/x$, where x is a real variable and a, b are real constants, show that y cannot lie between $\pm\sqrt{(4b/a)}$ if $ab > 0$, but that y can take all real values if $ab < 0$.

Sketch, in separate diagrams, the two curves $y = \frac{1}{2}x + 2/x$, $y = \frac{1}{2}x - 2/x$.

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CHAPTER IV

Arithmetical, Geometrical, Harmonic Progressions, Present Values, etc.

Arithmetical Progression. An *arithmetical progression* (A.P.) is a series of terms that increases by a constant amount which may be positive or negative. This constant amount is known as the *common difference* of the series and is usually denoted by d.

The following series are all arithmetical progressions.

$1, 2, 3, 4, \ldots$	common difference $= +1$.
$\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2, \ldots$	" " $= +\frac{1}{4}$.
$6, 3, 0, -3, \ldots$	" " $= -3$.
$-2, -1\frac{1}{2}, -1, -\frac{1}{2}, \ldots$	" " $= +\frac{1}{2}$.

To ascertain if a given series be an A.P. it is necessary to subtract from *each* term (except the first) the preceding term. If the results in all cases be the same, the given series will be an A.P. whose common difference is this common result.

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Theorem. *To find a formula for the* n*th term of an A.P., whose common difference is d and whose first term is a.*

$$\begin{aligned} \text{First term} &= a &&= a + (1 - 1)d. \\ \text{Second term} &= a + d &&= a + (2 - 1)d. \\ \text{Third term} &= a + 2d &&= a + (3 - 1)d. \end{aligned}$$

From these results it can be seen that the nth term is $a + (n - 1)d$.

The *arithmetic mean* (A.M.) between two quantities a and b is that quantity which, when inserted between a and b, forms with them three successive terms of an A.P.

Theorem. *To find the arithmetic mean (A.M.) between a and b.*

Let x be the required A.M. Then, a, x, b will be three successive terms of an A.P. The common difference of the A.P. is $(x - a)$ and also $b - x$

$$\therefore x - a = b - x$$
$$\text{i.e. } 2x = a + b$$
$$\therefore x = \frac{a + b}{2}.$$

More generally, the n arithmetic means between a and b are the n quantities that, when inserted between a and b, form with them $(n + 2)$ successive terms of an A.P.

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Theorem. *To find the n arithmetic means between a and b.*

Let d be the common difference of the A.P. formed. Then b is the $(n+2)$th term of the A.P.

$$\therefore b = a + (n+1)d,$$
$$\therefore d(n+1) = b - a,$$
$$\therefore d = \frac{(b-a)}{n+1}.$$

The required arithmetic means are $a+d$, $a+2d$, ..., $a+nd$, where $d = (b-a)/(n+1)$.

Theorem. *To find the sum S of n terms of an A.P. whose first term is a and whose common difference is d.*

The last term of the series will be $a + (n-1)d$,

$$\therefore S = a + (a+d) + (a+2d) + \ldots + [a + (n-1)d] \ldots (1).$$

Reversing the series,

$$S = [a + (n-1)d] + [a + (n-2)d] + [a + (n-3)d] + \ldots + a \ldots (2).$$

(1) + (2) gives,

$$2S = [2a + (n-1)d] + [2a + (n-1)d] + \ldots + [2a + (n-1)d] \text{ to } n \text{ terms}$$
$$= n[2a + (n-1)d].$$

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$$\therefore S = \frac{n}{2}[2a + (n-1)d].$$

Using l for the last term $[a + (n-1)d]$ the result can be written

$$S = \frac{n(a+l)}{2}.$$

The second result is more convenient than the first when the first and last terms are given.

When $a = 1$ and $d = 1$ the sum of the series becomes

$$1 + 2 + 3 + \ldots + n,$$

which is the sum of the first n *natural numbers*, and the value of the sum is $n(n+1)/2$.

When the sum S, the first term a, and the common difference d of an A.P. be given, and it is required to find the number of terms in the series, the formula

$$S = \frac{n}{2}[2a + (n-1)d]$$

is used giving rise to a quadratic in n from which two values of n can be found.

In certain types of problems of this nature only one value of n is admissible (n cannot be negative or fractional).

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EXAMPLE. The sum of an A.P. is 20, the first term being 8 and the common difference -2. Find the number of terms in the series.

Let n be the number of terms.

Using the formula

$$S = \tfrac{1}{2}n[2a + (n - 1)d]$$

with standard notation, in this case

$$20 = \tfrac{1}{2}n[16 - 2(n - 1)] = n(8 - n + 1) = 9n - n^2.$$
$$\therefore n^2 - 9n + 20 = 0,$$
$$\text{i.e. } (n - 4)(n - 5) = 0,$$
$$\therefore n = 4 \text{ or } 5.$$

EXAMPLE. Find the number of terms in an A.P. whose first term is 5, common difference 3, and sum 55.

Let n be the required number of terms.

$$\therefore 55 = \tfrac{1}{2}n[10 + 3(n - 1)]$$
$$(\text{using } S = \tfrac{1}{2}n[2a + (n - 1)d])$$
$$= \tfrac{1}{2}n(7 + 3n).$$
$$\therefore 110 = 7n + 3n^2, \text{ i.e. } 3n^2 + 7n - 110 = 0,$$
$$\therefore (3n + 22)(n - 5) = 0, \ \therefore n = -\frac{22}{3} \text{ or } 5.$$

But n must be a positive integer $\therefore n = 5$.

EXAMPLE (L.U.). The sum of the first n terms of a series is $2n^2 - n$. Find the nth term and show that the series is an A.P.

Using $n = 1$ in the sum for n terms, it is seen that the first term $= 2 - 1 = 1$.

Using $n = 2$, the sum of the first two terms $= 8 - 2 = 6$, therefore the second term is 5.

Using $n = 3$, the sum of the first three terms $= 18 - 3 = 15$, therefore the third term $= 15 - 6 = 9$.

Replacing n by $(n - 1)$ the sum of the first $(n - 1)$ terms is

$$2(n - 1)^2 - (n - 1) = 2n^2 - 4n + 2 - n + 1$$
$$= 2n^2 - 5n + 3.$$
$$\therefore n\text{th term} = \text{sum of first } n \text{ terms} - \text{sum of first } (n - 1) \text{ terms}$$
$$= 2n^2 - n - (2n^2 - 5n + 3)$$
$$= 2n^2 - n - 2n^2 + 5n - 3 = 4n - 3.$$

The series is 1, 5, 9, . . . $(4n - 3)$, which is an *A.P.* of common difference 4.

NOTE. When dealing with a problem involving three (five, or any odd number of terms) in A.P., it is advisable to use the *middle* term as a with d as the common difference.

EXAMPLE. The sum of five numbers in A.P. is 25 and the sum of their squares is 165. Find the numbers.

Let the middle term be a and the common difference of the series be d then the terms are $(a - 2d)$, $(a - d)$, a, $a + d$, $a + 2d$.

From the question,

$$(a - 2d) + (a - d) + a + (a + d) + (a + 2d) = 25$$
$$\text{i.e. } 5a = 25 \ \therefore a = 5.$$

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Also, $(a - 2d)^2 + (a - d)^2 + a^2 + (a + d)^2 + (a + 2d)^2 = 165$

i.e. $(a^2 - 4ad + 4d^2) + (a^2 - 2ad + d^2) + a^2$
$$+ (a^2 + 2ad + d^2) + (a^2 + 4ad + 4d^2) = 165$$
$$\therefore\ 5a^2 + 10d^2 = 165,$$
$$\text{i.e. } a^2 + 2d^2 = 33, \ \therefore\ 25 + 2d^2 = 33$$
$$\therefore\ 2d^2 = 8, \text{ i.e. } d^2 = 4, \ \therefore\ d = \pm 2.$$

Hence, the series is 1, 3, 5, 7, 9.

The Geometrical Progression. The geometrical progression (G.P.) is a series of terms that increase or decrease in a constant ratio. This constant ratio is known as the *common ratio* of the series and is usually denoted by r.

The following series are all geometrical progressions (G.P.).

$2, 4, 8, 16, \ldots$	common ratio	$+2.$
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$	,, ,,	$+\frac{1}{2}.$
$-\frac{1}{3}, 1, -3, 9, \ldots$	,, ,,	$-3.$
$-2, \frac{1}{2}, -\frac{1}{8}, \frac{1}{32}, \ldots$	,, ,,	$-\frac{1}{4}.$

If, in a given series, the ratio of each term to the preceding term is the same for all terms, the series must be a G.P. with this ratio as the common ratio.

Theorem. *To find the* n*th term of a G.P. whose common ratio is* r*, and first term* a.

$$\text{First term} = a = ar^0 = ar^{1-1}.$$
$$\text{Second term} = ar = ar^{2-1}.$$
$$\text{Third term} = ar^2 = ar^{3-1}.$$

From these it can be seen that the nth term is ar^{n-1}.

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The *geometric mean* (G.M.) between two quantities a and b is that quantity which, when inserted between a and b, forms with them three successive terms of a G.P.

Theorem. *To find the geometric mean between* a *and* b.

Let x be the required G.M. Then, a, x, b will be three successive terms of a G.P. The common ratio of the G.P. will be x/a and also b/x.
$$\therefore\ x/a = b/x$$
$$\therefore\ x^2 = ab$$
$$\text{i.e. } x = \pm\sqrt{ab}.$$

More generally, the *n geometric means* between a and b are the n quantities that, when inserted between a and b, form with them $(n + 2)$ terms (successive) of a G.P.

Theorem. *To find the* n *geometric means between* a *and* b.

Let r be the common ratio of the G.P. formed. Then, b is the $(n + 2)th$ term of the G.P.
$$\therefore\ b = ar^{n+1},$$
$$\text{i.e. } r^{n+1} = b/a$$
$$\therefore\ r = \sqrt[n+1]{b/a}.$$

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Using this value of r, the required geometric means will be

$$ar, ar^2, \ldots, ar^n.$$

Theorem. *To find the sum of the first n terms of a G.P. whose common ratio is r and first term a.*

Let S_n be the required sum. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \ldots\ldots\ldots\ldots (1),$$

$$\therefore\ rS_n = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n \ldots\ldots (2).$$

(1) − (2) gives, $S_n(1 - r) = a - ar^n$

$$= a(1 - r^n)$$

$$\therefore\ S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1}.$$

NOTE. The first form of the result is used when $|r| < 1$ and the second when $|r| > 1$, where the symbol $|r|$ is known as the *modulus of r*, and is here used to denote the numerical value of r when r is *real*, i.e. its value not taking into account the sign. Thus, $|-2| = 2$, $|-\frac{1}{3}| = \frac{1}{3}$, etc.

When $|r| < 1$, as n increases r^n gets smaller and smaller (n positive integer), and it can be said that r^n approaches zero as n approaches infinity. Similarly when $|r| > 1$, r^n approaches $\pm\infty$ as n approaches ∞.

The value that S_n approaches as n approaches infinity is known as its *sum to infinity* (S_∞).

From the previous result

$$S_n = a\frac{(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

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therefore as $n \to \infty$ (i.e. as n approaches infinity) $r^n \to 0$, if $|r| < 1$, and the second term becomes negligible, whilst $r^n \to \pm\infty$ when $|r| > 1$ and S_n is then infinite.

Thus, when $|r| < 1$, $S_\infty = \dfrac{a}{1 - r}$.

If the sum to infinity is finite, as in this case, the series is said to be *convergent*. When $|r| > 1$, $S_\infty = \pm\infty$ and the series is *divergent*.

EXAMPLE. Insert three geometric means between $2\frac{1}{4}$ and $\frac{4}{9}$.

Let r be the common ratio of the G.P. formed. Since $\frac{4}{9}$ is the fifth term of the G.P.

$$\frac{4}{9} = \frac{9}{4}r^4 \quad \therefore \quad r^4 = \frac{16}{81}$$

$$\therefore\ r^2 = \frac{4}{9}(r \text{ real}) \quad \therefore \quad r = \pm\frac{2}{3}$$

therefore required geometric means are $\frac{3}{2}, 1, \frac{2}{3}$ or $-\frac{3}{2}, 1, -\frac{2}{3}$.

EXAMPLE. In a geometrical progression the first term is 7, the last term 448, and the sum 889. Find the common ratio.

Let r be the common ratio, n the number of terms, and S_n the sum of n terms.

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$$\therefore\ S_n = 889 = 7\frac{(1-r^n)}{1-r} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also $$448 = 7r^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Using (2) in (1), $$889 = \frac{7-7r^n}{1-r} = \frac{7-448r}{1-r}$$

$$\therefore\ 889 - 889r = 7 - 448r$$

$$\therefore\ 882 = 441r \quad \therefore \quad r = 2.$$

EXAMPLE (L.U.). Find the sum of the first six terms of the geometric series whose third term is 27 and whose sixth term is 8.

Find how many terms of this series must be taken if their sum is to be within 1/10 % of the sum to infinity.

Let r be the common ratio of the series and a the first term.

$$\therefore\ ar^2 = 27 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$ar^5 = 8 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

(2) ÷ (1) gives, $r^3 = 8/27 \quad \therefore \quad r = 2/3.$

Using this in (1), $a \times \frac{4}{9} = 27 \quad \therefore \quad a = \frac{243}{4}.$

The sum of the first six terms of this series

$$= \frac{a(1-r^6)}{1-r} = \frac{243}{4}\,\frac{\left\{1-\left(\frac{2}{3}\right)^6\right\}}{1-\frac{2}{3}} = \frac{243}{4}\,\frac{\left\{1-\frac{2^6}{3^6}\right\}}{\frac{1}{3}}$$

$$= \frac{729}{4}\left\{1-\frac{2^6}{3^6}\right\} = \frac{729}{4} - \frac{729}{4}\times\frac{2^6}{3^6} = \frac{729}{4} - 16$$

$$= \frac{665}{4}$$

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Let n be the number of terms required so that their sum shall be 1/10% less than the sum to infinity S_∞.

Now, $S_\infty = \frac{a}{1-r}$, and $S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$

$$\therefore\ S_\infty - S_n = \frac{ar^n}{1-r}.$$

But $S_\infty - S_n = \frac{1}{10}$ per cent of S_∞ $\therefore\ \frac{S_\infty - S_n}{S_\infty} = \frac{1}{1{,}000}$

$$\therefore\ \frac{ar^n}{1-r} \div \frac{a}{1-r} = \frac{1}{1{,}000},\ \text{i.e. } r^n = \frac{1}{1{,}000}$$

$$\text{i.e. } \left(\frac{2}{3}\right)^n = \frac{1}{1{,}000}.$$

Taking logs to the base 10,

$$n[\log 2 - \log 3] = -3$$

$$\therefore\ n = \frac{3}{\log 3 - \log 2} = \frac{3}{0{\cdot}47712 - 0{\cdot}30103}$$

$$= \frac{3}{0{\cdot}17609} = 17{\cdot}03\ldots$$

therefore required number of terms is the next integer, i.e. 18.

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Interest, Annuities, Present Values, etc. If a sum of money be invested at interest the amount invested is known as the *principal* (£P) and the accumulated money (including interest) at the end of the period is known as the *amount*. There are two types of interest; (i) *simple interest* paid yearly, and (ii) *compound interest* which remains invested earning further interest.

If a certain sum of money £P be owing to a person and is to be repaid after a period of time, the amount which would be required to produce £P when invested for that period of time at a fixed rate of interest is known as its *present value* (P.V.) and the difference between the debt £P and its present value is the *true discount* on the debt.

When a banker discounts a bill (i.e. gives the value of the bill due at the end of a certain time less his discount) he deducts a percentage of the *face value* of the bill (value stated on the bill) according to the rate of interest prevailing. This discount (similar to simple interest) is known as the *banker's discount*, and is slightly larger than the ordinary true discount.

Theorem. *To find the present value £V, and the true discount £D of a given sum £P due in n years at r% simple interest (S.I.).*

£V invested at r% S.I. for n years will amount to £$V\left(1 + \frac{rn}{100}\right)$

$$\therefore\ V\left(1 + \frac{rn}{100}\right) = P$$

$$\therefore\ V = P\Big/\left(1 + \frac{rn}{100}\right)$$

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Now $$D = P - V = P - \frac{P}{1 + rn/100} = \frac{Prn}{100 + rn}.$$

The banker's discount in this case $= Prn/100$.

NOTE. If £P be invested at r% (paid yearly) compound interest for n years, the amount £A accruing is given by

$$A = P\left(1 + \frac{r}{100}\right)^n.$$

R is usually used to denote $1 + r/100$, and the above result becomes

$$A = PR^n.$$

If interest be paid m times a year (compound interest) in the above case, there will be mn periods of time in each of which the rate of interest is (r/m)%, and the result now becomes

$$A = P\left(1 + \frac{r}{100m}\right)^{mn}.$$

The compound interest in each case is *obtained by taking* the value of $(A - P)$.

Theorem. *To find the present value (P.V.) and true discount of £P in n years allowing compound interest (C.I.) at r% per annum.*

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If £V be the present value, then £V would have to be invested for n years at r% to produce an amount of £P.

$$\therefore P = VR^n, \text{ where } R = 1 + r/100$$
$$\therefore V = PR^{-n}.$$

If £D be the true discount

$$D = P - V = P - PR^{-n} = P(1 - R^{-n}).$$

An *annuity* is a fixed sum paid periodically under certain stated conditions, the payments being either paid once a year, or at more frequent intervals (once a year unless otherwise stated).

An *annuity certain* is one payable for a fixed term of years independent of any contingency (such as death).

A *life annuity* is only payable during the life of the person involved.

A *deferred annuity* (or reversion) is an annuity not beginning until after the lapse of a certain number of years. If deferred for n years the first payment is made at the end of $(n + 1)$ years.

If an annuity continues for ever it is called a *perpetuity*, and if it does not commence immediately it is known as a *deferred perpetuity*.

An annuity left unpaid for n years is said to be *forborne* for n years.

Theorem. *To find the amount of an annuity left unpaid for n years at r% S.I.*

Let £A be the annuity and £N the amount due in n years at r% S.I.

At the end of the first year £A is due, and in $(n - 1)$ years will amount to

$$£A\left\{1 + \frac{(n-1)r}{100}\right\}.$$

At the end of the second year £A is due, and in $(n - 2)$ years will amount to

$$£A\left\{1 + \frac{(n-2)r}{100}\right\}.$$

At the end of the third year £A is due, and in $(n - 3)$ years will amount to

$$£A\left\{1 + \frac{(n-3)r}{100}\right\},$$

and so on.

$$\text{Hence } N = A\left\{1 + \frac{(n-1)r}{100}\right\} + A\left\{1 + \frac{(n-2)r}{100}\right\} + A\left\{1 + \frac{(n-3)r}{100}\right\} + \ldots + A$$

$$= A\left\{n + \frac{r}{100} + \frac{2r}{100} + \ldots + \frac{(n-1)r}{100}\right\}$$

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$$= A\left\{n + \frac{r}{100}(1 + 2 + \ldots + \overline{n-1})\right\}$$

$$= A\left\{n + \frac{rn(n-1)}{200}\right\} \qquad \text{(Using an A.P.)}$$

Theorem. *To find the amount of annuity of £A left unpaid for n years at r% C.I.*

Let £N be the amount and $R = 1 + r/100$. At the end of the first year £A is due and in $(n-1)$ years will amount to £AR^{n-1}.

At the end of the second year £A is due and in $(n-2)$ years will amount to £AR^{n-2}.

At the end of the third year £A is due and in $(n-3)$ years will amount to £AR^{n-3}, and so on.

$$\therefore\ N = AR^{n-1} + AR^{n-2} + Ar^{n-3} + \ldots + A$$
$$= A[1 + R + R^2 + \ldots + R^{n-2} + R^{n-1}]$$
(G.P. common ratio R)
$$= \frac{A(R^n - 1)}{R - 1}.$$

NOTE. In finding the present value of an annuity it is always customary to use C.I., unless otherwise stated.

Theorem. *To find the present value (i.e. the amount that should be paid) for an annuity of £A to continue for n years, reckoning C.I. at r% per annum.*

Let $R = 1 + r/100$. www.dbraulibrary.org.in

The P.V. of £A paid in one year's time is £AR^{-1}.
The P.V. of £A paid in two years' time is £AR^{-2}.
The P.V. of £A paid in three years' time is £AR^{-3}.

. . . .

The P.V. of £A paid in n years' time is £AR^{-n}

$$\therefore\ V = AR^{-1} + AR^{-2} + \ldots + AR^{-n}$$
$$= AR^{-1}(1 + R^{-1} + R^{-2} + \ldots + R^{-n+1})$$
$$= AR^{-1}\frac{(1 - R^{-n})}{1 - R^{-1}} = \frac{A(1 - R^{-n})}{R - 1}.$$

In the case of a perpetuity $n \to \infty$, $R^{-1} < 1$, and thus $R^{-n} \to 0$

$$\therefore\ V = A/(R-1).$$

Depreciation. If a machine or piece of property depreciates at the rate of r% per annum and £P is its initial value, then the value at the end of the first year is

$$£P_1 = £P - £P\frac{r}{100} = £P\left(1 - \frac{r}{100}\right).$$

Its value £P_2 at the end of the second year is similarly

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$$£P_1\left(1-\frac{r}{100}\right) = £P\left(1-\frac{r}{100}\right)\left(1-\frac{r}{100}\right) = £P\left(1-\frac{r}{100}\right)^2.$$

Value at the end of the third year $= £P_3$

$$= £P_2\left(1-\frac{r}{100}\right)$$

$$= £P\left(1-\frac{r}{100}\right)^2\left(1-\frac{r}{100}\right)$$

$$= £P\left(1-\frac{r}{100}\right)^3.$$

Hence, at the end of n years its value will be

$$£P\left(1-\frac{r}{100}\right)^n.$$

EXAMPLE (L.U.). A company sets aside £5,000 out of its profits at the end of each year to form a reserve fund and invest the amount at 4% per annum C.I. What is the value of the fund at the end of ten years?

It will be assumed that the first £5,000 is invested at the *end* of the first year, as the question is not clear on this point.

Let $R = 1{\cdot}04$.

The first investment of £5,000 will in 9 years amount to $£5,000R^9$.
,, second ,, ,, ,, ,, ,, 8 ,, ,, ,, $£5,000R^8$.
,, third ,, ,, ,, ,, ,, 7 ,, ,, ,, $£5,000R^7$,
and so on.

The value of the fund at the end of ten years

$= £5,000R^9 + £5,000R^8 + \ldots + £5,000$

$= £5,000[1 + R + R^2 + \ldots + R^9]$

$= £5,000\left\{\dfrac{R^{10}-1}{R-1}\right\} = £5,000\left\{\dfrac{1{\cdot}04^{10}-1}{1{\cdot}04-1}\right\}$

$= £\dfrac{5,000}{0{\cdot}04}[1{\cdot}04^{10} - 1]$

log 1·04	=	0·01703
log $1{\cdot}04^{10}$	=	0·1703
∴ $1{\cdot}04^{10}$	=	1·480

$= £\dfrac{5,000}{0{\cdot}04} \times 0{\cdot}48 = £5,000 \times 12 = £60,000.$

EXAMPLE (L.U.). A manufacturer buys a machine for £5,000. Its value depreciates by 5% each year. At the end of the fifth year he decides to put by an annual sum which, with the sale of the old machine, will enable him to buy a new machine of equal value at the end of the fifteenth year, the last instalment being made at the end of the fifteenth year. Reckoning interest at 3% per annum, determine the annual sum he must put by.

Let $£x$ be the annual sum to be put by. The depreciated value of the machine at the end of the fifteenth year

$= £5,000\left(\dfrac{95}{100}\right)^{15}$

$= £5,000(0{\cdot}95)^{15}$

$= £2,316$

log 0·95	=	$\bar{1}{\cdot}97772$
15 log 0·95	=	$\bar{1}{\cdot}6658$
∴ $0{\cdot}95^{15}$	=	0·46324

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Let $R = 1{\cdot}03$.

The first instalment of £x becomes in 10 years £xR^{10}.
„ second „ „ „ „ „ 9 years £xR^9 and so on.
Therefore value of instalments at the end of the fifteenth year

$$= £xR^{10} + £xR^9 + \ldots + £x.$$
$$= £x(1 + R + R^2 + \ldots + R^{10})$$
$$= £x\frac{(R^{11} - 1)}{R - 1} = £x \times \frac{0{\cdot}3844}{0{\cdot}03}$$

$\log 1{\cdot}03 = 0{\cdot}01284$
$11 \log 1{\cdot}03 = 0{\cdot}14124$
$\therefore 1{\cdot}03^{11} = 1{\cdot}3844$

Hence, $2{,}316 + x \times \frac{0{\cdot}3844}{0{\cdot}03} = 5{,}000$

$$\therefore x \times \frac{38{\cdot}44}{3} = 2{,}684 \quad \therefore x = \frac{2{,}684 \times 3}{38{\cdot}44}$$

i.e. £x = £209·5 (by logarithms).

EXAMPLE (L.U.). Show that the sum of n terms of the series

$$a + ar + ar^2 + \ldots \text{ is } a(1 - r^n)/(1 - r)$$

A debt of £10,000 is to be repaid by ten equal annual instalments, the first to be paid two years after borrowing. Find the value of each instalment, reckoning compound interest at 4% per annum.

The summation has been proved as a theorem.
Let £x be the value of each instalment and $R = 1{\cdot}04$.

$\therefore$ present value of first instalment $= £xR^{-2}$,
„ „ „ second „ $= £xR^{-3}$,
. . . .
present value of tenth instalment $= £xR^{-11}$.

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Therefore present value of all instalments

$$= £x[R^{-2} + R^{-3} + \ldots + R^{-11}]$$
$$= £xR^{-2}[1 + R^{-1} + \ldots + R^{-9}] = £xR^{-2}\frac{[1 - R^{-10}]}{1 - R^{-1}}$$
$$= £xR^{-1}\frac{[1 - R^{-10}]}{R - 1}$$
$$\therefore x(1{\cdot}04)^{-1}\frac{[1 - (1{\cdot}04)^{-10}]}{0{\cdot}04} = 10{,}000$$

$$\text{i.e. } x = \frac{400}{1{\cdot}04^{-1} - (1{\cdot}04)^{-11}}$$
$$= \frac{400}{0{\cdot}96154 - 0{\cdot}64963}$$
$$= \frac{400}{0{\cdot}31191}$$
$$= 1{,}282$$

$\log 1{\cdot}04 = 0{\cdot}01703$
$-11 \log 1{\cdot}04 = \bar{1}{\cdot}81267$
$\therefore 1{\cdot}04^{-11} = 0{\cdot}64963$

Therefore value of each instalment = £1,282.

Harmonic Progressions (H.P.). A series of terms is said to be in *harmonic progression* if their inverses are in A.P.

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Thus, $a, b, c, \ldots$ are in harmonic progression if $1/a$, $1/b$, $1/c, \ldots$ are in A.P.

The only theorem required is the following.

Theorem. *To find the* harmonic mean *between the two quantities* a, b *(i.e. the quantity which, when inserted between them, forms with them three successive terms of an H.P.)*

Let x be the required harmonic mean. Then a, x, b are consecutive terms of an H.P. and $1/a, 1/x, 1/b$ are three consecutive terms of an A.P.

$$\therefore \frac{1}{b} - \frac{1}{x} = \frac{1}{x} - \frac{1}{a}$$

$$\text{i.e. } \frac{1}{a} + \frac{1}{b} = \frac{2}{x},$$

$$\therefore \frac{a+b}{ab} = \frac{2}{x}$$

$$\therefore \frac{x}{2} = \frac{ab}{a+b}$$

$$\text{i.e. } x = \frac{2ab}{a+b}.$$

Arithmetico-geometrical Progression. The series

$$a,\ (a+d)x,\ (a+2d)x^2,\ \ldots\ (a+\overline{n-1}d)x^{n-1}$$

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is an arithmetico-geometrical progression in x, i.e. a series of terms in which the coefficients of the powers of x are in A.P. and the x portions of the terms are in G.P.

Theorem. *To find the sum S_n of n terms of the above arithmetico-geometrical progression.*

NOTE. The method is similar to that for the summation of a G.P. from first principles.

$$S_n = a + (a+d)x + (a+2d)x^2 + \ldots + (a+\overline{n-1}d)x^{n-1} \ldots (1).$$

$$xS_n = ax + (a+d)x^2 + \ldots + (a+\overline{n-2}d)x^{n-1} + (a+\overline{n-1}d)x^n \ldots (2).$$

(1) − (2) gives,

$$\begin{aligned} S_n(1-x) &= a + dx + dx^2 + \ldots + dx^{n-1} - (a+\overline{n-1}d)x^n \\ &= a - (a+\overline{n-1}d)x^n + dx(1 + x + x^2 + \ldots + x^{n-2}) \\ &= a - (a+\overline{n-1}d)x^n + dx\frac{(1-x^{n-1})}{1-x}, \end{aligned}$$

$$\therefore S_n = \frac{a-(a+\overline{n-1}d)x^n}{1-x} + \frac{dx(1-x^{n-1})}{(1-x)^2}.$$

EXAMPLE. Find the sum of the series to n terms $1,\ 2.3,\ 3.3^2,\ 4.3^3, \ldots$ Let S_n be the required sum.

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$$\therefore\ S_n = 1 + 2.3 + 3.3^2 + 4.3^3 + \ldots + n.3^{n-1} \ldots\ldots (1)$$
$$3S_n = 1.3 + 2.3^2 + 3.3^3 + \ldots + (n-1).3^{n-1} + n3^n \ldots (2)$$

(1) − (2) gives,

$$\begin{aligned} -2S_n &= 1 + 1.3 + 1.3^2 + \ldots + 1.3^{n-1} - n.3^n \\ &= 1 - n.3^n + 3(1 + 3 + 3^2 + \ldots + 3^{n-2}) \\ &= 1 - n3^n + 3\frac{(3^{n-1} - 1)}{3 - 1} \\ &= 1 - n.3^n + \tfrac{3}{2}(3^{n-1} - 1) \\ &= -n3^n + \tfrac{1}{2}3^n - \tfrac{1}{2} \\ &= -\tfrac{1}{2}[3^n(2n - 1) + 1], \\ \therefore\ S_n &= \tfrac{1}{4}[3^n(2n - 1) + 1]. \end{aligned}$$

The Powers of the First n Natural Numbers. As stated previously, the first n natural numbers are 1, 2, 3, . . ., n.

NOTE. The method adopted in the summation of the second and third powers of the first n natural numbers is the method of *undetermined coefficients*, and consists of equating identically the given series to $A + Bn + Cn^2 + \ldots$ and then determining the values of the unknown constants A, B, C, etc., by the properties of identities. (This method is useful for many types of series.)

(a) The first n natural numbers form an A.P. whose sum S_1 has been shown to be $n(n + 1)/2$.

(b) Let the sum of the squares of the first n natural numbers be S_2 and

$$1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + Bn + Cn^2 + Dn^3 + \ldots\ldots (1).$$

Replacing n by $(n + 1)$ in this identity,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 + (n + 1)^2$$
$$\equiv A + B(n + 1) + C(n + 1)^2 + D(n + 1)^3 + \ldots\ldots (2).$$

(2) − (1) gives,

$$(n + 1)^2 \equiv B + C(2n + 1) + D(3n^2 + 3n + 1) + \ldots\ldots (3).$$

In the identity (3) the highest power of n on the left-hand side (L.H.S.) is the second, and therefore the highest power of n on the right-hand side (R.H.S.) must be the second and all other coefficients after D must vanish.

Equating coefficients of various powers of n in (3).

n^2 $\quad 1 = 3D, \qquad \therefore\ D = \tfrac{1}{3},$

n $\quad 2 = 3D + 2C$
$\quad = 1 + 2C, \qquad \therefore\ C = \tfrac{1}{2},$

unity $\quad 1 = B + C + D$
$\quad = B + \tfrac{1}{2} + \tfrac{1}{3}, \qquad \therefore\ B = \tfrac{1}{6}.$

$$\therefore\ 1^2 + 2^2 + 3^2 + \ldots + n^2 \equiv A + \tfrac{1}{6}n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3.$$

Using $n = 1$, $\quad 1^2 = A + \tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{3}$

$$\therefore\ A = 0.$$

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Thus, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

$$= \frac{n}{6}(1 + 3n + 2n^2)$$

$$= \frac{n(n+1)(2n+1)}{6}.$$

Alternative Method. It is known that, for all values of n,

$$(n+1)^3 - n^3 \equiv 3n^2 + 3n + 1 \ldots\ldots\ldots\ldots (1).$$

Replacing n by $(n-1)$, $(n-2)$, . . ., 2, 1 in succession,

$$n^3 - (n-1)^3 \equiv 3(n-1)^2 + 3(n-1) + 1 \ldots\ldots (2),$$
$$(n-1)^3 - (n-2)^3 \equiv 3(n-2)^2 + 3(n-2) + 1 \ldots\ldots (3),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$3^3 - 2^3 \equiv 3.2^2 + 3.2 + 1 \ldots\ldots\ldots\ldots (n-1),$$
$$2^3 - 1^3 \equiv 3.1^2 + 3.1 + 1 \ldots\ldots\ldots\ldots (n).$$

Adding these n identities,

$$(n+1)^3 - 1^3 \equiv 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + n$$

$$\equiv 3S_2 + 3\frac{n}{2}(n+1) + n,$$

$$\therefore\ 3S_2 \equiv (n+1)^3 - 1^3 - 3\frac{n}{2}(n+1) - n$$

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$$= n^3 + 3n^2 + 3n - 3\frac{n}{2}(n+1) - n$$

$$= \frac{n}{2}[2n^2 + 6n + 6 - 3n - 3 - 2]$$

$$= \frac{n}{2}[2n^2 + 3n + 1] = \frac{n}{2}(n+1)(2n+1),$$

$$\therefore\ S_2 = n(n+1)(2n+1)/6.$$

(*c*) Let S_3 be the sum of the cubes of the first n natural numbers, and

$$1^3 + 2^3 + 3^3 + \ldots + n^3 \equiv A + Bn + Cn^2 + Dn^3 + En^4 + \ldots (1).$$

Replacing n by $(n+1)$ in this identity,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 + (n+1)^3 \equiv A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots\ldots (2).$$

(2) $-$ (1) gives,

$$(n+1)^3 \equiv B + C(2n+1) + D(3n^2 + 3n + 1) + E(4n^3 + 6n^2 + 4n + 1) + \ldots (3).$$

The highest power of n on the L.H.S. of (3) is the third, and therefore all coefficients after E on the R.H.S. of (3) must vanish.

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Equating coefficients in identity (3),

n^3 $\quad 1 = 4E, \ \therefore E = \frac{1}{4},$

n^2 $\quad 3 = 6E + 3D = \frac{3}{2} + 3D, \ \therefore 3D = \frac{3}{2} \ \therefore D = \frac{1}{2},$

n $\quad 3 = 4E + 3D + 2C = 1 + \frac{3}{2} + 2C$

$\therefore \frac{1}{2} = 2C, \ \therefore C = \frac{1}{4},$

unity $1 = B + C + D + E = B + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, \ \therefore B = 0.$

Hence, $1^3 + 2^3 + 3^3 + \ldots + n^3 \equiv A + \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4.$

Using $n = 1$ in this,

$$1^3 = A + \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, \ \therefore A = 0,$$

$$\therefore S_3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$$

$$\equiv \frac{n^2}{4}(1 + 2n + n^2) \equiv \left\{\frac{n}{2}(n+1)\right\}^2 = S_1^2.$$

As in (*b*), the alternative method consists of using

$$(n+1)^4 - n^4 \equiv 4n^3 + 6n + 4n + 1$$

and replacing n by $(n-1)$, $(n-2)$, ..., etc., in succession, and then introducing the values of S_1 and S_2 already determined.

NOTE. The results of (*a*), (*b*), (*c*) should be memorised as they are essential for finding the sum of any other series involving the first n natural numbers.

When dealing with a problem involving the use of these three results the rth term of the series whose sum is required is written down and expanded, and the sum of this expression is taken from $r = 1$ to $r = n$, the summation being shown by the symbol

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$$\sum_{r=1}^{r=n}.$$

EXAMPLE. Find the sum of the series.

(i) $1^2 - 2^2 + 3^2 - 4^2 + \ldots$ to $2n$ terms,

(ii) $1.3^2 + 2.4^2 + 3.5^2 + \ldots$ to n terms,

(iii) $1^3 + 3^3 + 5^3 + \ldots$ to n terms.

(i) Let the required sum be S_{2n}

$\therefore S_{2n} = (1^2 + 2^2 + 3^2 + \ldots \text{ to } 2n \text{ terms})$

$\qquad -2(2^2 + 4^2 + 6^2 + \ldots \text{ to } n \text{ terms})$

$$= \sum_{r=1}^{r=2n} r^2 - 8(1^2 + 2^2 + 3^2 + \ldots \text{ to } n \text{ terms})$$

$$= \sum_{r=1}^{r=2n} r^2 - 8\sum_{r=1}^{r=n} r^2$$

$$= \frac{2n(2n+1)(4n+1)}{6} - \frac{8n(n+1)(2n+1)}{6} \qquad \left\{\begin{array}{l}\text{Using} \\ \text{formula} \\ \text{for } S_2.\end{array}\right\}$$

$$= \frac{n(2n+1)}{3}[(4n+1) - 4(n+1)] = -\frac{n(2n+1).3}{3}$$

$$= -n(2n+1).$$

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(ii) The rth term of the series $= r(r+2)^2 = r^3 + 4r^2 + 4r$, therefore sum of n terms of the series

$$= \sum_{r=1}^{n}(r^3 + 4r^2 + 4r)$$
$$= \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$$
$$= \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{4n(n+1)(2n+1)}{6} + 4 \times \frac{n}{2}(n+1)$$
$$= \frac{n}{12}(n+1)[3n(n+1) + 8(2n+1) + 24]$$
$$= \frac{n(n+1)}{12}[3n^2 + 19n + 32].$$

(iii) The rth term of the series is $(2r-1)^3 \equiv 8r^3 - 12r^2 + 6r - 1$, therefore sum of n terms

$$= \sum_{r=1}^{n}(8r^3 - 12r^2 + 6r - 1)$$
$$= 8\sum_{r=1}^{n} r^3 - 12\sum_{r=1}^{n} r^2 + 6\sum_{1}^{n} r - n$$
$$= 8\left\{\frac{n(n+1)}{2}\right\}^2 - 12\,\frac{n(n+1)(2n+1)}{6} + 6\frac{n(n+1)}{2} - n$$
$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$
$$= n[(2n^3 + 4n^2 + 2n) - (4n^2 + 6n + 2) + (3n + 3) - 1]$$
$$= n[2n^3 - n] = n^2[2n^2 - 1].$$

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NOTE. Example (i) can be solved more readily as follows:

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (2n-1)^2 - (2n)^2$$
$$= (1^2 - 2^2) + (3^2 - 4^2) + \ldots + [(2n-1)^2 - (2n)^2]$$
$$= -1.3 - 1.7 - 1.11 - \ldots - 1(4n-1)$$
$$= -1(3 + 7 + 11 + \ldots + 4n - 1) \equiv -\tfrac{1}{2}n(3 + \overline{4n-1})$$
$$= -\tfrac{1}{2}n(4n+2) = -n(2n+1).$$

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EXAMPLES IV

1. Prove that the geometrical progression

$$1 + \frac{2x}{3+x^2} + \left(\frac{2x}{3+x^2}\right)^2 + \ldots$$

is convergent for all values of x, and find the sum of the series to infinity.

2. (i) Find two numbers whose arithmetic mean is 39 and whose geometric mean is 15.

(ii) The sum to infinity of a convergent G.P. is k and the sum to infinity of squares of its terms is l.

Find the first term and the common ratio of the progression.

3. (i) Show that three unequal consecutive terms of an arithmetical progression cannot be in geometrical progression.

(ii) If

$$\frac{1}{y-x}, \quad \frac{1}{2(y-a)}, \quad \frac{1}{y-z}$$

are in A.P., prove that $(x - a)$, $(y - a)$, $(z - a)$ are in G.P.

4. Find the sum of n terms of an A.P. whose first term is a and second term b.

(i) Find the sum of all the integers between 1 and 200 excluding those that are multiples of 3 or 7.

(ii) Find the sum of n terms of the series $1 \cdot 3 + 5 \cdot 3^2 + 9 \cdot 3^3 + \ldots$

5. Find the sum of the first n terms of the series $1 + x + x^2 + x^3 + \ldots$

A man has initially £P invested in a security bearing r% per annum. At the end of each year he draws the interest and sells sufficient stock to make the total sum withdrawn £p. Prove that, after n years, the amount of his capital remaining is

$$PR^n - p\frac{R^n - 1}{R - 1},$$

where $R = 1 + r/100$, and show that if $p = P/20$ and $r = 4$, he can continue for forty-one years without his capital being exhausted.

6. If S_n denote the sum of the G.P., $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$ show, as a graph with n as abscissa and S_n as ordinate, the sum of n terms for values of n from 1 to 6. (Take 1 inch as unit for n and 5 inches as unit for S_n.)

Find the least number of terms of the series whose sum differs from the sum to infinity by less than 10^{-4}.

7. (i) Find the sum of n terms of the G.P. $3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$, and show that, however great n may be, the sum cannot exceed 6.

(ii) Sum the series $3 + \frac{5}{2} + \frac{7}{4} + \ldots + \frac{2n+1}{2^{n-1}}$.

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8. Find the sum of n terms of the series,

$$2 + \frac{2}{3} + \frac{2}{3^2} + \ldots$$

Prove that the sum cannot exceed 3 however great n may be.

Find the least number of terms whose sum exceeds 2·999.

9. Find the values of a and b in terms of n if

$$(x - n + 1)^3 - (x - n)^3 \equiv 3x^2 + ax + b$$

for all values of x.

By adding together equations of this form for $x = 0$ and $n = 1, 2, \ldots r$, obtain an expression for

$$\sum_{n=1}^{r} n^3.$$

10. (i) Prove that the arithmetic mean of two unequal positive numbers is greater than their geometric mean.

(ii) The sum of the first four terms of a geometric series is 34, and the sum of the first term and the fourth term is 49. Show that there are two series satisfying these conditions, and find the first two terms of each series.

If either series is convergent, find its sum to infinity.

11. Find the sum of the first n terms of a geometrical progression whose first two terms are a and b respectively.

The amplitude of the first oscillation of a pendulum is 15°. If the amplitude of each succeeding oscillation is 0·89 of the amplitude of the preceding

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oscillation, find after how many oscillations the amplitude will first be less than 1°.

12. Find an expression for the sum of n terms of the G.P.

$$1 + x + x^2 + x^3 + \ldots$$

The population of a town increases in such a way that, if it is p at the beginning of a year, then at the end of the year it is $(a + bp)$, where a and b are certain constants. Show that the population, after the lapse of n years from the time when it was p, is

$$\frac{a}{1-b} + \left(p - \frac{a}{1-b}\right)b^n.$$

13. If a man borrows £1,040 on January 1st, 1929, and agrees to repay the loan with 5% compound interest by means of fifteen equal annual instalments, the first instalment being due on January 1st, 1930, show that each instalment should be a little more than £100.

14. Find an expression for the sum, $1 + x + x^2 + \ldots + x^{n-1}$, and prove that $1 + x(1 + x) + x^2(1 + x + x^2) + \ldots + x^{n-1}(1 + x + x^2 + \ldots + x^{n-1})$

$$= \frac{(1 - x^n)(1 - x^{n+1})}{(1 - x)(1 - x^2)}.$$

For what values of x can one speak of the sum of an infinite number of terms of the second series; and when this sum exists, what is its value?

15. Find the sum of the first n terms of the A.P. whose first two terms are a and b.

The sum of an A.P. to n terms is $8n^2 - n$. Find the number of the term which has the value 263.

16. An insurance company agrees to pay the sum of £600 on January 1st, 1941, in return for ten annual payments of £50 16*s*. 0*d*., the first payment being made on January 1st, 1931.

If the percentage interest per annum allowed by the company is denoted by x, obtain an equation for x, and hence show that x is very nearly equal to 3.

17. A man invests £267 10*s*. 0*d*. on January 1st, 1932, and the same amount on January 1st in each succeeding year. If compound interest at 4% per annum be allowed, find how much is due to him on December 31st, 1942.

18. Prove that the arithmetic mean of two unequal positive numbers is greater than their geometric mean.

If a, b, c, d are four unequal positive numbers which are in A.P., prove that

$$\frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c} > \frac{4}{a + d}.$$

19. (i) Find the series in A.P. whose sum to n terms is $n(n + 1)$. (ii) If S be the sum of n terms of a G.P., P is the product of the terms, and R the sum of the reciprocals of the terms, prove that $(S/R)^{\frac{1}{2}n} = P$.

20. Explain clearly the method of induction as a means of establishing a formula that involves an arbitrary positive integer n. (See binomial theorem.)

Prove by induction that, if n be a positive integer, the sum

$$n \cdot 1 + (n - 1)2 \cdot + (n - 2) \cdot 3 + \ldots + 2(n - 1) + 1 \cdot n$$

is equal to $\frac{1}{6}n(n + 1)(n + 2)$.

21. Find the present value of £P due n years hence, if compound interest at the rate of r% per annum is charged.

A man arranges to buy a house by paying £100 down and nineteen additional amounts of £100 at intervals of one year. If compound interest at 5% per annum be charged, find the present value of the house.

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22. A company borrows £20,000 which is to be repaid with interest at 6% per annum in equal yearly instalments in ten years. Find, as accurately as the tables permit, the amount of each instalment, the first being paid one year after the money was borrowed.

23. Obtain an expression for the sum of n terms of a G.P., in terms of the first term a and the common ratio r.

Using the symbol S_n to denote this sum, prove that

$$\text{(i)}\quad S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$$

$$\text{(ii)}\quad r^{m-n} = \frac{S_{m+p} - S_m}{S_{n+p} - S_n}$$

24. If the tenth term of a G.P. is 25 and the twentieth term is $\frac{1}{2}$, find the first term, the common ratio, and the sum to infinity.

25. Find the present value of the lease of a house of which the annual rent is £75, the lease having thirty-five years to run. The rent is paid half-yearly, the first instalment being due six months hence, and interest is reckoned at 4% per annum.

26. Prove, by induction or otherwise, that the sum of the cubes of the first n natural numbers is $\frac{1}{4}n^2(n + 1)^2$.

If the arithmetic mean of n consecutive integers is a, prove that the sum of their cubes is $an[a^2 + \frac{1}{4}(n^2 - 1)]$.

27. A fund is started with £1,000 and at the end of each year £50 is withdrawn. If compound interest is reckoned at 4% per annum, show that the amount in the fund at the beginning of the $(n + 1)$th year is

$$£1{,}250[1 - \tfrac{1}{5}(1{\cdot}04)^n].$$

Find the amount in the fund after twenty years, and also how many years it will last.

28. Show that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

Find nine numbers in A.P. whose sum is 27 and the sum of whose squares is 456.

29. (i) The first term of an A.P. is 11 and the sums of the first five and first ten terms respectively are equal in magnitude but opposite in sign. Find the second term of the progression.

(ii) Sum the series $2 + 5x + 8x^2 + \ldots + (3n - 1)x^{n-1}$.

30. $(2a + 5b)$, $(a + b)$, $(a - b)$ are the first three terms of a G.P.

Find a in terms of b and find also in terms of b the sum of the first six terms of each of the two possible G.P.'s.

Show that one of these progressions may be summed to infinity and that the sum to infinity exceeds the sum of its first six terms by $b/54$.

31. A solid pyramid is to be built on a square base whose side is 54 feet, of cubical blocks of side 2 feet. Every layer is to be 2 feet high and is to be a square whose side is 4 feet shorter than that of the layer immediately below it. Determine the number of blocks required to build the pyramid.

32. Find the present value of £A due n years hence, if compound interest is payable at r% per annum.

A man arranges to purchase a house, valued now at £1,000, by paying £500 in ten years' time and spreading the remaining payments in ten equal annual instalments of £X, the first being paid now. If C.I. on all outstanding payments is payable at 4% per annum calculate the value of X.

33. (i) Prove that (*a*) $\log p^q = q \log p$.

(*b*) If $a^x = b^y = (ab)^{xy}$, then $x + y = 1$.

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(ii) A machine depreciates in value each year by 8% of its value at the beginning of the year. After how many years will its value first be less than half its original value.

34. (i) Find the number of sides in a polygon whose interior angles are in arithmetical progression, the smallest being 120° and the common difference 5°. (ii) Prove that, in a geometric series which has a sum to infinity, each term bears a constant ratio to the sum of all the following terms. If this ratio has the value p find the value of the common ratio in terms of p.

35. (i) Prove that the geometric mean of two positive, unequal numbers is less than their arithmetic mean.

(ii) Establish the identity

$$a^2 + b^2 + c^2 - bc - ca - ab \equiv \tfrac{1}{2}[(b - c)^2 + (c - a)^2 + (a - b)^2],$$

and deduce that

(iii) $a^2 + b^2 + c^2 > bc + ca + ab$;

(iv) $a^3 + b^3 + c^3 > 3abc$, where a, b and c are positive and unequal.

36. Obtain an expression for S_n, the sum of n terms of the geometrical progression $a + ar + ar^2 + \ldots$

When r is numerically less than unity, explain what is meant by the sum to infinity S_∞ and find this sum.

For the series $1 + \frac{1}{2} + \frac{1}{2}^2 + \ldots$, find the least value of n so that S_n may differ from S_∞ by less than 1/100,000.

37. (i) If the arithmetic mean of two positive quantities a and b is three times their geometric mean, prove that $a/b = 17 + 12\sqrt{2}$. (ii) The formula for the nth term of a certain series is known to be of the form $An + B^n$. If the first term is 5 and the second term is 10, find the values of A and B, and also the sum of the first 10 terms of the series.

38. Explain what is meant by the sum to infinity of the series

$$a + ar + ar^2 + ar^3 + \ldots$$

when r is numerically less than unity, and find this sum.

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The output of a coalmine in any year is 10% *less* than in the preceding year. Prove that the maximum possible output is ten times that of the first year.

It is decided to close the mine when 90% of the maximum possible output has been mined. For how many years will the mine be worked?

39. (i) The pth term of an arithmetical progression is q and the qth term is p. Find the first term and the common difference. (ii) Prove that the arithmetic mean of two unequal positive quantities is greater than their geometric mean, and use this fact to show that $\sqrt{70}$ lies between 8 and 8·5.

40. (i) Find the sum of all the positive integers less than 1,000 that are not multiples of 3.

(ii) The sixth term of a geometric series of positive numbers is 10 and the sixteenth term is 0·1. What is the eleventh term?

Find the sum to infinity correct to the nearest unit.

41. Find to n terms the sum of the following series:

(i) $3 \cdot 1^2 + 5 \cdot 2^2 + 7 \cdot 3^2 + \ldots$

(ii) $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots$

(iii) $1^3 - 2^3 + 3^3 - 4^3 + \ldots$ (to $2n$ terms).

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CHAPTER V

Permutations and Combinations, and the Binomial Theorem

Permutations and Combinations. Each of the arrangements that can be made by taking some or all of a number of different objects is known as a *permutation*.

Each of the groups or selections that can be made by taking some or all of a number of different objects is called a *combination*.

EXAMPLE. Consider the four letters *a, b, c, d.* The following are the *arrangements* of these letters taken two at a time. (Letters are all different.)

ab, ac, ad, bc, bd, cd, ba, ca, da, cb, db, dc

and each of these is a permutation of the four letters taken two at a time. Hence, there are twelve permutations of four things taken two at a time.

When considering the groups without the order of the letters being taken into account, the first six constitute the number of groups (or selections) of the four letters taken two at a time. Thus, there are six combinations of four objects taken two at a time.

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The number of permutations of n different objects taken r at a time is denoted by ${}^nP_r$ and the number of combinations of n different objects taken r at a time is denoted by ${}^nC_r$.

Theorem. *If one operation can be done in m ways, and (when this is done) a second operation can be performed in n ways, the number of ways of performing the two operations simultaneously is mn.*

If the first operation be performed in any one way, then the second operation can be done in n ways, and thus, since there are m ways of performing the first operation, each of these ways is associated with n ways of performing the second operation. Thus, there will be mn ways of doing the two operations simultaneously.

EXAMPLE. There are six trains travelling between Newcastle and London and back. In how many ways can a man travel from Newcastle to London by one train and return by a different train?

The man can travel on any one of the six trains to London, and has thus six ways of choosing his train to London.

On his return journey he has the choice of five trains, and each of these five ways of travelling can be associated with any one of the six ways of travelling from Newcastle. Hence, there are $6 \times 5 = 30$ ways of making the double journey.

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Theorem. *To find the value of ${}^nP_r$, i.e. the number of permutations of n different objects taken r at a time. (n and r must be positive integers with $n \geqslant r$.)*

This is equivalent to finding the number of ways of filling r different compartments from n different things.

The first compartment can be filled in n ways leaving a choice of $(n - 1)$ objects for the second compartment. The second compartment can therefore be filled in $(n - 1)$ ways, and thus the number of ways of filling the first two compartments is $n(n - 1)$. There are $(n - 2)$ objects from which to fill the third compartment, and therefore there are $(n - 2)$ ways of filling this compartment, each of which can be associated with the number of ways of filling the first two compartments. Hence, there are $n(n - 1)(n - 2)$ ways of filling the first three compartments.

Proceeding in this manner it can be seen that

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\ldots \text{ to } r \text{ factors} \\ &= n(n-1)(n-2)\ldots(n-r+1). \end{aligned}$$

Corollary. Using $r = n$ in this result

$$\begin{aligned} {}^nP_n &= n(n-1)(n-2)(n-3)\ldots 2.1 \\ &= n! \end{aligned}$$

NOTE. The symbol $r!$ (formerly written $\underline{|r}$) known as 'factorial r', denotes the product of the first r natural numbers.

EXAMPLE. Find the values of ${}^5P_3$, ${}^{10}P_4$, ${}^{14}P_2$.

$$\begin{aligned} {}^5P_3 &= 5 \times 4 \times 3 &&= 60. \\ {}^{10}P_4 &= 10 \times 9 \times 8 \times 7 &&= 5{,}040. \\ {}^{14}P_2 &= 14 \times 13 &&= 182. \end{aligned}$$

Theorem. *To find the value of ${}^nC_r$, i.e. the number of combinations of n different objects taken r at a time.*

Let the number of combinations be x. Then each combination will give rise to ${}^rP_r$ permutations, i.e. $r!$ permutations.

$$\therefore\ x \,.\, r! = {}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$$

$$\therefore\ x = {}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}.$$

Corollary 1.

$$\begin{aligned} {}^nC_r &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots\times 2\,.1}{r!(n-r)(n-r-1)\ldots\times 2\,.\,1} \\ &= \frac{n!}{r!(n-r)!}. \end{aligned}$$

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Corollary 2. ${}^nC_n$ is the number of ways of choosing n objects from n different things, and, since this can only be done in one way, by taking all the objects, it must follow that ${}^nC_n = 1$.

But, $${}^nC_r = n!/r!(n-r)!$$

$$\therefore {}^nC_n = \frac{n!}{n!(n-n)!} = \frac{1}{0!}$$

$$\therefore \frac{1}{0!} = 1, \text{ i.e. } 0! = 1 \text{ (compare } a^0 = 1).$$

Theorem. *To prove that* ${}^nC_{n-r} = {}^nC_r$.

Now $${}^nC_p = n!/p!(n-p)!$$
Using $p = (n-r)$ in this,

$${}^nC_{n-r} = \frac{n!}{(n-r)!(n-\overline{n-r})} = \frac{n!}{(n-r)!r!}$$
$$= {}^nC_r.$$

Replacing r by n in this result

$${}^nC_0 = {}^nC_n = 1$$

EXAMPLE. $${}^{11}C_8 = {}^{11}C_3 = \frac{11 \times 10 \times 9}{3!} = \frac{11 \times 10 \times 9}{1 \times 2 \times 3} = 165.$$

Theorem. *To find the number of ways in which $(m+n)$ different objects can be divided into two groups of m and n things.*

The number of ways required is equal to the number of ways of choosing m objects out of the $(m+n)$ objects, where m and n are not equal, i.e. there are ${}^{m+n}C_n$ ways

$$= \frac{(m+n)!}{m!\,.\,n!} \text{ ways.}$$

When $m = n$, the two groups are indistinguishable for any one selection, and can be interchanged without obtaining a new distribution. Hence, in this case, the number of ways is

$$\frac{(m+n)!}{m!n!2!} = \frac{(2m)!}{(m!)^2\,.\,2!}$$

Theorem. *To find the number of ways that $(m+n+p)$ different articles can be divided into three groups containing m, n, p things severally, and different from all the rest.*

The number of ways of dividing the $(m+n+p)$ objects into two groups of m and $(n+p)$ objects respectively will be

$$\frac{(m+n+p)!}{m!(n+p)!}$$

by the previous theorem.

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The number of ways of dividing the $(n + p)$ objects into two groups of n and p objects will be $(n + p)!/(n!p!)$, and each of these ways can be associated with the number of ways of performing the first operation. Hence, the number of ways of dividing the objects into three groups $(m + n + p)$ is

$$\frac{(m + n + p)!}{m!(n + p)!} \times \frac{(n + p)!}{n!p!} = \frac{(m + n + p)!}{m!n!p!}.$$

If $m = n = p$, any one selected set of groups can be rearranged in 3! ways without giving rise to a new grouping. Hence, the number of ways in this case, as given by the above result, must be divided by 3!, when $m = n = p$, and the required number of ways is then

$$\frac{(3m)!}{(m!)^3 \,.\, 3!}.$$

Similarly, when $m = n \neq p$ the number of ways is

$$\frac{(2m + p)!}{(m!)^2 p!2!}, \text{ etc.}$$

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EXAMPLE. The number of ways in which thirty schoolchildren can be divided into three equal groups is $30!/(10!)^3 \,.\, 3!$, and the number of ways in which these schoolchildren can be placed in three different schools in groups of ten in each school is $30!/(10!)^3$, since each set of groups can be arranged among the three different schools in 3! ways.

NOTE. In some problems it is advisable to make an analysis of the grouping before proceeding to find the number of selections.

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EXAMPLE (L.U.). Find from first principles the number of selections r at a time that can be made from n different things.

A committee of four is to be chosen from five Englishmen and three Americans. In how many ways can this be done so that the committee contains (*a*) at least one Englishman, (*b*) at least one of each nationality.

The bookwork has already been proved.

(*a*) In this case the committee possibilities are:

1 Englishman, 3 Americans.
2 Englishmen, 2 Americans.
3 Englishmen, 1 American.
4 Englishmen, 0 Americans.

The first of these gives rise to ${}^5C_1 \times 1 = 5$ ways (1 Englishman in ${}^5C_1$ ways and Americans in 1 way).

The second gives rise to

$${}^5C_2 \times {}^3C_2 = \frac{5 \times 4}{1 \times 2} \times \frac{3 \times 2}{1 \times 2} = 30 \text{ ways.}$$

(2 Englishmen in ${}^5C_2$ ways and 2 Americans in ${}^3C_2$ ways.)

The third gives rise to

$${}^5C_3 \times {}^3C_1 = \frac{5 \times 4 \times 3}{1 \times 2 \times 3} \times 3 = 30 \text{ ways.}$$

(3 Englishmen in ${}^5C_3$ ways and 1 American in ${}^3C_1$ ways.)

The fourth gives rise to ${}^5C_4 \times 1 = {}^5C_1 \times 1 = 5$ ways (4 Englishmen in ${}^5C_4$ ways).

Therefore total number of ways $= 5 + 30 + 30 + 5 = 70$.

(*b*) The number of ways of choosing the committee is the same as in (*a*) except that four Englishmen cannot be chosen, as at least one member of each nationality is required,

Therefore number of ways $= 5 + 30 + 30 = 65$.

EXAMPLE (L.U.). A cricket eleven has to be chosen from fifteen players of whom six can bowl and three keep wicket, while none can bowl and keep wicket. In how many ways can the eleven be chosen so as to contain at least four bowlers and two wicket-keepers?

The following are the possible distributions for the team (analysis of team).

	Bowlers	*Wicket-keepers*	*Batsmen only*
(*a*)	4	2	5
(*b*)	4	3	4
(*c*)	5	2	4
(*d*)	5	3	3
(*e*)	6	2	3
(*f*)	6	3	2

and these are chosen from 6 bowlers, 3 wicket-keepers, and 6 others.

For (*a*) there will be ${}^6C_4 \times {}^3C_2 \times {}^6C_5 = 15 \times 3 \times 6 = 270$ ways of picking the team (4 bowlers out of 6, 2 wicket-keepers out of 3, and 5 out of 6 batsmen).

For (*b*) there will be ${}^6C_4 \times 1 \times {}^6C_4 = 15 \times 15 = 225$ ways.

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For (*c*) there will be ${}^6C_5 \times {}^3C_2 \times {}^6C_4 = 6 \times 3 \times 15 = 270$ ways.

For (*d*) there will be ${}^6C_5 \times 1 \times {}^6C_3 = 6 \times 1 \times 20 = 120$ ways.

For (*e*) the number of ways of choosing the eleven is

$$1 \times {}^3C_2 \times {}^6C_3 = 3 \times 20 = 60.$$

For (*f*) the number of ways of picking the eleven is

$$1 \times 1 \times {}^6C_2 = 15.$$

Therefore the total number of ways of choosing the team

$$= 270 + 225 + 270 + 120 + 60 + 15 = 960.$$

Permutations and Combinations when Some of the Objects are Alike. Two objects are said to be *like* when they are indistinguishable from one another to the eye. Previously only cases have been dealt with where all the objects were different, and now the problems of dealing with like objects will be considered.

Theorem. *To find the number of ways in which n objects can be arranged amongst themselves taken all at a time, when p are exactly alike of one kind, q exactly alike of a second kind, r exactly alike of a third kind, and all the rest are different.*

Let x be the required number of permutations. If the p like objects be replaced by p unlike objects different from any of the

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rest, from *any one* of the x permutations $p!$ permutations can be obtained without altering the position of the remaining things. Hence, if the change be made in each of the x permutations $x.p!$ permutations are obtained.

Similarly if the q like objects, in addition, be replaced by q different things, the total number of permutations will be

$$[x(p!)](q)! = x.(p)!(q)!.$$

And further, if the r like objects be replaced by unlike things the total number of permutations will be $(x.p!q!).r! = x.p!q!r!$. But now all the objects will be different, and the number of arrangements $= n!$,

$$\therefore\ x.p!q!r! = n!,$$

$$\therefore\ x = \frac{n!}{p!q!r!}.$$

This theorem can be extended to cover any number of groups of like objects.

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Example (L.U.). A signaller has six flags, of which one is blue, two are white, and three are red. He sends messages by hoisting flags on a flagpole, the message being conveyed by the order in which the colours are arranged. Find how many different messages he can send, (i) by using exactly six flags, (ii) by using exactly five flags.

(i) The number of messages is equal to the number of permutations of the flags taken all together, and, by the previous theorem, the number of messages

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$$= \frac{6!}{3!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 60.$$

(ii) In this case the possible selection of five flags must be first considered and is given as follows:

	Blue	*White*	*Red*
(*a*)	1	2	2
(*b*)	1	1	3
(*c*)	0	2	3

By the previous theorem:

For (*a*) there will be $\frac{5!}{2!2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 2} = 30$ messages.

For (*b*) there will be $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{6} = 20$ messages.

For (*c*) there will be $\frac{5!}{2!3!} = \frac{120}{12} = 10$ messages.

Therefore the total number of messages that can be sent with five flags

$$= 30 + 20 + 10 = 60.$$

EXAMPLE. Find the number of arrangements of the letters of the word '*committee*' taken all at a time.

In the word there are nine letters consisting of 2 m's, 2 t's, 2 e's, 1 c, 1 o, and 1 i, therefore required number of arrangements

$$= \frac{9!}{2!2!2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 2} = 45{,}360.$$

Theorem. *To find the number of arrangements of n things taken r at a time, when each may be repeated up to r times in any permutation.*

This is a case of filling a number (r) of compartments with the objects, when any of the n objects can be used as many times as is liked in any arrangement.

The first compartment can be filled in n ways, and, when that is filled, there are n ways of filling the second compartment since all the objects can be used again. Hence, the first two compartments can be filled in $n \times n = n^2$ ways.

Similarly there are n ways of filling the third compartment, and therefore the number of ways of filling the first three compartments is $n^2 \times n = n^3$.

Proceeding in this manner, it can be seen that the r compartments can be filled in n^r ways.

EXAMPLE. How many numbers of three digits can be formed from the numbers 1, 2, 3, 4, 5, when each of these can be repeated up to five times?

How many of these have two or more equal digits?

From the previous theorem there will be 5^3 numbers having the three digits, i.e. 125 numbers.

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There will be ${}^5P_3$ of these numbers having all different digits (i.e. the number of arrangements of five different objects taken three at a time), i.e. $5 \times 4 \times 3 = 60$ numbers.

Hence, there will be $125 - 60 = 65$ numbers having two or more equal digits.

Theorem. *To find the total number of ways of making a selection by taking some or all of n objects.*

Every object can be dealt with in two ways, for it may be taken or left, and since the ways of dealing with one particular object is associated with the ways of dealing with each of the other objects, the number of ways of dealing with the n objects is

$$2 \times 2 \times 2 \times \ldots \text{ to } n \text{ factors} = 2^n.$$

This, however, includes the case in which all the objects are left, which is not permissible. Hence, the required number of selections is $2^n - 1$.

EXAMPLE. There are five people living in a house. In how many ways can one or more of them leave the house?

This is a particular case of the last theorem and hence the required number of ways is $2^5 - 1 = 32 - 1 = 31$.

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EXAMPLE (L.U.). (*a*) Prove that $3n$ unlike things can be divided into n sets each containing three of the things in

$$\frac{(3n)!}{n!6^n}$$

different ways.

(*b*) n like objects are to be placed in three unlike boxes so that there is at least one object in each box. Find the number of different ways this can be done.

(*a*) Consider the objects being put into n unlike boxes, three in each.

The first box can be filled in ${}^{3n}C_3$ ways leaving $(3n - 3)$ objects from which to fill the second box, giving ${}^{3n-3}C_3$ ways of filling the second box.

The third box can similarly be filled in ${}^{3n-6}C_3$ ways, the fourth in ${}^{3n-9}C_3$ ways, and so on, and finally there will be three objects left to fill the last box, which can therefore be filled in one way only.

Hence, since these ways of filling the boxes can be associated with one another, the total number of ways of filling the boxes is

$$ {}^{3n}C_3 \times {}^{3n-3}C_3 \times {}^{3n-6}C_3 \times \ldots \times {}^{6}C_3 \times 1$$

$$= \frac{(3n)!}{(3n-3)!3!} \times \frac{(3n-3)!}{(3n-6)!3!} \times \frac{(3n-6)!}{(3n-9)!3!} \times \ldots \times \frac{6!}{3!3!} \times 1$$

$$= \frac{(3n)!}{(3!)^n} = \frac{(3n)!}{6^n}$$

The n boxes can be arranged amongst themselves in $n!$ ways. Hence, if x be the number of selections, the number of arrangements will be $n!x$.

$$\therefore\ n!x = \frac{(3n)!}{6^n} \qquad \therefore\ x = \frac{(3n)!}{n!6^n}$$

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(*b*) Let the boxes be lettered A, B, C. Since there must be at least one object in each box, the first box can be filled in $(n - 2)$ ways (i.e. there can be any number from 1 to $(n - 2)$ balls in box A). If there be one ball in A the remaining $(n - 1)$ balls can be divided amongst B and C in $(n - 2)$ ways (there can be any number from 1 to $(n - 2)$ balls in B and the remainder in C).

Similarly, if there be two balls in A, the remaining balls can be distributed amongst B and C in $(n - 3)$ ways; if there be three balls in A, the remaining balls can be distributed amongst B and C in $(n - 4)$ ways, and so on, until when there are $(n - 2)$ balls in A there will only be one way of filling B and C (one ball each).

Hence, the required number of ways is

$$(n - 2) + (n - 3) + (n - 4) + \ldots + 2 + 1$$
$$= \frac{(n-2)}{2}[(n - 2) + 1] = \frac{(n-2)(n-1)}{2}.$$

The Binomial Theorem and its Applications. Any expression, such as $(a + x)$ involving two terms is known as a binomial expression, and thus $(a + x)^n$ is a binomial function, and the statement of its expansion in powers of x is known as the *binomial theorem*. (n is the index.)

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The method used in proving the binomial theorem for a positive integral index is known as the *method of induction*, and in this method (which can be applied to various other theorems and problems), it is assumed that the expansion holds true for the power n. Next it is proved by algebra that the result is true when n is replaced by $(n + 1)$. The expansion is then proved to be valid for $n = 2$. This being the case, it will also be true for $n = 3$, and therefore $n = 4$, and so on for all positive integral values of n.

Proof of the Binomial Theorem by Induction for Positive Integral Indices. It is assumed that, if n be a positive integer, then

$$(a + x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \dots + {}^nC_{r-1}a^{n-r+1}x^{r-1} + {}^nC_ra^{n-r}x^r + \dots + x^n \dots (1).$$

Multiplying (1) by $(a + x)$, and showing only necessary terms,

$$\begin{aligned}(a + x)^{n+1} &= a^{n+1} + {}^nC_1a^nx + {}^nC_2a^{n-1}x^2 + \dots + {}^nC_ra^{n-r+1}x^r + \dots + ax^n \\ &\quad + a^nx + {}^nC_1a^{n-1}x^2 + \dots + {}^nC_{r-1}a^{n-r+1}x^r + \dots + x^{n+1} \\ &= a^{n+1} + ({}^nC_1 + 1)a^nx + ({}^nC_2 + {}^nC_1)a^{n-1}x^2 + \dots \\ &\quad + ({}^nC_r + {}^nC_{r-1})a^{n-r+1}x^r + \dots + x^{n+1} \dots (2).\end{aligned}$$

Now ${}^nC_1 + 1 = n + 1 = {}^{n+1}C_1$,

and
$$\begin{aligned}{}^nC_r + {}^nC_{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\ &= n!\left\{\frac{1}{r!(n-r)!} + \frac{1}{(r-1)!(n-r+1)!}\right\} \\ &= \frac{n!}{r!(n-r+1)!}\left\{\frac{(n-r+1)!}{(n-r)!} + \frac{r!}{(r-1)!}\right\} \\ &= \frac{n!}{r!(n-r+1)!}[(n-r+1)+r] \\ &= \frac{(n+1).n!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)!} \\ &= {}^{n+1}C_r.\end{aligned}$$

Using these results (2) becomes

$$(a + x)^{n+1} = a^{n+1} + {}^{n+1}C_1a^nx + {}^{n+1}C_2a^{n-1}x^2 + \dots + {}^{n+1}C_ra^{n-r+1}x^r + \dots + x^{n+1} \dots .(3)$$

If n be replaced by $(n + 1)$ in (1) the equation (3) is obtained. Thus, if the binomial theorem be true for the index n it must also be true for the index $(n + 1)$ .... (A)

By algebra,

$$\begin{aligned}(a + x)^2 &= a^2 + 2ax + x^2 \\ &= a^2 + {}^2C_1ax + x^2 \qquad (2 = {}^2C_1) \\ (a + x)^3 &= a^3 + 3a^2x + 3ax^2 + x^3 \\ &= a^3 + {}^3C_1a^2x + {}^3C_2ax^2 + x^3. \qquad (3 = {}^3C_1 = {}^3C_2)\end{aligned}$$

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But these are the results of replacing n by 2 and 3 respectively in equation (1), and therefore the binomial theorem is true for $n = 2$ and $n = 3$.

Hence, by statement (*A*), since the theorem is true for $n = 3$, it must be true for $n = 4$, and since it is true for $n = 4$ it must be true for $n = 5$, and so on for all positive integral values of n.

Thus, for all positive integral values of n

$$(a + x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \dots + {}^nC_ra^{n-r}x^r + \dots + x^n.$$

The notation ${}^nC_r$ is best reserved for the proof and the expansion of the binomial theorem should be remembered in the following equivalent form, since it also applies to the case when n is not a positive integer.

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}a^{n-r}x^r + \dots$$

No proof is given for the cases when n is negative or fractional as the proofs are beyond the scope of this book, but it is assumed that the binomial theorem in its last form is valid when n is fractional or negative and the numerical value of x is less than the numerical value of a.

NOTE. When one part of the binomial expression is negative it is advisable to consider the negative quantity as $+x$ in $(a + x)^n$. In this manner the formula, as stated, can be used with all positive signs between the terms.

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EXAMPLE. Find the expansions of (*a*) $(2 - 3y)^4$, (*b*) $(3 - z)^5$.

By the binomial theorem

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!}a^{n-r}x^r + \dots$$

(i) Let $a = 2$ and $x = -3y$, $n = 4$.

$$\therefore (2 - 3y)^4 = (2)^4 + 4(2)^3(-3y) + \frac{4 \times 3}{2!}(2)^2(-3y)^2 + \frac{4 \times 3 \times 2}{3!}(2)(-3y)^3 + (-3y)^4$$

$$= 16 - 96y + 216y^2 - 216y^3 + 81y^4.$$

(ii) Let $a = 3$, $x = (-z)$, $n = 5$.

$$\therefore (3 - z)^5 = 3^5 + 5(3)^4(-z) + \frac{5 \times 4}{2!}(3)^3(-z)^2 + \frac{5 \times 4 \times 3}{3!}(3)^2(-z)^3 + \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4}(3)(-z)^4 + (-z)^5$$

$$= 243 - 405z + 270z^2 - 90z^3 + 15z^4 - z^5.$$

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Points to be noted relating to the binomial theorem are the following:

(i) When n is a positive integer there are a finite number $(n + 1)$ terms in the expansion, and in all other cases there are an infinite number.

(ii) The $(r + 1)$th term in the binomial expansion is known as the *general term*, and is seen to be

$$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}a^{n-r}x^r,$$

which has r factors in both numerator and denominator of its coefficient, the factors decreasing by unity in succession in each case.

(iii) The sum of the powers of a and x in each term is equal to n.

(iv) If $a = 1$ the expansion becomes

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

which is simpler to remember and manipulate than the general expansion.

Hence, in many cases, it is advisable to base the expansion of $(a + x)^n$ on the expansion of $(1 + y)^n$ as follows:

$$(a+x)^n = [a(1+x/a)]^n = a^n(1+y)^n,$$

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where $y = x/a$.

(v) When there are an infinite number of terms in the binomial expansion the expansion is only valid if the numerical value of x ($|x|$) is less than that of a ($|a|$) for $(a + x)^n$ in ascending powers of x. ($|x|$ stands for 'modulus x'.)

EXAMPLE. (i) Find the fifth term in the expansion of

$$\left(x^3 - \frac{1}{2x}\right)^6.$$

(ii) Find the middle term in the expansion of $(2a - 3b)^{10}$.

(iii) Find the coefficient of x^{10} in the expansion of

$$\left(2x^2 - \frac{3}{x}\right)^{11}.$$

(iv) Find the term not containing x in the expansion of

$$\left(x - \frac{1}{3x}\right)^8.$$

(i) The fifth term in the expansion of $(a + y)^n$ is

$$\frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}y^4. \qquad (r = 4)$$

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Using $n = 6,\ a = x^3,\ y = -\dfrac{1}{2x}$, the required term is

$$\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} \cdot (x^3)^2\left(-\frac{1}{2x}\right)^4 = 15x^6 \times \frac{1}{16x^4}$$
$$= \frac{15}{16}x^2.$$

(ii) The middle term will be the sixth term since there will be eleven terms in the expansion. Therefore the required term (as in (i)) is,

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}(2a)^5(-3b)^5$$
$$= 252 \times 32a^5 \times (-243b^5)$$
$$= -1{,}959{,}552a^5b^5.$$

(iii) The $(r + 1)$th term in the expansion of $(2x^2 - 3/x)^{11}$ is

$$\frac{11 \times 10 \times \ldots (11 - r + 1)}{r!}(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r$$
$$= \frac{11 \times 10 \times \ldots (12 - r)}{r!} \cdot 2^{11-r} \cdot x^{22-2r} \cdot x^{-r}(-3)^r$$
$$= \frac{11 \times 10 \times \ldots (12 - r)}{r!} \cdot 2^{11-r}(-3)^r \cdot x^{22-3r}.$$

If this be the term in x^{10}, then $22 - 3r = 10$, i.e. $12 = 3r \ \therefore\ r = 4$ and the required coefficient is

$$\frac{11 \times 10 \times \ldots (12 - 4)}{4!} \cdot 2^7(-3)^4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} \times 128 \times 81$$

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$$= 330 \times 128 \times 81 = 3{,}421{,}440.$$

(iv) The $(r + 1)$th term in the expansion is

$$\frac{8 \times 7 \times \ldots (8 - r + 1)}{r!} \cdot x^{8-r}\left(\frac{-1}{3x}\right)^r$$
$$= (-1)^r \cdot \frac{8 \times 7 \times \ldots (9 - r)}{3^r \cdot (r)!}x^{8-2r}.$$

The term will not contain x if $8 - 2r = 0$, i.e. r $= 4$, therefore required term is

$$(-1)^4 \cdot \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1 \times 3^4} = \frac{70}{81}.$$

EXAMPLE. Find, in ascending powers of x, the following binomial expansions to three terms.

$$\text{(i) } \frac{1}{\sqrt[3]{8 - 3x}}, \quad \text{(ii) } \sqrt{4 + 5x}, \quad \text{(iii) } \frac{1}{(2 + x)^2}.$$

(i) $$\frac{1}{\sqrt[3]{8 - 3x}} = \frac{1}{(8 - 3x)^{\frac{1}{3}}} = \frac{1}{\left\{8\left(1 - \frac{3x}{8}\right)\right\}^{\frac{1}{3}}} = \frac{1}{8^{\frac{1}{3}}\left\{1 - \frac{3x}{8}\right\}^{\frac{1}{3}}}$$
$$= \tfrac{1}{2}\left(1 - \frac{3x}{8}\right)^{-\frac{1}{3}}$$

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$$= \tfrac{1}{2}\left\{1 + \left(-\tfrac{1}{3}\right)\left(-\frac{3x}{8}\right) + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}\left(-\frac{3x}{8}\right)^2 + \ldots\right\}$$

$$= \tfrac{1}{2}[1 + \tfrac{1}{8}x + \tfrac{1}{32}x^2 + \ldots]$$

$$= \tfrac{1}{2} + \tfrac{1}{16}x + \tfrac{1}{64}x^2 \text{ to three terms.}$$

(ii) $\sqrt{4+5x} = (4+5x)^{\frac{1}{2}} = \left\{4\left(1+\frac{5x}{4}\right)\right\}^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{5x}{4}\right)^{\frac{1}{2}}$

$$= 2\left\{1 + \tfrac{1}{2}\left(\frac{5x}{4}\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{5x}{4}\right)^2 + \ldots\right\}$$

$$= 2\left\{1 + \frac{5x}{8} - \frac{25x^2}{128} + \ldots\right\}$$

$$= 2 + \frac{5x}{4} - \frac{25x^2}{64} \text{ to three terms.}$$

(iii) $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+\frac{1}{2}x)]^3} = \dfrac{1}{2^3(1+\frac{1}{2}x)^3} = \tfrac{1}{8}(1+\tfrac{1}{2}x)^{-3}$

$$= \tfrac{1}{8}\left\{1 + (-3)(\tfrac{1}{2}x) + \frac{(-3)(-4)}{2!}(\tfrac{1}{2}x)^2 + \ldots\right\}$$

$$= \tfrac{1}{8}\left\{1 - \frac{3x}{2} + \frac{3}{2}x^2 + \ldots\right\}$$

$$= \frac{1}{8} - \frac{3x}{16} + \frac{3x^2}{16} \text{ to three terms.}$$

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NOTE 1. In the expansion of $(1-y)^{-n}$ where n is positive all the terms will be positive, as the $(r+1)$th term

$$= \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}(-y)^r$$

$$= (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}(-1)^r y^r$$

$$= (-1)^{2r}\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}y^r$$

$$= \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}y^r.$$

NOTE 2. The following expansions, which can be readily obtained by using the binomial theorem, should be memorised:

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots,$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots,$$

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots,$$

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots.$$

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In each case the expansion is only true if $-1 < x < 1$, which can be written $|x| < 1$.

Approximations using the Binomial Theorem. When x be small compared with a, a first approximation to $(a + x)^n$ will be given by the first two terms of the binomial expansion, i.e. $a^n + na^{n-1}x$. A second and better approximation is given by the first three terms in the binomial expansion, i.e.

$$a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2,$$

and so on for further approximations.

The approximation chosen in any particular problem is dependent upon the degree of accuracy required in the result, and it can be usually ascertained, in the process of working, which is the first term that can be neglected.

EXAMPLE. Evaluate the following to four decimal places.

$$(i)\ \frac{1}{\sqrt{4{\cdot}08}}, \qquad (ii)\ (1{\cdot}98)^5, \qquad (iii)\ \sqrt[3]{7{\cdot}76}.$$

NOTE. It is always advisable to base approximation questions on the expansion of $(1 + y)^n$, where *y is small.*

$$(i)\ \frac{1}{\sqrt{4{\cdot}08}} = \frac{1}{2\sqrt{1{\cdot}02}} = \tfrac{1}{2}(1 + 0{\cdot}02)^{-\frac{1}{2}}$$

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$$= \tfrac{1}{2}\left\{1 + (-\tfrac{1}{2})(0{\cdot}02) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(0{\cdot}02)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(0{\cdot}02)^3 + \ldots\right\}$$

$$= \tfrac{1}{2}(1 - 0{\cdot}01 + 0{\cdot}00015 + \text{negligible terms})$$

$$= \tfrac{1}{2}(0{\cdot}99015) = 0{\cdot}4951 \text{ to four decimal places.}$$

$$(ii)\ (1{\cdot}98)^5 = 2^5(0{\cdot}99)^5 = 32(1 - 0{\cdot}01)^5$$

$$= 32[1 + 5(-0{\cdot}01) + 10(-0{\cdot}01)^2 + 10(-0{\cdot}01)^3 + 5(-0{\cdot}01)^4 + (-0{\cdot}01)^5]$$

$$= 32[1 - 0{\cdot}05 + 0{\cdot}001 - 0{\cdot}00001 + 0{\cdot}00000005 - \ \ ]$$

(since the quantity inside the bracket is multiplied by 32 it must be taken to seven decimal places to obtain a result correct to four decimal places).

$$\therefore (1{\cdot}98)^5 = 32 - 1{\cdot}6 + 0{\cdot}032 - 0{\cdot}00032 + \text{terms not affecting the fifth decimal place}$$

$$= 30{\cdot}43168 = 30{\cdot}4317 \text{ to four decimal places.}$$

$$(iii)\ \sqrt[3]{7{\cdot}76} = (8 - 0{\cdot}24)^{\frac{1}{3}} = 8^{\frac{1}{3}}(1 - 0{\cdot}03)^{\frac{1}{3}}$$

$$= 2\left\{1 + \tfrac{1}{3}(-0{\cdot}03) + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!}(-0{\cdot}03)^2 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!}(-0{\cdot}03)^3 + \ldots\right\}$$

$$= 2[1 - 0{\cdot}01 - 0{\cdot}0001 - 0{\cdot}000002].$$

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(working to six decimal places inside the brackets since it is multiplied by a number between 1 and 10 outside the brackets).

$$\therefore \sqrt[3]{7\cdot76} = 2(0\cdot989898) = 1\cdot9798 \text{ to 4 decimal places.}$$

NOTE. When dealing with the expansions or approximations obtained from a fraction $f(x)/\varphi(x)$, it is advisable to put the given fraction in the form $f(x) \times [\varphi(x)]^{-1}$ before attempting to use the binomial theorem. The degree of the approximation will either be stated in the problem or implied in it.

Thus, if a required result be wanted as far as x^2, all terms in x^3, x^4, etc., will be usually neglected at each stage.

EXAMPLE. If x be so small that x^2 and higher powers of x may be neglected, find the values of:

$$\text{(i)}\ \frac{(8+3x)^{\frac{2}{3}}}{(1+2x)\sqrt{4-7x}}, \qquad \text{(ii)}\ \frac{(1+\frac{2}{3}x)^{-3}(4-5x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}.$$

(i) $\dfrac{(8+3x)^{\frac{2}{3}}}{(1+2x)\sqrt{4-7x}}$

$$= 8^{\frac{2}{3}}(1+\tfrac{3}{8}x)^{\frac{2}{3}}(1+2x)^{-1}\,.\,\tfrac{1}{2}\left(1-\frac{7x}{4}\right)^{-\frac{1}{2}}$$

$$= 4[1+\tfrac{2}{3}(\tfrac{3}{8}x)+\ldots][1-2x+\ldots] \times \tfrac{1}{2}\left\{1+(-\tfrac{1}{2})\left(-\frac{7x}{4}\right)+\ldots\right\} \quad (8^{\frac{2}{3}} = 2^2 = 4)$$

$$= 2[1+\tfrac{1}{4}x+\ldots][1-2x+\ldots][1+\tfrac{7}{8}x+\ldots]$$

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$$= 2[1+\tfrac{1}{4}x-2x+\tfrac{7}{8}x+\ldots]$$

$$= 2[1-\tfrac{7}{8}x+\ldots] = 2-\frac{7}{4}x \text{ (neglecting } x^2\text{, etc.).}$$

(ii) $\dfrac{(1+\frac{2}{3}x)^{-3}\times(4-5x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}$

$$= (1+\tfrac{2}{3}x)^{-3} \times 4^{\frac{1}{2}}\left(1-\frac{5x}{4}\right)^{\frac{1}{2}} \times 4^{-\frac{3}{2}}(1+\tfrac{1}{4}x)^{-\frac{3}{2}}$$

$$= 4^{-1}[1+(-3)(\tfrac{2}{3}x)+\ldots]\left\{1+\tfrac{1}{2}\left(-\frac{5x}{4}\right)+\ldots\right\}[1+(-\tfrac{3}{2})(\tfrac{1}{4}x)+\ldots]$$

$$= \tfrac{1}{4}[1-2x+\ldots][1-\tfrac{5}{8}x+\ldots][1-\tfrac{3}{8}x+\ldots]$$

$$= \tfrac{1}{4}[1-2x-\tfrac{5}{8}x-\tfrac{3}{8}x-\ldots]$$

$$= \tfrac{1}{4}[1-3x+\ldots]$$

$$= \tfrac{1}{4}-\tfrac{3}{4}x \text{ (neglecting } x^2\text{, etc.).}$$

EXAMPLE. Find the first three terms in the expansion of

$$\frac{[(1+x)^{\frac{1}{3}}+\sqrt{1+5x}]}{(1-x)^2}.$$

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$$\frac{[(1+x)^{\frac{3}{4}} + \sqrt{1+5x}]}{(1-x)^2}$$

$$= \left\{\left[1 + \tfrac{3}{4}x + \frac{\frac{3}{4}(-\frac{1}{4})}{2!}x^2 + \ldots\right] + \left[1 + \tfrac{1}{2}(5x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(5x)^2 + \ldots\right]\right\}(1-x)^{-2}$$

$$= [1 + \tfrac{3}{4}x - \tfrac{3}{32}x^2 + 1 + \tfrac{5}{2}x - \tfrac{25}{8}x^2 + \ldots][1 + 2x + 3x^2 + \ldots]$$
$$= [2 + \tfrac{13}{4}x - \tfrac{103}{32}x^2 + \ldots][1 + 2x + 3x^2 + \ldots]$$
$$= (2 + \tfrac{13}{4}x - \tfrac{103}{32}x^2 + \ldots) + (4x + \tfrac{13}{2}x^2 + \ldots) + 6x^2 + \ldots$$
$$= 2 + \tfrac{29}{4}x + \tfrac{297}{32}x^2 \text{ to three terms.}$$

Extension of the Binomial Theorem. The binomial theorem can be modified to apply to a *trinomial expression* $(a + b + c)^n$ and further extended to apply to multinomial expressions (more than three terms). In the case of the trinomial expression $(a + b + c)^n$ the quantity $(b + c)$ is chosen as a single term initially and the expression to be expanded will be written as

$$[a + (b + c)]^n = a^n + na^{n-1}(b + c) + \frac{n(n-1)}{2!}a^{n-2}(b + c)^2 + \ldots$$

The quantities $(b + c)^2$, $(b + c)^3$, etc., will then each be expanded by the binomial theorem, and the required expansion obtained after simplification.

NOTE. If a, b, or c be equal to unity, that term is generally kept as the single term and the remaining two combined as a single term.

EXAMPLE. (i) Expand $(1 - 3x + x^2)^4$ in powers of x as far as the term containing x^4.

(ii) Find the coefficient of x^5 in the expansion of $(1 + x + x^3)^9$.

(i) $(1 - 3x + x^2)^4$
$= [1 + x(x - 3)]^4$
$= 1 + 4x(x - 3) + 6[x(x - 3)]^2 + 4[x(x - 3)]^3 + [x(x - 3)]^4$
$= 1 + 4x^2 - 12x + 6x^2(x^2 - 6x + 9) + 4x^3(x^3 - 9x^2 + 27x - 27) + x^4(81 - \ldots)$
$= 1 + 4x^2 - 12x + 6x^4 - 36x^3 + 54x^2 + 108x^4 - 108x^3 + 81x^4 + \ldots$
$= 1 - 12x + 58x^2 - 144x^3 + 195x^4$ as far as x^4.

(ii) $(1 + x + x^3)^9 = [(1 + x) + x^3]^9 = (1 + x)^9 + 9(1 + x)^8 . x^3 + 36(1 + x)^7 . x^6 + \ldots$

Considering only the first two terms in this expansion, since further terms contain x^6 and higher powers of x, the coefficient of x^5 is

$${}^9C_5 + 9 \times {}^8C_2 = 126 + 9 \times 28 = 378.$$

EXAMPLE (L.U.). (i) In the expansion of $(1 + x^2 + ax^3)^4$, the coefficient of x^4 is equal to the coefficient of x^6. Find a.

(ii) The fifth, sixth, and seventh coefficients in the expansion of $(1 + x)^n$ in ascending powers of x are in arithmetical progression, find the possible values of n.

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(i) $(1 + x^2 + ax^3)^4 = [1 + x^2(1 + ax)]^4$

$$= 1 + 4x^2(1 + ax) + 6x^4(1 + ax)^2 + 4x^6(1 + ax)^3 + x^8(1 + ax)^4$$

The coefficient of x^4 in the expansion is 6, and the coefficient of x^6 is $6a^2 + 4$ ($6a^2$ from expansion of third term and 4 from expansion of the fourth term).

Hence, $6a^2 + 4 = 6 \quad \therefore \quad 3a^2 = 1$

$$\therefore a = \pm 1/\sqrt{3} = \pm\sqrt{3}/3.$$

(ii) The coefficients of the fifth, sixth, and seventh terms in the expansion of $(1 + x)^n$ are respectively ${}^nC_4$, ${}^nC_5$, and ${}^nC_6$, and since these are in arithmetical progression

$${}^nC_5 - {}^nC_4 = {}^nC_6 - {}^nC_5 \quad \therefore 2{}^nC_5 = {}^nC_4 + {}^nC_6$$

i.e. $$\frac{2n(n-1)\ldots(n-4)}{5!} = \frac{n(n-1)\ldots(n-3)}{4!} + \frac{n(n-1)\ldots(n-5)}{6!}$$

Cancelling through by $\dfrac{n(n-1)(n-2)(n-3)}{4!}$

($n \neq 0, 1, 2, 3$ or expansion would not have 7 terms)

$$\frac{2(n-4)}{5} = 1 + \frac{(n-4)(n-5)}{5 \cdot 6}$$

$$\therefore 12(n-4) = 30 + (n-4)(n-5)$$

$$\therefore 12n - 48 = 30 + n^2 - 9n + 20$$

$$\text{i.e. } n^2 - 21n + 98 = 0$$

$$\therefore (n-7)(n-14) = 0$$

$$\therefore n = 7 \text{ or } 14.$$

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Theorems connected with the Binomial Theorem

Theorem. *To find the greatest coefficient in the expansion of $(1 + x)^n$, where n is a positive integer.*

The coefficient of the $(r + 1)$th term (i.e. of x^r) is ${}^nC_r$, and that of the rth term is ${}^nC_{r-1}$. Now,

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \times \frac{(r-1)!}{n(n-1)\ldots(n-r+2)}$$

$$= \frac{n-r+1}{r} = \frac{n+1}{r} - 1.$$

Now the $(r + 1)$th coefficient will be greater than the rth coefficient if $(n + 1)/r - 1 > 1$, i.e. $(n + 1)/r > 2$,

$$\text{i.e. } \frac{n+1}{2} > r \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But r must be an integer, and therefore, when n is *even*, the greatest coefficient is given by the greatest value of r consistent with (1), i.e. $r = n/2$, and hence the greatest coefficient is ${}^nC_{n/2}$.

Similarly if n be *odd* the greatest coefficient is given when $r = \frac{1}{2}(n - 1)$ or $\frac{1}{2}(n + 1)$ and the coefficient itself will be ${}^nC_{\frac{1}{2}(n+1)} = {}^nC_{\frac{1}{2}(n-1)}$.

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Theorem. *To find the numerically greatest term in the expansion of $(a+x)^n$, where n is a positive integer.*

Let T_{r+1} be the $(r+1)$th term in the expansion, and T_r the rth term, then

$$T_{r+1} = {}^nC_r a^{n-r}x^r, \quad T_r = {}^nC_{r-1}a^{n-r+1}x^{r-1}$$

$$\therefore \left|\frac{T_{r+1}}{T_r}\right| = \left|\frac{{}^nC_r}{{}^nC_{r-1}} \cdot \frac{a^{n-r}}{a^{n-r+1}} \cdot \frac{x^r}{x^{r-1}}\right|$$

$$= \left|\frac{n-r+1}{r} \cdot \frac{x}{a}\right|$$

$$= \left|\frac{n+1}{r} - 1\right| \left|\frac{x}{a}\right|$$

(The 'modulus' sign $|\ |$ means the numerical value, i.e. the sign is not taken into account.)

Thus $|T_{r+1}| > |T_r|$ if $\left\{\frac{n+1}{r} - 1\right\}\left|\frac{x}{a}\right| > 1$,

$$\text{i.e. } \frac{n+1}{r} - 1 > \left|\frac{a}{x}\right|,$$

($(n+1)/r - 1$ must be positive since $n > r$.)

$$\text{i.e. } \frac{n+1}{r} > 1 + \left|\frac{a}{x}\right|,$$

$$\text{i.e. } \frac{n+1}{1 + |a/x|} > r \qquad (1)$$

Thus T_{r+1} will be the greatest term if r has the greatest value consistent with the inequality (1), where r is a positive integer.

NOTE. $|-\frac{1}{2}| = \frac{1}{2}$, $|-3| = 3$, etc.

Theorem. *To prove that* $c_0 + c_1 + c_2 + \ldots + c_n = 2^n$. (*In this and following theorems* c_r *denotes* ${}^nC_r$.)

Now
$$c_0 = c_n = 1$$
$$\therefore (1+x)^n = 1 + c_1x + c_2x^2 + \ldots + x^n$$
$$= c_0 + c_1x + c_2x^2 + \ldots + c_nx^n.$$

Using $x = 1$ in this result

$$2^n = c_0 + c_1 + c_2 + \ldots + c_n.$$

Theorem. *To prove that*

$$c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots = 2^{n-1}$$

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n.$$

Putting $x = -1$ in this

$$0 = c_0 - c_1 + c_2 - c_3 + \ldots$$

$$\therefore c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots$$
$$= \tfrac{1}{2}[c_0 + c_1 + c_2 + c_3 + \ldots]$$
$$= \tfrac{1}{2}(2^n) \text{ (by previous theorem)}$$
$$= 2^{n-1}.$$

Theorem. *To find the value of* $c_0 + 2c_1 + 3c_2 + \ldots + (n+1)c_n$.

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NOTE. The easiest method is by calculus (see later chapter), as shown.

$$(1 + x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n.$$

Differentiating this equation with respect to x

$$n(1 + x)^{n-1} = c_1 + 2c_2x + 3c_3x^2 + \ldots + nc_nx^{n-1}.$$

Using $x = 1$ in this

$$n.2^{n-1} = c_1 + 2c_2 + 3c_3 + \ldots + nc_n \ldots\ldots\ldots (1)$$

But $$2^n = c_0 + c_1 + c_2 + \ldots + c_n \ldots\ldots\ldots\ldots (2)$$

Adding (i) and (2)

$$2^{n-1}(2 + n) = c_0 + 2c_1 + 3c_2 + \ldots + (n + 1)c_n.$$

Theorem. *To find the value of* $(c_0)^2 + (c_1)^2 + \ldots + (c_n)^2$.

Consider the expansion of the product of $(1 + x)^n$ and $(x + 1)^n$.

$$(1 + x)^n(x + 1)^n = (c_0 + c_1x + \ldots + c_rx^r + \ldots + c_nx^n)(c_0x^n + c_1x^{n-1} + \ldots + c_n)$$

$$\therefore (1 + x)^{2n} = (c_0 + c_1x + \ldots + c_nx^n)(c_0x^n + c_1x^{n-1} + \ldots + c_n) \ldots\ldots (1)$$

The coefficient of x^n on the right-hand side (R.H.S.) of (1), picking it out term by term, is

$$c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2,$$

and the coefficient on the left-hand side of (1) is ${}^{2n}C_n$.

Hence,

$$c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}.$$

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NOTE. If it were required to find the value of

$$c_0^2 - c_1^2 + c_2^2 - c_3^2 + \ldots,$$

it would be necessary to consider the expansion of $(1 - x)^n(x + 1)^n$, and if n be *even* the result is ${}^nC_{n/2}$. When n is *odd* the result is zero as $(1 - x)^n(x + 1)^n = (1 - x^2)^n$ which does not contain any odd powers in its expansion.

EXAMPLE (L.U.). Show that the binomial expansion of $(x^p + 1/x^{2p})^{3n}$, where n is a positive integer, always contains a term which is independent of x. Find the value of this term for $n = 4$.

Find the greatest term in the expansion of $(x + 1/x^2)^{12}$ when $x = \frac{8}{5}$.

The $(r + 1)$th term in the expansion of $(x^p + 1/x^{2p})^n$ where n is a positive integer is

$${}^{3n}C_r(x^p)^{3n-r}\left(\frac{1}{x^{2p}}\right)^r = \frac{{}^{3n}C_rx^{3pn-pr}}{x^{2pr}} = {}^{3n}C_rx^{3p(n-r)}.$$

Hence, when $r = n$ there will be a term independent of x, and this is always possible since n is a positive integer, therefore there will always be a term in the expansion which is independent of x and its value will be ${}^{3n}C_n$.

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When $n = 4$ the value of this term will be

$$^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495.$$

Let T_{r+1} be the $(r+1)$th term in the expansion of $(x + 1/x^2)^{12}$ and T_r be the rth term in this expansion.

Then $T_{r+1} = {}^{12}C_r x^{12-3r}$ (by first part of question),

$T_r = {}^{12}C_{r-1} x^{15-3r}$.

$$\therefore \frac{T_{r+1}}{T_r} = \frac{^{12}C_r}{^{12}C_{r-1}} \cdot \frac{x^{12-3r}}{x^{15-3r}} = \frac{12 - r + 1}{r} \cdot \frac{1}{x^3}$$

$$= \frac{13 - r}{r} \cdot \frac{3^3}{2^3}, \text{ when } x = \tfrac{2}{3}.$$

Therefore $T_{r+1} > T_r$ as long as $\dfrac{13 - r}{r} \times \dfrac{27}{8} > 1$,

i.e. as long as $351 - 27r > 8r$,
i.e. as long as $351 > 35r$.

Therefore $(r+1)$th term will be the greatest if $r = 10$, i.e. the 11th term is the greatest, and its value is

$$^{12}C_{10}(\tfrac{2}{3})^{12-30} = {}^{12}C_2 \cdot (\tfrac{3}{2})^{18}$$

$$= \frac{12 \cdot 11}{2 \cdot 1} \times \frac{3^{18}}{2^{18}} = 11 \times \frac{3^{19}}{2^{17}}.$$

EXAMPLE (L.U.). If $(2 - 3x)^{20} = c_0 - c_1 x + c_2 x^2 - \ldots + c_{20} x^{20}$ prove that, (i) $c_0 - c_1 + c_2 - c_3 + \ldots + c_{20} = 1$, (ii) $c_{12} >$ any other coefficient.

(i) www.dbraulibrary.org.in $(2 - 3x)^{20} = c_0 - c_1 x + c_2 x^2 - \ldots + c_{20} x^{20}$

Using $x = 1$ in this,

$$(-1)^{20} = c_0 - c_1 + c_2 - c_3 + \ldots + c_{20}$$

$\therefore c_0 - c_1 + c_2 - c_3 + \ldots + c_{20} = 1.$

(ii) Now c_r is the coefficient of the $(r+1)$th term in the expansion of $(2 - 3x)^{20}$ neglecting the sign, therefore $c_r = {}^{20}C_r(2)^{20-r}(3)^r$.

Also, $c_{r-1} = {}^{20}C_{r-1}(2)^{20-r+1}(3)^{r-1}$

$$\therefore \frac{c_r}{c_{r-1}} = \frac{^{20}C_r}{^{20}C_{r-1}} \cdot \frac{3}{2} = \frac{20 - r + 1}{r} \cdot \frac{3}{2} = \frac{21 - r}{r} \cdot \frac{3}{2},$$

$$\therefore c_r > c_{r-1} \text{ as long as } \frac{21 - r}{r} \cdot \frac{3}{2} > 1,$$

i.e. as long as $63 - 3r > 2r$,
i.e. as long as $63 > 5r$.

Therefore the greatest value of c_r will be given by the greatest value of r consistent with the above inequality, i.e. when $r = 12$

$\therefore c_{12} >$ any other coefficient.

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EXAMPLES V

1. (i) Four people draw simultaneously a card each from an ordinary pack of fifty-two cards. In how many different ways can the draw result in four cards of the same suit? (ii) How many different arrangements can be made by using all the letters of the word *syzygy*?

2. Obtain a formula (i) for the number of permutations, (ii) for the number of combinations of n unlike things taken r at a time.

There are ten articles, two of which are alike and the rest all different. In how many ways can a selection of five articles be made?

3. Two straight lines intersect at O. Points $A_1, A_2, \ldots, A_n$ are taken on one line, and $B_1, B_2, \ldots, B_n$ on the other.

Prove that the number of triangles that can be drawn with three of the points for vertices is

(i) $n^2(n-1)$, if the point O be not used,
(ii) n^3, if O may be used.

4. (i) Find from first principles the number of permutations of n different things taken three at a time. (ii) In how many ways can an escort of four soldiers be chosen from nine soldiers, and in how many of these escorts will a particular soldier be included?

5. How many numbers, each of four digits, can be formed from the digits 1, 2, 3, 4 when each digit can be repeated four times?

Calculate the sum of all these numbers.

6. Prove that the number of different permutations of n things of which a are of one kind, b of another kind, and the rest all different, is $n!/(a! \,.\, b!)$.

Find the number of different permutations of the letters of the word *parallel* taken all together, and find also the number of such permutations in which no two l's are consecutive.

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7. How many 'words', each consisting of one vowel and two consonants, can be formed from the letters of the word *permutations*, where, in each case, the vowel must occur in the middle?

How many of these words will begin with the letter p and how many with the letter t?

8. Prove that if ${}^nC_r$ denote the number of combinations of n different things when they are taken r together, $r \,.\, {}^nC_r = n \,.\, {}^{n-1}C_{r-1}$. (If any formula for ${}^nC_r$ be used it must be proved.)

How many points of intersection are made by m straight lines and n circles drawn in a plane if each line (straight or curved) intersects every other line, none of the circles touches a straight line or another circle, and no three lines have a point in common?

9. How many numbers are there having four digits, and such that each digit is one of the numbers 1, 2, 3, 4, 5?

How many of these numbers have two or more equal digits? How many of them have two equal digits and two other digits which are different from each other and from the equal digits?

10. Find the number of combinations of n dissimilar things taken r at a time.

Out of nine articles five are alike and the rest unlike. How many different permutations of three articles can be made from the nine?

11. Find the number of permutations of n things taken all at a time, of which p things are alike of one kind, q alike of another, and the rest all different.

Find the number of ways in which ten labels, five alike of one kind and five alike of another, can be attached to ten out of fifteen different boxes.

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12. How many different selections of r objects can be made from n different objects?

In how many ways can the letters of the word *infinitesimal* be arranged, when all are taken at a time?

13. Find the number of permutations ${}^nP_r$ of n different things taken r at a time, and deduce the number of combinations ${}^nC_r$ of n different things taken r at a time.

Prove that
$$\begin{aligned} {}^7C_3 &= {}^6C_3 + {}^6C_2 = {}^5C_3 + {}^5C_2 \cdot {}^2C_1 + {}^5C_1 \\ &= {}^4C_3 + {}^4C_2 \cdot {}^3C_1 + {}^4C_1 \cdot {}^3C_2 + {}^3C_3. \end{aligned}$$

14. (i) If ${}^nC_r$ denotes the number of combinations of n unlike things r at a time, prove that ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$. (ii) There are four ladies and four gentlemen ready to play tennis, but only one court is available. In how many ways is it possible to arrange for one 'mixed doubles' to be played, i.e. a lady and gentleman to play a second lady and gentleman?

15. n points divide the circumference of a circle into n equal parts. Find the number of triangles that can be made with three of the points for vertices.

If n be even and not divisible by three, prove that $n(n - 2)/2$ of the triangles are isosceles, but that, when n is even and divisible by three, the number of isosceles triangles is $n(3n - 10)/6$.

16. Prove that, if n be a positive integer,
$$(1 + x)^n = 1 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + x^n,$$
where $c_r = n!/[r!(n - r)!]$.

Find the term independent of x in the expansion of $(2x + 1/x^2)^{12}$ in descending powers of x, and find the greatest term in the expansion when $x = \frac{2}{3}$.

17. By using the relation $(1 - x^2)^n = (1 - x)^n(1 + x)^n$, or otherwise, prove that, if n be a positive integer, and
$$(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n,$$
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then $a_0^2 - a_1^2 + a_2^2 + \ldots + (-1)^n a^2_n = 0$ if n be odd, but is equal to
$$(-1)^{n/2}\frac{(n + 2)(n + 4)\ldots(2n)}{2 \times 4 \times \ldots \times n}$$
if n be even.

18. If the coefficients of x^{r-1}, x^r, x^{r+1} in the binomial expansion of $(1 + x)^n$ are in A.P., prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Find three consecutive coefficients of the expansion of $(1 + x)^{14}$ which form an A.P.

19. (i) Write down and simplify the coefficient of x^9 in the expansion of $(5 - 2x)^{11}$.

(ii) If T_r be the rth term in the expansion of $(1 + x)^n$ in ascending powers of x, prove that
$$\frac{T_{r+2}}{T_r} = \frac{(n - r + 1)(n - r)}{r(r + 1)}x^2.$$

If C_r be the coefficient of x^{r-1} in the expansion of $(1 + 2x)^{10}$ in ascending powers of x, find r when
$$\frac{C_{r+2}}{C_r} = 4.$$

20. If $(1 + x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$, prove that
$$c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots = 2^{n-1}.$$
Prove also that, when n is even,
$$c_0 + 2^2c_2 + 2^4c_4 + \ldots + 2^nc_n = \tfrac{1}{2}(3^n + 1).$$

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21. If $(1 + x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, find the value of c_r / c_{r-1}, and show that

(i) $\dfrac{c_1}{c_0} + 2\dfrac{c_2}{c_1} + 3\dfrac{c_3}{c_2} + \ldots + n\dfrac{c_n}{c_{n-1}} = \tfrac{1}{2}n(n+1)$,

(ii) $(c_0 + c_1)(c_1 + c_2) \ldots (c_{n-1} + c_n) = c_0 c_1 \ldots c_{n-1} \cdot \dfrac{(n+1)^n}{n!}$.

22. Find the positive integral value of n which makes the ratio of the coefficient of x^4 to that of x^3 in the expansion of $(1 + 2x + 3x^2)^n$ in a series of powers of x equal to $121/28$.

23. (i) In the expansion of $(3 + 2x)^{10}$ in ascending powers of x the ratio of one of the coefficients to the preceding coefficient is $\frac{4}{5}$. Express these two coefficients in prime factors. (ii) In the expansion of $(1 + x^2)(1 + x)^n$ in ascending powers of x, when n is a positive integer, the coefficient of x^3 is six times that of x. Find n and for this value the coefficient of x^4.

24. Write down the coefficient of x^{16} in the expansion of $(1 - 2x)^{18}$ in ascending powers of x.

If $(1 + ax + bx^2)(1 - 2x)^{18}$ be expressed in ascending powers of x, determine a and b if the coefficients of x^3 and x^4 are both zero.

25. (i) Prove that the ratio of the $(r + 1)$th term to the rth term in the expansion of $(1 + x)^n$ by the binomial theorem is $(n - r + 1)x/r$.

Find which is the greatest term in the expansion of $(4 + 3x)^{12}$ when $x = \frac{3}{8}$.

(ii) Prove that the coefficients of x^2 and x^3 in the expansion of $(2 + 2x + x^2)^n$ in ascending powers of x are $n^2 \cdot 2^{n-1}$ and $\frac{1}{3}n(n^2 - 1)2^{n-1}$.

26. (i) Write down the $(r + 1)$th term in the binomial expansion of $(4 + 3x)^{16}$ and find the greatest coefficient in the expansion. (ii) If θ be an acute angle such that $\cos\theta = 1 - x$, where x is so small that x^2 is negligible compared with unity, prove that $\cos 2\theta = 1 - 4x$ and $\cos 3\theta = 1 - 9x$, approximately. (Hint: $\cos 2\theta = 2\cos^2\theta - 1$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

27. Obtain the expansion of $(1 + x)^n$ in powers of x, when n is a positive integer.

By comparing the coefficients of x^r on both sides of the identity

$$(1 + x)^n \equiv (1 + x)^2(1 + x)^{n-2},$$

prove that ${}^nC_r = {}^{n-2}C_r + 2 \cdot {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$.

28. (i) The expansion of $(2 + ax)^{11}$ is $c_0 + c_1 x + c_2 x^2 + \ldots + c_{11}x^{11}$. If c_8 be greater than any other coefficient, prove that $4 < a < 6$.

(ii) Find the expansion in powers of x of $(3 - 2x + x^2)^4$.

29. (i) If x be so small that x^3 and higher powers can be neglected, show that

$$(1 - \tfrac{3}{2}x)^5(2 + 3x)^6 = 64 + 96x - 720x^2$$

(ii) Expand $(1 - \frac{3}{2}x - x^2)^5$ in ascending powers of x as far as the term in x^4.

30. Find the coefficient of x in the expansion of $(2x - 3/x)^{31}$ as a series of descending powers of x; determine also the greatest term in the expansion when $x =$. Express both results in prime factors.

31. Find the term independent of x in the binomial expansion of

$$(2x^3 - 1/x)^{12}.$$

Find, without using tables, the value of $(1{\cdot}01)^{10} + (0{\cdot}99)^{10}$ correct to eight places of decimals.

32. State the binomial theorem in the case when the index is a positive integer.

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If n be a positive integer, prove that the coefficients of x^2 and of x^3 in the expansion of $(x^2 + 2x + 2)^n$ are

$$2^{n-1} \cdot n^2 \text{ and } \frac{2^{n-1}}{3} n(n^2 - 1)$$

respectively.

33. If $(1 + x^2)^2(1 + x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$, and if c_0, c_1, c_2 are in A.P., show that there are two possible values of n, and find them.

Write out the complete expansion of the expression in powers of x for these values of n.

34. Find the first three terms in the expansion of $(1 + x)^{m+n}(1 - x)^{m-n}$, where m and n are both positive integers and $m > n$.

If the coefficients of x and of x^2 in the expansion of $(1 + x)^{pa}(1 - x)^{p(a-1)}$, where p and a are positive integers, are equal to one another, find an expression for p in terms of a.

35. State and prove the binomial theorem for a positive integral index.

Obtain the coefficients of x and of x^5 in the expansion of $(1/x - 2x + 3x^3)^5$.

36. Obtain the expansion of $(1 + x)^n$ in ascending powers of x, n being a positive integer.

Show that it is impossible for three consecutive terms of this expansion to be in geometric progression.

37. (i) If ${}^nC_r$ be the coefficient of x^r in the expansion of $(1 + x)^n$, where n is a positive integer, prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

(ii) If $\dfrac{{}^{n-1}C_{r-1}}{a} = \dfrac{{}^{n-1}C_r}{b} = \dfrac{{}^{n-1}C_{r+1}}{c}$, prove that

$$r = \frac{a(b + c)}{b^2 - ac}, \text{ and } n = \frac{(a + b)(b + c)}{b^2 - ac}.$$

38. Prove that

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$$(1 + x)^n = \sum_{r=0}^{n} {}^nC_r x^r.$$

where r is a positive integer.

Prove that the term independent of x in the expansion of $(x^3 + 1/x)^{4n}$ by the binomial theorem is ${}^{4n}C_{3n}$. Show that this term is the greatest term if

$$\frac{n + 1}{3n} > x^4 > \frac{n}{3n + 1}.$$

39. In the expansion of $(1 + 2x + ax^2)^n$ in ascending powers of x, the third term is zero. Find a, and the coefficients of x^3 and x^4 in terms of n.

40. Obtain the expansion of $(1 + x)^n$, where n is a positive integer, in ascending powers of x.

Prove that, if the coefficients of three consecutive powers of x in the expansion of $(1 + x)^n$ are in A.P., $(n + 2)$ must be the square of an integer.

Find the coefficients when $n = 7$.

41. Obtain the formula for the expansion of $(1 + x)^n$ in a series of ascending powers of x when n is a positive integer; give the coefficient of x^r where r is a positive integer less than n.

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is double the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

42. Obtain the expansion of $(a + b)^n$, for a positive integral value of n.

Show $(\sqrt{2} + \sqrt{3})^n + (\sqrt{2} - \sqrt{3})^n$ is an integer if n be even, or an integral multiple of $\sqrt{2}$ if n be odd.

Evaluate $(\sqrt{2} + \sqrt{3})^8 + (\sqrt{2} - \sqrt{3})^8$.

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43. State and prove the formula for the expansion of $(1 + x)^n$ in ascending powers of x, n being a positive integer, and give the general term involving x^r, where r is an integer less than n.

Find the coefficient of x^9 in the expansion of $(5a^2 - 4x^3)^6$, and the term independent of x in that of $(3x - 2/x^2)^{15}$.

44. Obtain the expansion of $(1 + x)^n$ in the form

$$c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + c_nx^n,$$

where n is a positive integer, giving an expression for the coefficient c_r.

Show that the values of the coefficients increase to a maximum, and then diminish, but that three consecutive coefficients cannot be in geometrical progression.

45. (i) Write down the general term in the expansion of $(x^2 + 1/x)^n$ in descending powers of x, and show therefrom that the powers of x which occur are alternately even and odd. Show that, if a term involving x^9 occurs, then n must be a multiple of 3. (ii) Find the expansion of $(a + b)^5(a - b)^3$ in a series of ascending powers of b.

46. If $(x + a_1)(x + a_2) \ldots (x + a_n) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$, prove that $p_r - a_1p_{r-1} + a_1{}^2p_{r-2} + \ldots + (-1)^ra_1{}^r$ is equal to the sum of the products r at a time of $a_2, a_3, \ldots, a_n$.

Deduce that ${}^{n-1}C_r = {}^nC_r - {}^nC_{r-1} + {}^nC_{r-2} + \ldots + (-1)^r$.

47. Prove that the coefficient of x^r in the expansion of $(1 + x)^n$, when n is a positive integer, is $n(n - 1) \ldots (n - r + 1)/r!$.

If x be small, prove that the value of $(2 + \frac{3}{2}x)^4(2 + \frac{3}{8}x^2)^2 - (2 + x)^6$ is approximately $20x^3$.

48. (i) Find the term independent of x in the expansion of

$$(x - 1/x)^{10}(x + 1/x)^4.$$

(ii) If the expansion of $(1 + cx)^n$, where $n > 3$ and $c \neq 0$ is

$$1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots$$

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find n, c if $16(a_2 - a_1) = 3a_3 = a_4$.

49. (i) Find the coefficient of x^7 in the expansion of $(2x^2 - 1/4x)^{11}$.

(ii) If $(1 + x)^{14} = c_0 + c_1x + c_2x^2 + \ldots + c_rx^r + \ldots + c_{14}x^{14}$, find the value of r for which $c_r = 2c_{r+1}$, and find c_r and c_{r+1}.

When $x = \frac{2}{3}$ in this expansion, prove that two consecutive terms are equal and that these terms are greater than any other.

50. (i) Given that $nC_r : nC_{r+1} : nC_{r+2} = 1 : 2 : 3$, find the values of n and r. (ii) There are ten candidates for three vacancies, and an elector can vote for any number of candidates not greater than the number of vacancies. In how many ways can an elector vote?

51. (i) How many numbers not exceeding 10,000 can be made without using the digits 8 and 9? (ii) In how many ways can eight different beads be strung on a necklace, if all the beads are to be used? (iii) A bag contains six white and two black balls. One ball is drawn out of the bag. What is the probability that it will be (*a*) white, (*b*) black?

52. State the formula for nC_r, the number of selections of r objects that can be made from n different objects, and explain why $nC_r = nC_{n-r}$.

A party of nine persons is to travel in two cars, one of which will not hold more than seven persons, and the other not more than four. In how many ways can the party travel?

53. A boy has eleven different objects, five of which are black and six are white. In how many ways can he arrange *three* objects in a row with a black one in the middle?

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If the eleven objects are put in a bag, what is the probability of drawing out two in the order white, black?

54. (i) How many permutations can be made of the letters of the word *Canada*, each of the six letters being used once in each permutation?

In how many of these permutations will the three a's be together? In how many will two a's be together, but not three?

(ii) Given that $nC_r = \dfrac{n!}{r!(n-r)!}$, prove that

$$nC_r + nC_{r-1} = n+1C_r,$$
$$nC_{r+1} + 2 \cdot nC_r = n+2C_{r+1}.$$

55. If $C_{r+1} = C_r = kC_{r-1}$, where C_{r-1}, C_r, C_{r+1} have their usual meaning in the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots$, show that (i) n is odd, (ii) $3 \geqq k > 1$, and find n and r in terms of k.

56. Prove that

$$(x+a)^n = x^n + {}^nC_1x^{n-1}a + \ldots + {}^nC_rx^{n-r}a^r + \ldots + a^n,$$

when n is a positive integer.

Find the first four terms in the expansion of $(r^2 - 2ar\mu + a^2)^{-1}$ in powers of r^{-1}, where $|\mu| < 1$ and $|a| < |r|$.

57. Write down the first four terms in the expansion of $(1-x)^{-n}$ by the binomial theorem, n being a positive integer.

Show that the expansions of the expressions (i) $(1-2x)^{-1} + (1-x)^{-2}$ and (ii) $\left(1 - \dfrac{10}{7}x\right)^{-1} + \left(1 - \dfrac{9}{7}x\right)^{-2}$ agree for the first three terms.

If the coefficient of x^n in the expansion of the first expression is denoted by $f(n)$, show that $f(n+1) = 2f(n) - n$.

58. Write down the expansion of $(1+x)^{10}$ as far as the term involving x^4, and use it to evaluate $1{\cdot}01^{10}$ correct to five decimal places.

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Find correct to three decimal places the sum to ten terms of the series:

$$(a)\ 1 + 1{\cdot}01 + 1{\cdot}01^2 + 1{\cdot}01^3 + \ldots$$
$$(b)\ 1 - 1{\cdot}01 + 1{\cdot}01^2 - 1{\cdot}01^3 + \ldots$$

59. (i) Write down the series for the expansion of $(1+x)^n$ in ascending powers of x, where n is a positive integer.

Use the expansion to show that to four decimal places $(1{\cdot}01)^{12}$ exceeds $(1{\cdot}02)^6$ by an amount $0{\cdot}0007$.

(ii) Find the coefficient of x^5 and of x^7 in the expansion of $(1 + x^2 + x^3)^n$ in ascending powers of x.

60. (i) Find the coefficient of x^4 in the expansion of $(2 - x/3)^8$ by the binomial theorem. (ii) If y is so small that its second and higher powers may be neglected, prove that the expression $(1-5y)^{\frac{1}{2}}(1+2y)^{-\frac{2}{3}}$ is approximately equal to $(6-19y)/6$.

61. (i) Find the coefficient of x^7 in the expansion, in ascending powers of x, of $(1+3x)/(1+2x)^{\frac{1}{2}}$. (ii) Find $\sqrt{101}$ correct to six decimal places by using the binomial expansion of $(100+x)^{\frac{1}{2}}$.

62. Write down the series for the expansion of $(1+x)^n$ in ascending powers of x, when n is a positive integer.

Show that $(1+\sqrt{3})^n + (1-\sqrt{3})^n$ is always rational providing n is a positive integer.

Show also that $(\sqrt{2}+\sqrt{3})^n + (\sqrt{2}-\sqrt{3})^n$ is rational when n is an even integer, and evaluate $(\sqrt{2}+\sqrt{3})^6 + (\sqrt{2}-\sqrt{3})^6$.

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63. (i) Write down the general term in the expansion of $(1 + x)^{p/q}$, where $0 < x < 1$, p, q are positive integers and $p > q$, and explain why the terms must ultimately alternate in sign.

Show that $\frac{\sqrt{17}}{4} = 1 + \frac{1}{2} \cdot \frac{1}{2^4} - \frac{1}{2.4} \cdot \frac{1}{2^8} + \frac{1.3}{2.4.6} \cdot \frac{1}{2^{12}} \ldots$

(ii) A candidate taking an examination consisting of three papers each carrying a maximum of 100 marks.

Find the number of ways in which he can score 200 marks on the three papers.

64. (i) Find the coefficient of x^3 in the expansion of $(1 + 3x + 2x^2)^{10}$.

(ii) Expand $(1 + x)^{-1}(4 + x^2)^{-\frac{1}{2}}$ in ascending powers of x as far as and including the term in x^3. For what values of x is the expansion valid?

65. (i) Write down the expansion of $(1 + x)^7$ by the binomial theorem, and use it to calculate the value of $(0{\cdot}998)^7$, correct to six decimal places.

(ii) Find the coefficient of x^3 in the expansion of

$$\left(2x + \frac{1}{2x}\right)^{11}.$$

66. (i) Find the number of ways of arranging seven red, four white, and two blue counters in a row.

In how many of these arrangements will the two blue counters come together?

(ii) Prove the binomial theorem for the expansion of $(1 + x)^n$ where n is a positive integer.

Find the coefficient of x^3 in the expansion of $(1 - 5x + 2x^2)^5$.

67. (i) In how many ways can three consonants and two vowels be chosen from the word *logarithms*, and in how many of these will the letter *i* occur?

(ii) Write down the $(r + 1)$th term in the expansion of $(a + bx)^n$ as a series in ascending powers of x, where n is a positive integer.

Find a positive value of a which will make the coefficient of x^5 equal to that of x^{15} in the expansion of $(2x^2 + a/x^3)^{10}$.

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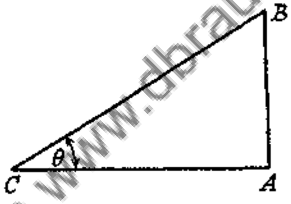
CHAPTER VI

Trigonometry

Circular Functions, Heights and Distances, Solution of Trigonometrical Equations, Compound Angles, etc.

Circular Functions of the Acute Angle. If ABC be a triangle right-angled at A, with the acute angle $ACB = \theta$, then the following ratios are known as the circular or trigonometric functions:



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$$\frac{AB}{BC}, \quad \frac{CA}{BC}, \quad \frac{AB}{CA}, \quad \frac{BC}{AB}, \quad \frac{BC}{CA}, \quad \frac{CA}{AB},$$

and are denoted by sine θ, cosine θ, tangent θ, cosecant θ, secant θ, and cotangent θ, or more briefly $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$ respectively.

Tables known as Mathematical Tables can be obtained in which the particular circular function can be obtained for any given angle.

From the definitions

$$\operatorname{cosec}\theta = 1/\sin\theta, \ \sec\theta = 1/\cos\theta, \ \tan\theta = \sin\theta/\cos\theta,$$
$$\cot\theta = 1/\tan\theta = \cos\theta/\sin\theta.$$

Also, since $BC \geqslant AB$ and $BC \geqslant AC$,

$$\sin\theta \leqslant 1, \ \cos\theta \leqslant 1, \ \operatorname{cosec}\theta \geqslant 1, \text{ and } \sec\theta \geqslant 1,$$

and since $AB + CA \geqslant BC$,

$$\frac{AB}{BC} + \frac{CA}{BC} \geqslant 1,$$

$$\text{i.e. } \sin\theta + \cos\theta \geqslant 1.$$

Using Pythagoras' theorem on the above triangle,

$$AB^2 + CA^2 = BC^2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

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Dividing through (1) by BC^2

$$\frac{AB^2}{BC^2} + \frac{CA^2}{BC^2} = 1,$$

$$\text{i.e. } \left(\frac{AB}{BC}\right)^2 + \left(\frac{CA}{BC}\right)^2 = 1$$

$$\therefore (\cos\theta)^2 + (\sin\theta)^2 = 1$$

$(\cos\theta)^2$ is written $\cos^2\theta$, and $(\sin\theta)^2 = \sin^2\theta$, etc.

$$\therefore \cos^2\theta + \sin^2\theta = 1 \ldots\ldots\ldots\ldots\ldots (2)$$

This result is the *fundamental identity* in trigonometry and is extremely important.

Dividing through (2) by $\cos^2\theta$,

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta},$$

$$\text{i.e. } 1 + \tan^2\theta = \sec^2\theta \ldots\ldots\ldots\ldots\ldots (3)$$

Dividing through (2) by $\sin^2\theta$,

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta},$$

$$\text{i.e. } \cot^2\theta + 1 = \operatorname{cosec}^2\theta \ldots\ldots\ldots\ldots\ldots (4)$$

The identities (2), (3), (4) can be used when θ is acute, and therefore when all the circular functions are positive, to find the various trigonometrical functions when one of them is given.

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Thus, if $\sec\theta = 2$, using (3) (with θ acute)

$1 + \tan^2\theta = \sec^2\theta = 4 \quad \therefore \tan^2\theta = 3$

$\therefore \tan\theta = \sqrt{3}$, and $\cot\theta = 1/\tan\theta = 1/\sqrt{3}$.

$\cos\theta = 1/\sec\theta = \frac{1}{2}$,

$\sin\theta = \tan\theta \times \cos\theta = \sqrt{3} \times \frac{1}{2} = \sqrt{3}/2$

$\operatorname{cosec}\theta = 1/\sin\theta = 2/\sqrt{3}$.

Radian Measure. The angle subtended at the centre of a circle by an arc of the circle equal in length to its radius is known as a *radian*.

Now the angle subtended by the circumference at the centre of a circle is one complete turn, i.e. 360°. Since the circumference of a circle is $2\pi \times$ its radius it will subtend 2π radians ($2\pi^c$) at its centre

$$\therefore 2\pi \text{ radians} = 360°,$$

$$\text{i.e. } 1 \text{ radian} = \frac{180°}{\pi} \sim 57° \, 18'.$$

$$\text{Also, } 1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi^c}{180}.$$

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Thus, to convert radians to degrees, it is necessary to multiply the angle by $180°/\pi$, and to convert from degrees to radians multiply by

$$\frac{\pi^c}{180}.$$

Theorem. *To find the length of arc of a circle radius r subtending an angle θ radians at its centre.*

Since the size of the angle at the centre of a circle is proportional to the arc subtending it, it follows that (S = length of arc subtending $\angle\theta$),

$$\frac{S}{\theta} = \frac{\text{circumference of circle}}{\text{number of radians subtended at the centre by its circumference}}.$$

$$\therefore \frac{S}{\theta} = \frac{2\pi r}{2\pi}, \text{ where } r = \text{radius of circle.}$$

$$\therefore S = r\theta.$$

Theorem. *If the area of a circle be taken as πr^2, to find the area of a sector subtending θ^c at the centre of the circle (radius r).*

Let A be the area required.

Since the area of a sector is proportional to the angle it subtends at the centre, it follows that

$$\frac{A}{\theta} = \frac{\text{area of circle}}{2\pi} = \frac{\pi r^2}{2\pi} = \frac{r^2}{2},$$

$$\therefore A = \tfrac{1}{2}r^2\theta.$$

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EXAMPLE. Find the number of seconds in the angle subtended at the centre of a circle of radius five miles by an arc of length 2 feet.

Let θ be the required angle in radians, r the radius.

$$\therefore \text{arc} = r\theta = 5 \times 1{,}760 \times 3 \times \theta \text{ feet}$$

$$\therefore 2 = 5 \times 1{,}760 \times 3\theta$$

$$\therefore \theta = \frac{2}{5 \times 1{,}760 \times 3} \text{ radians} = \frac{2}{5 \times 1{,}760 \times 3} \times \frac{180°}{\pi}$$

$$= \frac{3}{5 \times 44} \times \frac{1}{\pi} \times 60 \times 60 \text{ seconds}$$

$$= \frac{540}{11\pi} = 15{\cdot}6 \text{ seconds.}$$

EXAMPLE. The development of a cone is a sector of a circle of radius r and angle $\frac{1}{3}(2\pi^c)$. Find the semi-vertical angle of the cone.

Let x be the radius of the cone and θ its semi-vertical angle. Its slant length $= r$, and the circumference $2\pi x$ of the base of the cone is equal to the arc $\frac{2\pi}{3}r$ of the sector.

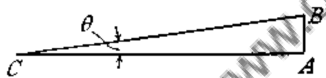
$$\therefore\ 2\pi x = \frac{2\pi}{3}r, \quad \therefore\ x = \frac{r}{3},$$

$$\therefore\ \sin\theta = x/r = \tfrac{1}{3}r/r = \tfrac{1}{3} = 0{\cdot}33333,$$

$$\therefore\ \theta = 19^\circ\ 28'.$$

Trigonometric Functions of the Angles 0°, 30°, 45°, 60°, 90°. The most important trigonometric functions are sine, cosine, and tangents, and these will be the only ones considered in this case, since those for the other three trigonometric functions can be obtained from them by inverting the results.

(i) *The angle* 0°. Consider the triangle ABC right-angled at A with $\angle BCA = \theta$, where θ is very small.



As θ approaches zero, A, B tend to coincide, i.e. as θ approaches zero, AB approaches zero, and CB tends to equality with CA.

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$$\sin\theta = \frac{AB}{BC} \quad \therefore\ \sin 0^\circ = \frac{0}{BC} = 0.$$

$$\cos\theta = \frac{CA}{BC} \quad \therefore\ \cos 0^\circ = \frac{CA}{CA} = 1.$$

$$\tan\theta = \frac{AB}{CA} \quad \therefore\ \tan 0^\circ = \frac{0}{CA} = 0.$$

(ii) *The angles* 30° *and* 60°. Consider the equilateral triangle ABC of side a, with AD drawn perpendicular to BC. (Left-hand figure on next page.)

By geometry D is the mid-point of BC and AD bisects $\hat{B}$ ...

$$\therefore\ BAD = 30^\circ.$$

From Pythagoras' theorem,

$$AB^2 = AD^2 + BD^2 \quad \therefore\ a^2 = AD^2 + (\tfrac{1}{2}a)^2$$

$$\text{i.e. } a^2 = AD^2 + \tfrac{1}{4}a^2, \quad \therefore\ AD^2 = \tfrac{3}{4}a^2,$$

$$\text{i.e. } AD = \frac{\sqrt{3}}{2}a.$$

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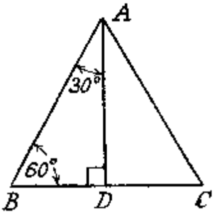
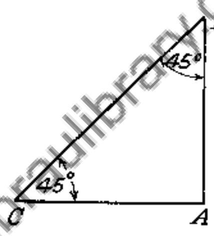
$$\sin 60^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2}; \qquad \sin 30^\circ = \frac{BD}{AB} = \frac{\frac{1}{2}a}{a} = \tfrac{1}{2};$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{\frac{1}{2}a}{a} = \tfrac{1}{2}; \qquad \cos 30^\circ = \frac{AD}{AB} = \frac{\frac{\sqrt{3}}{2}a}{a} = \frac{\sqrt{3}}{2};$$

$$\tan 60^\circ = \frac{AD}{BD} = \frac{\frac{\sqrt{3}}{2}a}{\frac{1}{2}a} = \sqrt{3}; \qquad \tan 30^\circ = \frac{BD}{AD} = \frac{\frac{1}{2}a}{\frac{\sqrt{3}}{2}a} = \frac{1}{\sqrt{3}}.$$





(iii) *The angle* 45°. Consider the isosceles triangle ABC right-angled at A with $AB = AC = a$. (Right-hand figure.)

Then $A\widehat{B}C = B\widehat{C}A = 45^\circ$.

By Pythagoras $\quad BC^2 = AB^2 + CA^2 = a^2 + a^2 = 2a^2$

$$\therefore\ BC = \sqrt{2}a$$

$$\sin 45^\circ = \frac{AB}{BC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$$\cos 45^\circ = \frac{AC}{BC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{AB}{AC} = \frac{a}{a} = 1.$$

(iv) *The angle* 90°. Using the diagram and notation of (i), as θ approaches zero $C\widehat{B}A \to 90^\circ$ ($\to$ denotes 'approaches')

$$\sin C\widehat{B}A = \frac{AC}{BC} \quad \therefore \sin 90^\circ = \frac{AC}{AC} = 1$$

$$\cos C\widehat{B}A = \frac{AB}{BC} \quad \therefore \cos 90^\circ = \frac{0}{BC} = 0$$

$$\tan C\widehat{B}A = \frac{CA}{AB} \quad \therefore \tan 90^\circ = \frac{CA}{0} = \infty.$$

The complete set of results is given in the following table.

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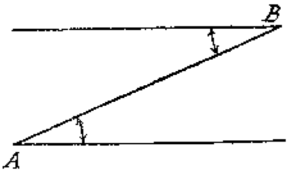
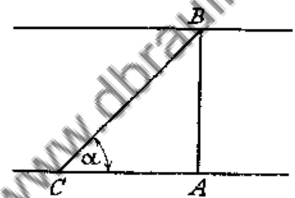
Angle	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
Cosine	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$\frac{1}{2}$	0
Tangent	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Heights and Distances. (Left-hand figure.) If a point B be viewed from a point A below it, the angle that AB makes with the horizontal through A in the same vertical plane as AB is known as the *angle of elevation* of B from A.

If the point A be viewed from the point B above it, then the angle that BA makes with the horizontal line through B in the same vertical plane as AB is known as the *angle of depression* of B seen from A.

By geometry, the above angles of elevation and depression are equal.





Problem. (Right-hand figure.) *To find the distance across a river where the distance cannot be measured directly.*

Take some visible point B on the opposite side of the river (banks parallel), and find a point A directly opposite it by means of a theodolite.

Move to a point C on the bank and measure the distance AC and $A\widehat{C}B = \alpha$ by means of instruments.

Then $AB = AC \tan \alpha$.

Theorem. *To find the height of a tower, when the foot of the tower is inaccessible.*

Let A be the foot of the tower and D the top. Let B and C be two points on the same level as A such that C, B, A, are collinear. By means of instruments (theodolite) measure the angles $DBA = \alpha$ and $DCA = \beta$. The distance $BC = a$ is also measured. Let x be the height of the tower. From the diagram

$$AB = x \cot \alpha, \; AC = x \cot \beta,$$
$$\therefore \; AC - AB = x \cot \beta - x \cot \alpha,$$
$$\text{i.e. } a = x(\cot \beta - \cot \alpha),$$
$$\therefore \; x = \frac{a}{\cot \beta - \cot \alpha}.$$

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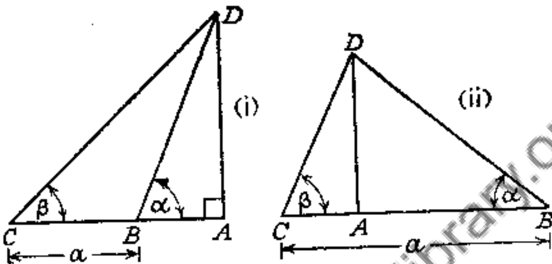
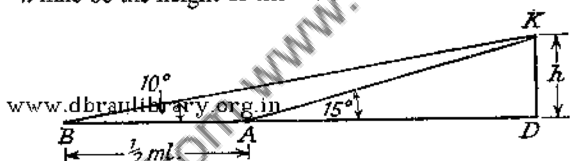
When C and B are on opposite sides of A as shown in diagram (ii)

$$AC + AB = x \cot \beta + x \cot \alpha$$
$$\text{i.e. } a = x(\cot \beta + \cot \alpha)$$
$$\therefore x = \frac{a}{\cot \beta + \cot \alpha}$$



EXAMPLE. Two observers, ½ mile apart, observe a kite in the same vertical plane and are both on the same side of it. The angles of elevation of the kite are 15° and 10°. Find the height of the kite.

Let K be the kite, A the first observer and B the second observer. Let $KD = h$ mile be the height of the kite.



From the diagram $BD = h \cot 10°$, $AD = h \cot 15°$,

$$\therefore BD - AD = h(\cot 10° - \cot 15°)$$
$$\text{i.e. } \tfrac{1}{2} = h(5{\cdot}6713 - 3{\cdot}7321)$$
$$\therefore 1{\cdot}9392h = 0{\cdot}5 \text{ i.e. } h = \frac{0{\cdot}5}{1{\cdot}9392}$$
$$\therefore h = \frac{1}{3{\cdot}8784} = 0{\cdot}2578 \text{ mile (using tables of reciprocals).}$$

EXAMPLE (L.U.). At a point A at the bottom of a hill the elevation of the top of a tower on the hill is 51° 18′. At a point B on the side of the hill and in the same vertical plane as A and the tower, the elevation is 71° 40′. AB makes an angle 20° with the horizontal and the distance $AB = 52$ feet. Determine the height of the top of the tower above A. (Left-hand figure on opposite page.)

C is the top of the tower and CL its height above A. BM is the perpendicular from B on CL, $x = CL$.

$$AL = x \cot 51° 18'.$$
$$BM = CM \cot 71° 40'$$
$$= (x - 52 \sin 20°) \cot 71° 40'.$$

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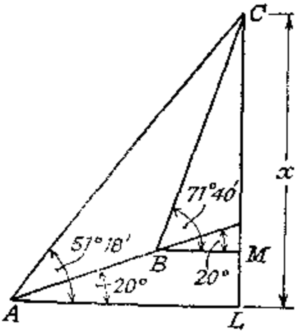
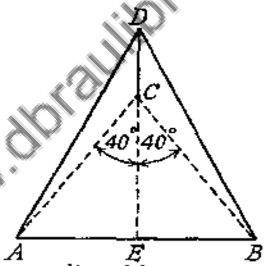
$AL - BM$
$= x \cot 51° 18' - (x - 52 \sin 20°) \cot 71° 40',$
i.e. $52 \cos 20°$
$= x[\cot 51° 18' - \cot 71° 40']$
$+ 52 \sin 20° \cot 71° 40',$
$\therefore x[\cot 51° 18' - \cot 71° 40']$
$= 52[\cos 20° - \sin 20° \cot 71° 40'],$
$\therefore x[0{\cdot}80115 - 0{\cdot}33147]$
$= 52[0{\cdot}93969 - 0{\cdot}34202 \times 0{\cdot}33147],$
$\therefore x \times 0{\cdot}46968 = 52[0{\cdot}82632],$

$$\therefore x = \frac{52 \times 0{\cdot}82632}{0{\cdot}46968}$$
$$= 91{\cdot}48 \text{ feet}$$

using 5-figure log tables.

$\log 52 = 1{\cdot}71600$
$\log 0{\cdot}82632 = \bar{1}{\cdot}91715$
$\qquad 1{\cdot}63315$
$\log 0{\cdot}46968 = \bar{1}{\cdot}67180$
$\qquad 1{\cdot}96135$





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Example (L.U.). Two points A, B of a straight horizontal road are at a distance 400 feet apart. A vertical flag-pole, 100 feet high, is at equal distances from A and B, and the angle subtended by AB at the foot C of the pole (which is in the same horizontal plane as the road) is 80°. (Right-hand figure.)

Find (i) the distance from the road to the foot of the pole;
(ii) the angle subtended by AB at the top of the pole.

D is the top of the pole and CE the perpendicular from C on AB.

By geometry, E is the mid-point of AB, CE bisects $\angle ACB$ and DE is perpendicular to AB ($AD = DB$ by symmetry).

From right-angled triangle AEC,

$$EC = AE \cot 40° = 200 \times 1{\cdot}19175 = 238{\cdot}4 \text{ feet.}$$

By Pythagoras, $DE^2 = DC^2 + EC^2 = 100^2 + 200^2 \cot^2 40°$
$= 100^2[1 + 4 \times 1{\cdot}19175^2]$
$= 100^2[1 + 4 \times 1{\cdot}4201] = 100^2 \times 6{\cdot}6804$
$\therefore DE = 100\sqrt{6{\cdot}6804} = 100 \times 2{\cdot}585 = 258{\cdot}5$
$\therefore \cot A\widehat{D}E = DE/AE = 258{\cdot}5/200 = 1{\cdot}2925$
$\therefore \angle ADE = 37° 44' \quad \therefore \angle ADB = 75° 28'.$

Projections. The *projection* of a straight line AB on another straight line XY in the same plane is the distance between the feet of the perpendiculars drawn from A and B on to the straight line XY.

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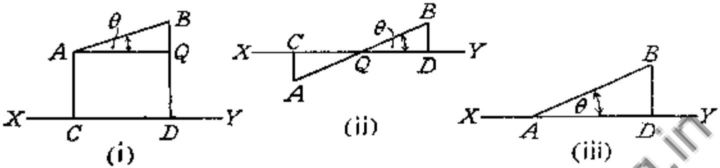
Let θ be the angle between AB and XY, and C, D be the feet of the perpendiculars from A and B on XY. The diagrams show the three different cases with A on XY in case (iii) and AQ perpendicular to BD in (i).



From diagram (i) $AQ = CD = AB\cos\theta.$

From diagram (ii) $CD = CQ + QD = AQ\cos\theta + QB\cos\theta$

$= (AQ + QB)\cos\theta = AB\cos\theta.$

From diagram (iii) $AD = AB\cos\theta.$

Hence, in all three cases, the projection of AB on XY in the same plane is $AB\cos\theta$, where θ is the angle between AB and XY.

This result also holds for the case of *two skew lines* (lines not in the same plane) AB and XY, where the angle θ between the skew lines is defined as the angle between two straight lines drawn parallel to the two skew lines through any point in space.

The *projection of a line AB on a plane* π is the distance between the feet of the perpendiculars from A and B on to the plane.

When the plane π is horizontal the projection is known as the *plan* of the line AB, and when the plane is vertical the projection is known as the *elevation* of AB.

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The *angle between a line and a plane* is defined as the angle between the line and its projection on the plane.

From this definition and from the definition of the projection of a straight line AB on a straight line XY, it follows that, if θ be the angle between AB and a given plane, the projection of AB on that plane is $AB\cos\theta$.

The *angle between two planes* is the angle between two straight lines, one in each plane, drawn through a point on the line of intersection of the two planes and perpendicular to that line of intersection.

A *line of greatest slope* in a given plane is a line that is perpendicular to any horizontal line in that plane.

The *projection of a plane area A* on a given plane π is the area formed by drawing perpendiculars on to the plane π from all points on the boundary of A.

Consider a unit square $PQRS$ and its projection on a plane π with which it makes an angle θ, the sides PQ and RS being parallel to the plane π. (Left-hand figure on next page.)

The projections of PQ and RS on the plane π will each be one

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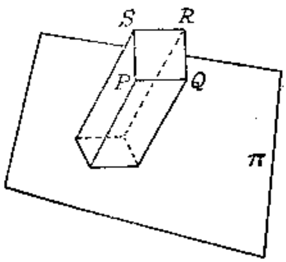
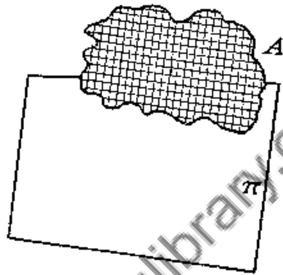
unit, and the projections of PS and RQ will be $\cos\theta$ units. The projected area will therefore be a rectangle whose area equals $\cos\theta$ square units.

Next consider the area A and its projection on a plane π with which it makes an angle θ. Divide the area A into small unit squares by means of lines parallel to the plane π and others perpendicular to these. (Right-hand figure.)





The number of small unit squares (small to efface irregularities) will be A, and the projection of each unit square on the plane π will have an area of $\cos\theta$ square units. Hence, the projected area of A will be $A\cos\theta$ *square units.*

Definition. *A right pyramid* is a pyramid having its vertex directly over the centroid of the base. (A pyramid is formed by joining any point in space to the vertices of a plane polygon, the polygon being known as the *base* of the pyramid and the point in space is the *vertex*.)

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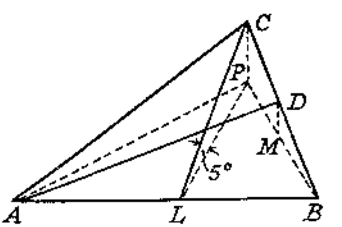
EXAMPLE (L.U.). On a plane inclined at an angle of 5° to the horizontal an equilateral triangle ABC is drawn with AB horizontal and 10 inches in length. If P be the foot of the perpendicular from C on the horizontal plane through AB, find $\angle APB$.



If D be the middle point of BC, find the angles of inclination of AD and AC to the horizontal plane.

L is the mid-point of AB.

By geometry, $PA = PB$, and PL bisects $\angle APB$ and is at right angles to AB. $\angle CLP = 5°$ and $CL = 10 \times \frac{1}{2}\sqrt{3} = 5\sqrt{3}$ inches.

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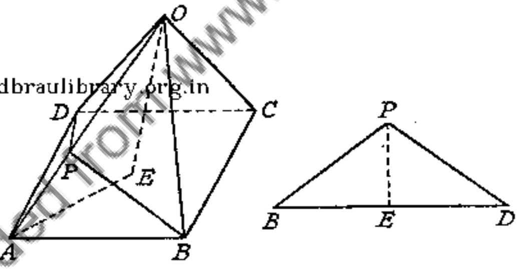
$$\therefore\ PL = CL \cos 5° = 5\sqrt{3} \cos 5°,$$
$$\cot \angle APL = PL/AL = 5\sqrt{3} \cos 5°/5 = \sqrt{3} \cos 5°,$$
$$\log (\cot \angle APL) = \tfrac{1}{2} \log 3 + \log (\cos 5°) = 0{\cdot}23856 + \bar{1}{\cdot}99834$$
$$= 0{\cdot}23690,$$
$$\therefore\ \angle APL = 30° \, 6' \quad \therefore\ \angle APB = 60° \, 12'.$$

If DM be the perpendicular from D on the plane ABP, by geometry $DM = \frac{1}{2}CP = \frac{1}{2} \times 5\sqrt{3} \sin 5°$ inches, and $AD = 5\sqrt{3}$ inches

$$\therefore\ \sin \angle DAM = \frac{DM}{AD} = \frac{\frac{1}{2}5\sqrt{3} \sin 5°}{5\sqrt{3}} = \tfrac{1}{2} \sin 5°$$
$$= \tfrac{1}{2} \times 0{\cdot}08716 = 0{\cdot}04358,$$
$$\therefore\ \angle DAM = 2° \, 30'$$
$$\sin \angle CAP = CP/AC = 5\sqrt{3} \sin 5°/10 = \tfrac{1}{2}\sqrt{3} \sin 5°.$$
$$\therefore\ \log (\sin \angle CAP) = \tfrac{1}{2} \log 3 + \log (\sin 5°) - \log 2$$
$$= 0{\cdot}23856 + \bar{2}{\cdot}94030 - 0{\cdot}30103$$
$$= \bar{2}{\cdot}87783,$$
$$\therefore\ \angle CAP = 4° \, 19\tfrac{1}{2}'.$$

EXAMPLE (L.U.). A pyramid (right) $OABCD$ stands on a square base $ABCD$ of side $2a$ and is of height $4a$.

Find the angle OA makes with the base and the angle between the planes OAB, OAD.



E is the centre of the square $ABCD$. The angle OA makes with $ABCD$ is $\angle OAE$. ($\angle OEA = 90°$.)

Now $AE = \sqrt{2}a$ ($\frac{1}{2}$ diagonal of square)

$$\therefore\ \cot \angle OAE = AE/OE$$
$$= \sqrt{2}a/4a = \tfrac{1}{4}\sqrt{2}$$
$$= 1{\cdot}4142/4$$
$$= 0{\cdot}35355$$
$$\therefore\ \angle OAE = 70° \, 32'.$$

BP is the perpendicular from B to OA, and by geometry DP is perpendicular to OA and $BP = DP$ by symmetry. But E is the mid-point of BD, therefore PE is perpendicular to BD and bisects $\angle BPD$.

By Pythagoras,

$$OA^2 = OE^2 + AE^2 = 16a^2 + 2a^2 = 18a^2$$
$$\therefore\ OA = a\sqrt{18} = 3\sqrt{2}a.$$

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If h be the perpendicular from O on AB,

$$h^2 = OA^2 - (\tfrac{1}{2}AB)^2 \quad \text{(using Pythagoras).}$$
$$= 18a^2 - a^2 = 17a^2.$$
$$\therefore\ h = \sqrt{17}a.$$

$\therefore$ twice area of $\triangle OAB = h \times AB = \sqrt{17}a \times 2a.$

Also, twice area of $\triangle OAB = BP \times OA = BP \times 3\sqrt{2}a,$

$$\therefore\ BP \times 3\sqrt{2}a = 2\sqrt{17}a^2,$$
$$\therefore\ BP = \tfrac{2}{3}\sqrt{\frac{17}{2}}a,$$
$$\therefore\ \sin \angle BPE = \frac{BE}{BP} = \frac{\sqrt{2}a}{\tfrac{2}{3}\sqrt{\frac{17}{2}}a} = \frac{3}{\sqrt{17}} = \frac{3\sqrt{17}}{17}$$
$$= \frac{12{\cdot}3693}{17} = 0{\cdot}72761$$
$$\therefore\ \angle BPE = 46^\circ\ 41',$$
$$\therefore\ \angle BPD = 2\angle BPE = 93^\circ\ 22',$$

i.e. the angle between the planes OAB, OAD is $93^\circ\ 22'$.

EXAMPLE. The vertex V of a right pyramid stands over a regular pentagonal base $ABCDE$. If the length of a slant edge VA be 20 inches and an edge AB of the base be 10 inches, calculate the height and the volume of the pyramid and the dihedral angle between the lateral face and the base.

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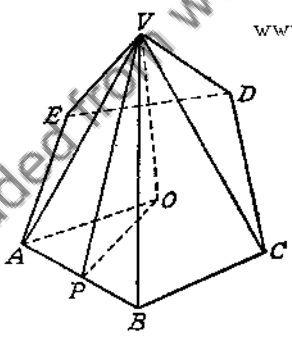

O is the centroid of the base and P the mid-point of AB. Then VP and OP are perpendicular to AB.

Since $ABCDE$ is a regular pentagon, $\angle EAB = 108^\circ$, therefore $\angle OAB = 54^\circ$ (by geometry).

From $\triangle OAP$, $\quad OA = AP \sec 54^\circ = 5 \sec 54^\circ$
$$= 5 \times 1{\cdot}70130 = 8{\cdot}5065.$$

By Pythagoras,

$$VO^2 = VA^2 - OA^2$$
$$= 400 - 72{\cdot}361 = 327{\cdot}639$$

$\therefore\ VO = 18{\cdot}10$ inches correct to 4 significant figures.

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The volume V of the pyramid is given by

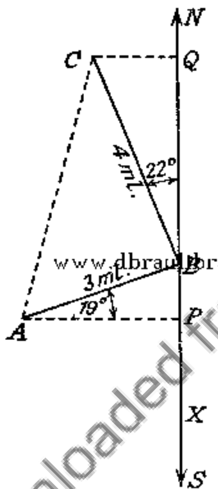
$$\begin{aligned} V &= \tfrac{1}{3}\text{ area of base } \times \text{ height} \\ &= \tfrac{5}{3} \times \triangle OAB \times VO \\ &= \tfrac{5}{3} \times 5 \times OP \times 18{\cdot}10 \\ &= \tfrac{5}{3} \times 25 \tan 54^\circ \times 18{\cdot}10. \\ \therefore \log V &= \log 125 + \log \tan 54^\circ \\ &\qquad + \log 18{\cdot}1 - \log 3 \\ &= 3{\cdot}01621 \\ \therefore V &= 1{,}038 \text{ cubic inches.} \end{aligned}$$

$$\begin{aligned} \log 125 &= 2{\cdot}09691 \\ \log \tan 54^\circ &= 0{\cdot}13874 \\ \log 18{\cdot}10 &= 1{\cdot}25768 \\ &\ 3{\cdot}49333 \\ \log 3 &= 0{\cdot}47712 \\ &\ 3{\cdot}01621 \end{aligned}$$

$$\tan \angle VPO = VO/OP = \frac{18{\cdot}1}{5 \tan 54^\circ} = \frac{3{\cdot}62}{\tan 54^\circ}$$

$$\begin{aligned} \therefore \log(\tan \angle VPO) &= \log 3{\cdot}62 - \log(\tan 54^\circ) = 0{\cdot}55871 - 0{\cdot}13874 \\ &= 0{\cdot}41997, \\ \therefore \angle VPO &= 69^\circ 11'. \end{aligned}$$

EXAMPLE (L.U.). *A, B, C* are three farms on a plain. The distance and bearing of *B* from *A* are three miles E. 19° N.; of *C* from *B* four miles, N. 22° W. Find how far E. or W., and how far N. or S., *C* is of *A*. What is the bearing of *C* from *A*?



BX is the N.-S. line through *B* and *AP*, *CQ* are the perpendiculars from *A* and *C* on *BX*.

Then *C* is E. of *A* a distance

$$\begin{aligned} AP - CQ &= 3 \cos 19^\circ - 4 \sin 22^\circ \\ &= 2{\cdot}83656 - 1{\cdot}49844 \\ &= 1{\cdot}338 \text{ miles.} \end{aligned}$$

C is N. of *A* a distance

$$\begin{aligned} PB + BQ &= 3 \sin 19^\circ + 4 \cos 22^\circ \\ &= 0{\cdot}97671 + 3{\cdot}70872 \\ &= 4{\cdot}685 \text{ miles.} \end{aligned}$$

If *C* be θ° E. of N. of *A*, then,

$$\begin{aligned} \tan \theta &= 1{\cdot}338/4{\cdot}685 \\ \log(\tan \theta) &= \log 1{\cdot}338 - \log 4{\cdot}685 \\ &= 0{\cdot}12645 - 0{\cdot}67072 = \bar{1}{\cdot}45573 \\ \therefore \theta &= 15^\circ 56' \end{aligned}$$

$\therefore$ *C* is 15° 56′ E. of N. of *A*.

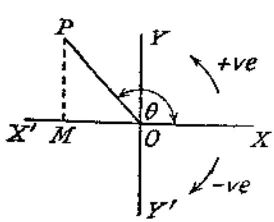
Circular Functions of the General Angle. The *general angle* is an angle of any size including negative angles.

Let *OX*, *OY* be the usual perpendicular axes used in graphical work, and let the *same* scales be used along the axes of x and of y.

If P be the point (x, y), $OP = r$ (always positive), and $\angle POX = \theta$, the following are the definitions of the trigonometric functions when θ is the general angle.

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x},$$

$$\operatorname{cosec}\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}.$$



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It is to be noted that the position OP can be arrived at by turning through any number of complete turns plus the angle θ, and all positive angles are obtained by turning in an anti-clockwise direction from OX, and negative angles by turning in a clockwise direction from OX.

From the above definitions the following identities are readily obtained:

$$\operatorname{cosec}\theta = 1/\sin\theta, \qquad \sec\theta = 1/\cos\theta,$$

$$\tan\theta = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

If PM be the ordinate of P, from right-angled triangle OPM, using Pythagoras,

$$OM^2 + PM^2 = OP^2, \text{ i.e. } x^2 + y^2 = r^2.$$

$$\therefore \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1, \text{ i.e. } \cos^2\theta + \sin^2\theta = 1 \quad \ldots\ldots(1),$$

which is the same fundamental identity as obtained for an acute angle θ. It will be found that all the identities relating to an acute angle θ are also true for the general angle θ.

Dividing through the identity (1) by $\cos^2\theta$ and $\sin^2\theta$ in succession,

$$\left.\begin{aligned} 1 + \tan^2\theta &= \sec^2\theta \\ \cot^2\theta + 1 &= \operatorname{cosec}^2\theta \end{aligned}\right\}$$

The signs of the circular functions of θ in the four quadrants $XOY, X'OY, X'OY', XOY'$ (first, second, third, and fourth quadrants respectively) are obtained in the following table.

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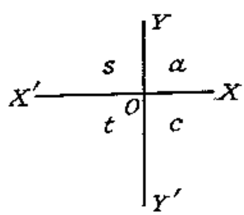
Quadrant	x	y	r	$\sin\theta = y/r$	$\cos\theta = x/r$	$\tan\theta = y/x$
1st	+	+	+	+	+	+
2nd	−	+	+	+	−	−
3rd	−	−	+	−	−	+
4th	+	−	+	−	+	−

To remember these results it is advisable to follow the diagram, using the word 'cast' showing what functions are positive in the various quadrants.



(a = all, s = sine, t = tan, c = cos.)

The signs for $\operatorname{cosec}\theta$, $\sec\theta$, $\cot\theta$ can be obtained from the table, being the same as for $\sin\theta$, $\cos\theta$, $\tan\theta$ respectively, since they are the inverses of these functions.

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Since the position of OP is unaltered if any multiple of 2π radians be added or subtracted from the general angle, it follows that any multiple of 2π radians (360°) can be added to or subtracted from the angle of a trigonometric function without altering its value.

Example. Find (i) sin 792°, (ii) tan (−336°), using tables.

$$\sin 792° = \sin(792° - 2 \times 360°) = \sin 72° = 0{\cdot}95106$$
$$\tan(-336°) = \tan(-336° + 360°) = \tan 24° = 0{\cdot}44523.$$

Theorem. *To find the sine, cosine, and tangent of the angles* (i) $90° - \theta$, (ii) $90° + \theta$, (iii) $180° - \theta$, (iv) $180° + \theta$, (v) $270° - \theta$, (vi) $270° + \theta$, (vii) $360° - \theta$, *or* $-\theta$.

In all that follows P is the point (x, y), $OP = r$, and $\theta = \angle POX$ is considered acute, but the results are valid for all values of θ.

(i) $(90° - \theta)$. $OP_1 = r$ and $\angle P_1OX = 90° - \theta$. From congruent triangles $\qquad P_1 \equiv (y, x)$

$$\left.\begin{aligned} \therefore \sin(90° - \theta) &= \frac{y \text{ co-ordinate of } P_1}{r} = \frac{x}{r} = \cos\theta \\ \cos(90° - \theta) &= \frac{x \text{ co-ordinate of } P_1}{r} = \frac{y}{r} = \sin\theta \\ \tan(90° - \theta) &= \frac{\sin(90° - \theta)}{\cos(90° - \theta)} = \frac{\cos\theta}{\sin\theta} = \cot\theta \end{aligned}\right\}$$

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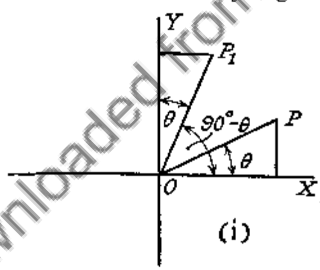
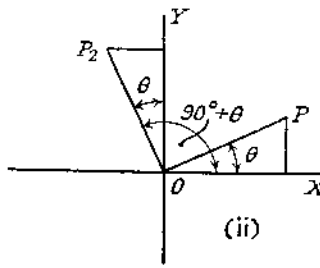


(i)



(ii)

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(ii) $(90° + \theta)$. $OP_2 = r$ and $\angle P_2OX = 90° + \theta$.

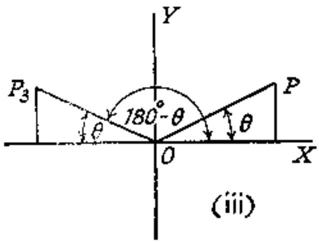
From congruent triangles and the fact that P_2 lies in the second quadrant, $\qquad P_2 \equiv (-y, x)$

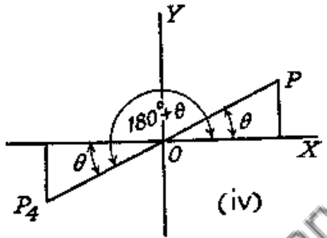
$$\left.\begin{aligned} \therefore \sin(90° + \theta) &= y \text{ co-ordinate of } P_2/r = x/r = \cos\theta \\ \cos(90° + \theta) &= x \text{ co-ordinate of } P_2/r = -y/r = -\sin\theta \\ \tan(90° + \theta) &= \frac{\sin(90° + \theta)}{\cos(90° + \theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta \end{aligned}\right\}$$

(iii) $(180 - \theta)$. $OP_3 = r$, $\angle P_3OX = 180° - \theta$.

By symmetry, $P_3 \equiv (-x, y)$

$$\left.\begin{aligned} \therefore \sin(180^\circ - \theta) &= y/r = \sin\theta \\ \cos(180^\circ - \theta) &= -x/r = -\cos\theta \\ \tan(180^\circ - \theta) &= \frac{\sin\theta}{-\cos\theta} = -\tan\theta \end{aligned}\right\}$$





(iv) $(180^\circ + \theta)$. $OP_4 = r$, $\angle P_4OX = 180^\circ + \theta$.

From the diagram, $P_4 \equiv (-x, -y)$

$$\left.\begin{aligned} \therefore \sin(180^\circ + \theta) &= -y/r = -\sin\theta \\ \cos(180^\circ + \theta) &= -x/r = -\cos\theta \\ \tan(180^\circ + \theta) &= \frac{-\sin\theta}{-\cos\theta} = \tan\theta \end{aligned}\right\}$$

(v) $(270^\circ - \theta)$. $OP_5 \equiv r$, $\angle P_5OX = 270^\circ - \theta$.

From the diagram, $P_5 = (-y, -x)$

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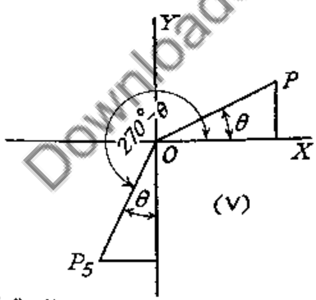
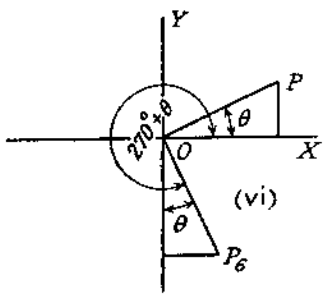
$$\left.\begin{aligned} \therefore \sin(270^\circ - \theta) &= -x/r = -\cos\theta \\ \cos(270^\circ - \theta) &= -y/r = -\sin\theta \\ \tan(270^\circ - \theta) &= \frac{-\cos\theta}{-\sin\theta} = \cot\theta \end{aligned}\right\}$$





(vi) $(270^\circ + \theta)$. $OP_6 = r$, $\angle P_6OX = 270^\circ + \theta$.

From the diagram, $P_6 \equiv (y, -x)$

$$\left.\begin{aligned} \therefore \sin(270^\circ + \theta) &= -x/r = -\cos\theta \\ \cos(270^\circ + \theta) &= y/r = \sin\theta \\ \tan(270^\circ + \theta) &= \frac{-\cos\theta}{\sin\theta} = -\cot\theta. \end{aligned}\right\}$$

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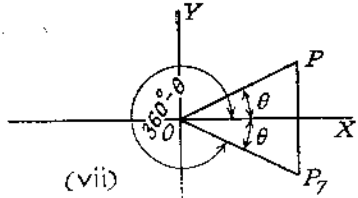
(vii) $360° - \theta$ or $(-\theta)$. $OP_7 = r$, and $\angle P_7OX = 360° - \theta$, or $-\theta$.
By symmetry, $P_7 \equiv (x, -y)$

$$\left.\begin{aligned} \therefore \sin(360° - \theta) &= \sin(-\theta) = -y/r = -\sin\theta \\ \cos(360° - \theta) &= \cos(-\theta) = x/r = \cos\theta \\ \tan(360° - \theta) &= \tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta \end{aligned}\right\}$$



The most important of these results are:

$$\left.\begin{aligned} \sin(180° - \theta) &= \sin\theta \\ \cos(180° - \theta) &= -\cos\theta \\ \tan(180° - \theta) &= -\tan\theta \end{aligned}\right\} \quad \left.\begin{aligned} \sin(180° + \theta) &= -\sin\theta \\ \cos(180° + \theta) &= -\cos\theta \\ \tan(180° + \theta) &= \tan\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin(360° - \theta) &= \sin(-\theta) = -\sin\theta \\ \cos(360° - \theta) &= \cos(-\theta) = \cos\theta \\ \tan(360° - \theta) &= \tan(-\theta) = -\tan\theta \end{aligned}\right\},$$

as these are sufficient, with the aid of tables (which only give trigonometric functions for angles from 0° to 90°), to find the value of the trigonometric function of any angle. The cosec, sec, and cot results are similar to the sin, cos, and tan results respectively.

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EXAMPLE. Find the sine, cosine, and tangent of the angles (i) 150°, (ii) 255°, (iii) 295°, (iv) −140°.

(i) $\sin(150°) = \sin(180° - 30°) = +\sin 30° = +0{\cdot}5.$
$\cos(150°) = \cos(180° - 30°) = -\cos 30° = -\frac{1}{2}\sqrt{3}.$
$\tan(150°) = \tan(180° - 30°) = -\tan 30° = -1/\sqrt{3}.$

(ii) $\sin(255°) = \sin(180° + 75°) = -\sin 75° = -0{\cdot}96593.$
$\cos(255°) = \cos(180° + 75°) = -\cos 75° = -0{\cdot}25882.$
$\tan(255°) = \tan(180° + 75°) = +\tan 75° = +3{\cdot}73205.$

(iii) $\sin(295°) = \sin(360° - 65°) = -\sin 65° = -0{\cdot}90631.$
$\cos(295°) = \cos(360° - 65°) = +\cos 65° = +0{\cdot}42262.$
$\tan(295°) = \tan(360° - 65°) = -\tan 65° = -2{\cdot}14451.$

(iv) $\sin(-140°) = -\sin 140° = -\sin(180° - 40°) = -\sin 40° = -0{\cdot}64279.$
$\cos(-140°) = \cos 140° = \cos(180° - 40°) = -\cos 40° = -0{\cdot}76604.$
$\tan(-140°) = -\tan 140° = -\tan(180° - 40°) = +\tan 40° = +0{\cdot}83910.$

The following are the graphs of the six trigonometric functions as obtained from previous results and the use of tables, the graphs being taken between $x = -360°$ and $x = +360°$ in each case.

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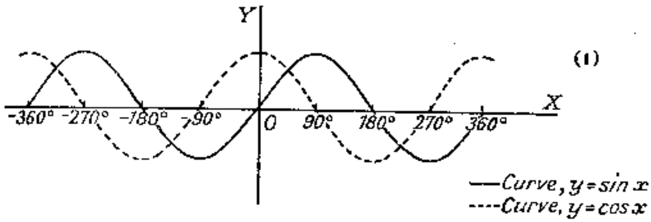

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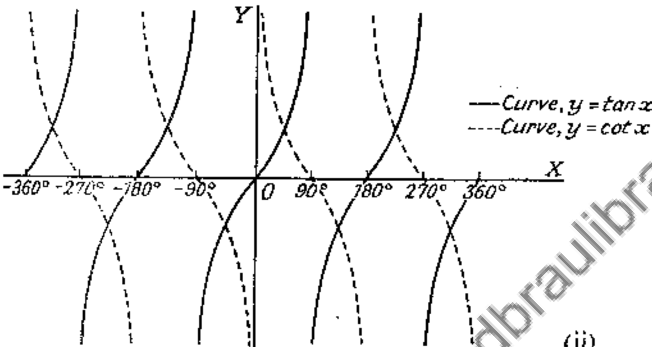

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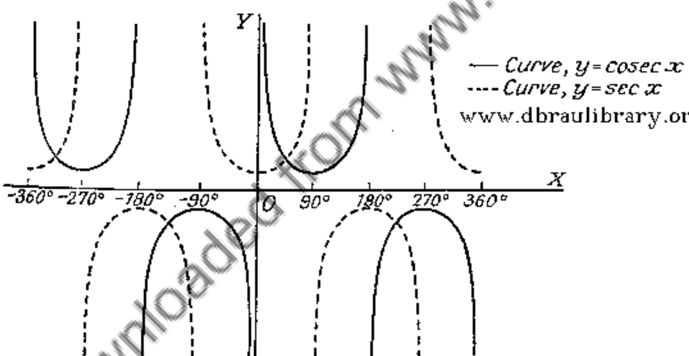

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Inverse Trigonometric Functions. The inverse trigonometric functions are angles that are defined as follows:

(i) $\sin^{-1} a$ is the angle between *−90° and +90°* satisfying the equation $\sin \theta = a$.

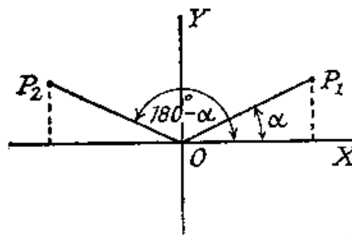
(ii) $\cos^{-1} b$ is the angle between *0 and 180°* satisfying the equation $\cos \theta = b$.

(iii) $\tan^{-1} c$ is the angle between *−90° and +90°* satisfying the equation $\tan \theta = c$.

NOTE. The angles are chosen so that their trigonometric functions cover all positive and negative values of a, b, c.

Theorem. *To find the general values of the angles satisfying the equations,* (i) *sin* $\theta = a$, (ii) *cos* $\theta = b$, (iii) *tan* $\theta = c$, *where* a, b, c *are constants.*

(i) *sin* $\theta = a$. Let $\alpha = \sin^{-1} a$ and OP_1 be the position for an angle α, and OP_2 be the position for an angle $180° - \alpha$.



Since $\sin(180° - \alpha) = \sin\alpha = a$, it follows that both OP_1 and OP_2 will be the positions for all angles θ satisfying $\sin\theta = a$.

In the position OP_1

$$\theta = 2m\pi + \alpha \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and in the position OP_2

$$\theta = 2p\pi + (\pi - \alpha)$$
$$= (2p + 1)\pi - \alpha \ldots\ldots\ldots\ldots\ldots (2),$$

where m and p are any integers.

These results are combined in the single result

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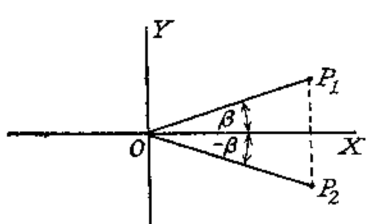
$$\theta = n\pi + (-1)^n\alpha,$$

where n is any integer, for when $n = 2m$ (even) the result (1) is obtained, and when $n = (2p + 1)$ (odd) the result (2) is obtained.

NOTE. π represents π radians $= 180°$, and α is taken positive but the result holds true if α is negative.

(ii) *cos* $\theta = b$. Let $\beta = \cos^{-1} b$, $P_1OX = \beta$, and $P_2OX = -\beta$.

Since $\cos(-\beta) = \cos\beta = b$, the angles in the positions OP_1 and OP_2 satisfy the equation $\cos\theta = b$.



In the position OP_1, $\theta = 2m\pi + \beta$, and in the position OP_2, $\theta = 2p\pi - \beta$, where m and p are any integers.

These are combined in the single result $\theta = 2n\pi \pm \beta$, where n is y integer. (β taken acute but result is true if β obtuse.)

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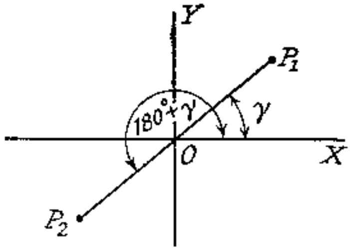
(iii) $\tan\theta = c$. Let $\gamma = \tan^{-1} c$, $P_1OX = \gamma$, $P_2OX = (180° + \gamma)$. Since $\tan(180° + \gamma) = \tan\gamma = c$, the angles in the positions OP_1 and OP_2 will satisfy the equation $\tan\theta = c$.



In the position OP_1, $\theta = 2m\pi + \gamma$, and in the position OP_2, $\theta = 2p\pi + (\pi + \gamma) = (2p + 1)\pi + \gamma$, where m and p are any integers.

These results are combined to give $\theta = n\pi + \gamma$, where n is any integer. (γ taken positive but result valid if γ negative.)

NOTE. These results are stated for cases when α, β, γ are taken in radians. When these angles are in degrees the quantity π in the results should be replaced in each case by 180°.

The corresponding results for the other three trigonometric functions are the same as for their inverses.

Certain trigonometric equations are reducible to quadratic equations in $\sin\theta$, $\cos\theta$, or $\tan\theta$ by making use of the various trigonometric identities, and these can then be solved for the respective function, and the general values of the angle then obtained.

EXAMPLE. Find the general solutions of the equations

(i) $8\sin^2\theta + 6\cos\theta - 9 = 0$;
(ii) $2\sec^2\theta = 1 + 3\tan\theta$;
(iii) $2\sin^4\theta + \sin^2\theta - 1 = 0$.

(i) Using $\sin^2\theta = 1 - \cos^2\theta$, the equation becomes,

$$8 - 8\cos^2\theta + 6\cos\theta - 9 = 0$$
$$\text{i.e. } 8\cos^2\theta - 6\cos\theta + 1 = 0,$$
$$\therefore (2\cos\theta - 1)(4\cos\theta - 1) = 0$$
$$\therefore \cos\theta = \tfrac{1}{2} \ldots\ldots\ldots\ldots (1),$$
$$\text{or } \cos\theta = \tfrac{1}{4} \ldots\ldots\ldots\ldots (2)$$

From (i) $\theta = m \,.\, 360° \pm 60°$
From (ii) $\theta = n \,.\, 360° \pm 75° 31'$,

where m and n are any integers.

(ii) Using $\sec^2\theta = 1 + \tan^2\theta$, the equation becomes,

$$2 + 2\tan^2\theta = 1 + 3\tan\theta,$$
$$\text{i.e. } 2\tan^2\theta - 3\tan\theta + 1 = 0,$$
$$\therefore (2\tan\theta - 1)(\tan\theta - 1) = 0,$$
$$\therefore \tan\theta = 1 \ldots\ldots\ldots\ldots (1)$$
$$\text{or } \tan\theta = \tfrac{1}{2} \ldots\ldots\ldots\ldots (2).$$

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From (1), $\theta = m \cdot 180° + 45°$
From (2), $\theta = n \cdot 180° + 26° 34'$ $\Big\}$,
where m and n are any integers.

(iii) Factorising the given equation,

$$(2 \sin^2 \theta - 1)(\sin^2 \theta + 1) = 0$$
$$\therefore \sin^2 \theta = \tfrac{1}{2} \text{ or } -1$$

Now $\sin^2 \theta \neq -1$, $\therefore \sin^2 \theta = \frac{1}{2}$, $\therefore \sin \theta = \pm 1/\sqrt{2}$.
From $\sin \theta = 1/\sqrt{2}$, $= m\pi + (-1)^m \frac{1}{4}\pi$.
From $\sin \theta = -1/\sqrt{2}$, $\theta = n\pi + (-1)^n(-\frac{1}{4}\pi)$.
These results can be classed together under the formula $\theta = p\pi \pm \frac{1}{4}\pi$, where p is any integer.

Certain other types of equations, as shown in the following examples, can be solved by making use of the general solutions of the equations $\sin \theta = a$, $\cos \theta = b$, and $\tan \theta = c$.

Example. Solve the equations (i) $\sin 2x = \sin 3x$, (ii) $\cos mx = \sin nx$ giving the general solutions, and also all solutions for x between $x = 0$ and π in (i).

(i)
$$\sin 3x = \sin 2x$$
$$= \sin [n\pi + (-1)^n 2x],$$
where n is any integer. Therefore $3x = n\pi + (-1)^n \cdot 2x$.

When n is *even* and equal to $2m$,

$$3x = 2m\pi + 2x, \quad \therefore x = 2m\pi$$

When n is *odd* and equal to $(2p + 1)$,

$$3x = (2p + 1)\pi - 2x, \quad \therefore 5x = (2p + 1)\pi,$$
$$x = \frac{(2p + 1)}{5}\pi.$$

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Therefore the general solutions are $x = 2m\pi$ or $\dfrac{(2p + 1)}{5}\pi$.
where m and p are any integers.

For values of x between 0 and π, not including these end values, $p = 0$ or 1 (m cannot take any values), therefore values of x between 0 and π are $x = \frac{1}{5}\pi$ and $3\pi/5$.

(ii)
$$\cos mx = \sin nx = \cos (\tfrac{1}{2}\pi - nx)$$
$$= \cos [2p\pi \pm (\tfrac{1}{2}\pi - nx)], \; p \text{ any integer}$$
$$\therefore mx = 2p\pi \pm (\tfrac{1}{2}\pi - nx).$$

Using the negative sign,

$$(m - n)x = (4p - 1)\tfrac{1}{2}\pi, \quad \therefore x = \frac{4p - 1}{m - n} \cdot \tfrac{1}{2}\pi.$$

Using the positive sign,

$$(m + n)x = (4p + 1)\tfrac{1}{2}\pi \quad \therefore x = \frac{4p + 1}{m + n} \cdot \tfrac{1}{2}\pi,$$

where p is any integer.

Trigonometric Functions of the Compound Angles $(A \pm B)$. In the following proofs $\angle A$ and $\angle B$ are taken to be acute and $\angle(A + B)$ is taken to be acute ($\angle A > \angle B$), but the results are valid for all

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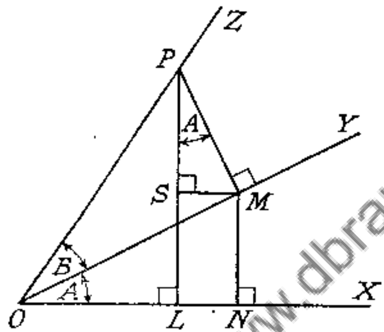
angles as can be shown by applying the results for the trigonometric functions of $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° - \theta$ to the identities obtained.

Theorem. *To find the expansions of the sine, cosine, and tangent of the angle $(A + B)$, where $\angle A$ and $\angle B$ are acute and $\angle(A + B)$ is acute.*

In the diagram $\angle XOY = \angle A$, $\angle YOZ = \angle B$, P is any point on OZ; PL, PM are the perpendiculars from P on OX, OY respectively; MN, MS are the perpendiculars from M on OX and PL respectively.



Since PL and PM are perpendicular to OX and OY respectively,

$$\angle MPS = \angle A.$$

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From the construction, $SLNM$ is a rectangle, and

$$\therefore SL = MN;\ SM = LN.$$

$$\text{Now } \sin(A + B) = \frac{PL}{PO} = \frac{PS + SL}{PO} = \frac{PS}{PO} + \frac{MN}{PO}$$

$$= \frac{MN}{PO} + \frac{PS}{PO}$$

$$= \frac{MN}{OM} \cdot \frac{OM}{PO} + \frac{PS}{PM} \cdot \frac{PM}{PC}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\cos(A + B) = \frac{OL}{OP} = \frac{ON - LN}{OP} = \frac{ON}{OP} - \frac{SM}{OP}$$

$$= \frac{ON}{OM} \cdot \frac{OM}{OP} - \frac{SM}{PM} \cdot \frac{PM}{OP}$$

$$= \cos A \cos B - \sin A \sin B.$$

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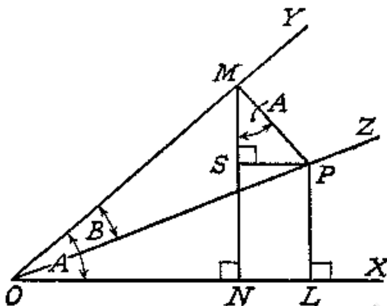
$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Theorem. *To find the expansions of the sine, cosine, and tangent of the angle $(A - B)$, where $\angle A$ and $\angle B$ are acute with $\angle A > \angle B$.*



In the diagram

$$\angle XOY = \angle A;$$
$$\angle YOZ = \angle B;$$

P is any point OZ and PL, PM are the perpendiculars from P on OX and OY respectively; MN is the perpendicular from M on OX and PS the perpendicular from P on MN.

Since PM and MN are perpendicular to OY and OX respectively, $\angle PMS = \angle A$.

By construction, $SNLP$ is a rectangle, $\therefore SN = PL$, $NL = SP$.

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$$\sin(A-B) = \frac{PL}{OP} = \frac{SN}{OP} = \frac{MN - MS}{OP} = \frac{MN}{OP} - \frac{MS}{OP}$$

$$= \frac{MN}{OM}\cdot\frac{OM}{OP} - \frac{MS}{MP}\cdot\frac{MP}{OP}$$

$$= \sin A\cos B - \cos A\sin B.$$

$$\cos(A-B) = \frac{OL}{OP} = \frac{ON + NL}{OP} = \frac{ON}{OP} + \frac{SP}{OP}$$

$$= \frac{ON}{OM}\cdot\frac{OM}{OP} + \frac{SP}{MP}\cdot\frac{MP}{OP}$$

$$= \cos A\cos B + \sin A\sin B.$$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} - \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} + \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

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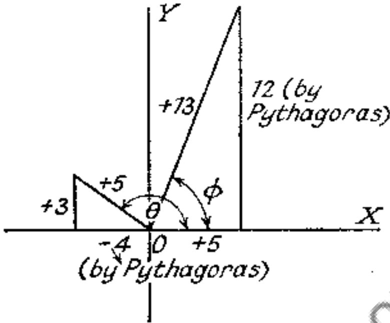
EXAMPLE. Find the values of

$\sin(\theta \pm \varphi)$, $\cos(\theta \pm \varphi)$,

$\tan(\theta \pm \varphi)$, given $\sin\theta = \frac{3}{5}$, $\cos\varphi = \frac{5}{13}$, where θ is obtuse and φ is acute.

From the right-angled triangles shown in the diagram,



$$\cos\theta = -\frac{4}{5},$$
$$\tan\theta = -\frac{3}{4},$$
$$\sin\varphi = \frac{12}{13},$$
$$\tan\varphi = \frac{12}{5}$$

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$$
$$= \frac{3}{5} \times \frac{5}{13} + \left(\frac{-4}{5}\right) \times \frac{12}{13} = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}.$$

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$$
$$= \frac{3}{5} \times \frac{5}{13} - \left(\frac{-4}{5}\right) \times \frac{12}{13} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}.$$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$$
$$= \left(\frac{-4}{5}\right)\left(\frac{5}{13}\right) - \frac{3}{5} \times \frac{12}{13} = \frac{-20}{65} - \frac{36}{65} = \frac{-56}{65}.$$

$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$
$$= \left(\frac{-4}{5}\right)\left(\frac{5}{13}\right) + \frac{3}{5} \times \frac{12}{13} = \frac{-20}{65} + \frac{36}{65} = \frac{16}{65}.$$

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$$\tan(\theta + \varphi) = \frac{\sin(\theta + \varphi)}{\cos(\theta + \varphi)} = \frac{-33/65}{-56/65} = \frac{33}{56},$$

$$\tan(\theta - \varphi) = \frac{\sin(\theta - \varphi)}{\cos(\theta - \varphi)} = \frac{63/65}{16/65} = \frac{63}{16}.$$

Check. $$\tan(\theta + \varphi) = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = \frac{-\frac{3}{4} + 12/5}{1 - (-\frac{3}{4})(12/5)} = \frac{33/20}{56/20} = \frac{33}{56}.$$

$$\tan(\theta - \varphi) = \frac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi} = \frac{-\frac{3}{4} - 12/5}{1 + (-\frac{3}{4})(12/5)} = \frac{-63/20}{-16/20} = \frac{63}{16}.$$

NOTE. The results for the compound angles $(A \pm B)$ can be used in expanding the sine, cosine, and tangent of the angles $(A \pm B \pm C)$.

EXAMPLE. Expand $\sin(A + B + C)$ in sines and cosines of the angles A, B, and C.

$$\sin(A + B + C) = \sin[(A + B) + C] = \sin(A + B)\cos C + \cos(A + B)\sin C$$
$$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$$
$$= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C.$$

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Example. Without the use of tables, find the values of cos 15° and tan 75°.

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \tfrac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

$$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$$
$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

Example. Simplify the following:

(i) $\cos x + \sin y \sin(x - y)$;
(ii) $\cos 3\theta \cos(\theta + \varphi) - \sin 3\theta \sin(\theta + \varphi)$;
(iii) $\sin(x - y) \cos 2y + \cos(x - y) \sin 2y$;
(iv) $\dfrac{\tan(\theta - \varphi) + \tan(\theta + \varphi)}{1 - \tan(\theta - \varphi)\tan(\theta + \varphi)}$.

(i) Using $\sin(x - y) = \sin x \cos y - \cos x \sin y$,

$$\begin{aligned}\cos x + \sin y \sin(x - y) &= \cos x + \sin y(\sin x \cos y - \cos x \sin y)\\ &= \cos x(1 - \sin^2 y) + \sin x \sin y \cos y\\ &= \cos x \cos^2 y + \sin x \sin y \cos y \qquad (1 - \sin^2 y = \cos^2 y)\\ &= \cos y(\cos x \cos y + \sin x \sin y)\\ &= \cos y \cos(x - y). \qquad (\cos(x - y) = \cos x \cos y + \sin x \sin y)\end{aligned}$$

(ii) Now $\cos A \cos B - \sin A \sin B = \cos(A + B)$

$$\begin{aligned}\therefore \cos 3\theta \cos(\theta + \varphi) - \sin 3\theta \sin(\theta + \varphi) &= \cos[3\theta + (\theta + \varphi)]\\ &= \cos(4\theta + \varphi).\end{aligned}$$

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(iii) Now $\sin A \cos B + \cos A \sin B = \sin(A + B)$,

$$\begin{aligned}\therefore \sin(x - y)\cos 2y + \cos(x - y)\sin 2y &= \sin[(x - y) + 2y]\\ &= \sin(x + y).\end{aligned}$$

(iv) $$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A + B)$$

$$\therefore \frac{\tan(\theta - \varphi) + \tan(\theta + \varphi)}{1 - \tan(\theta - \varphi)\tan(\theta + \varphi)} = \tan[(\theta - \varphi) + (\theta + \varphi)] = \tan 2\theta.$$

Example (L.U.). Express $\cos(A + B + C + D)$ in terms of the sines and cosines of A, B, C, D, and hence, or otherwise, obtain the identity $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

$$\begin{aligned}&\cos(A + B + C + D)\\ &= \cos(x + y), \text{ where } x = A + B,\ y = C + D.\\ &= \cos x \cos y - \sin x \sin y\\ &= \cos(A + B)\cos(C + D) - \sin(A + B)\sin(C + D)\\ &= (\cos A \cos B - \sin A \sin B)(\cos C \cos D - \sin C \sin D)\\ &\qquad - (\sin A \cos B + \cos A \sin B)(\sin C \cos D + \cos C \sin D)\\ &= \cos A \cos B \cos C \cos D - \sin A \sin B \cos C \cos D\\ &\qquad - \cos A \cos B \sin C \sin D - \sin A \cos B \sin C \cos D\\ &\qquad - \sin A \cos B \cos C \sin D - \cos A \sin B \sin C \cos D\\ &\qquad - \cos A \sin B \cos C \sin D + \sin A \sin B \sin C \sin D.\end{aligned}$$

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Putting $A = B = C = D = \theta$ in this result,

$$
\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta \\
&= \cos^4\theta - 6\cos^2\theta\,(1 - \cos^2\theta) + (1 - \cos^2\theta)^2 \\
&= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + (1 - 2\cos^2\theta + \cos^4\theta) \\
&= 8\cos^4\theta - 8\cos^2\theta + 1.
\end{aligned}
$$

EXAMPLE (L.U.). If α, β, γ, δ be four angles, prove that

(i) $\sin\alpha\sin(\beta - \gamma) + \sin\beta\sin(\gamma - \alpha) + \sin\gamma\sin(\alpha - \beta) = 0;$

(ii) $\sin(\beta - \gamma)\sin(\alpha - \delta) + \sin(\gamma - \alpha)\sin(\beta - \delta) + \sin(\alpha - \beta)\sin(\gamma - \delta) = 0.$

(i)
$$
\begin{aligned}
\text{L.H.S.} &= \sin\alpha[\sin\beta\cos\gamma - \cos\beta\sin\gamma] \\
&\quad + \sin\beta[\sin\gamma\cos\alpha - \cos\gamma\sin\alpha] \\
&\quad + \sin\gamma[\sin\alpha\cos\beta - \cos\alpha\sin\beta] \\
&= \sin\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta\sin\gamma \\
&\quad + \cos\alpha\sin\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma \\
&\quad + \sin\alpha\cos\beta\sin\gamma - \cos\alpha\sin\beta\sin\gamma \\
&= 0.
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\text{L.H.S.} &= (\sin\beta\cos\gamma - \cos\beta\sin\gamma)(\sin\alpha\cos\delta - \cos\alpha\sin\delta) \\
&\quad + (\sin\gamma\cos\alpha - \cos\gamma\sin\alpha)(\sin\beta\cos\delta - \cos\beta\sin\delta) \\
&\quad + (\sin\alpha\cos\beta - \cos\alpha\sin\beta)(\sin\gamma\cos\delta - \cos\gamma\sin\delta) \\
&= \sin\alpha\sin\beta\cos\gamma\cos\delta - \sin\alpha\cos\beta\sin\gamma\cos\delta \\
&\quad - \cos\alpha\sin\beta\cos\gamma\sin\delta + \cos\alpha\cos\beta\sin\gamma\sin\delta \\
&\quad + \cos\alpha\sin\beta\sin\gamma\cos\delta - \sin\alpha\sin\beta\cos\gamma\cos\delta \\
&\quad - \cos\alpha\cos\beta\sin\gamma\sin\delta + \sin\alpha\cos\beta\cos\gamma\sin\delta \\
&\quad + \sin\alpha\cos\beta\sin\gamma\cos\delta - \sin\alpha\cos\beta\cos\alpha\sin\delta \\
&\quad - \cos\alpha\sin\beta\sin\gamma\cos\delta + \cos\alpha\sin\beta\cos\gamma\sin\delta \\
&= 0.
\end{aligned}
$$

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EXAMPLE (L.U.). Express $\tan(A + B)$ in terms of $\tan A$ and $\tan B$.

Prove that, if $A + B + C = 180°$,

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C.$$

Also, prove that

$$\tan\theta\tan(\theta + 60°) + \tan\theta\tan(\theta - 60°) + \tan(\theta + 60°)\tan(\theta - 60°) = -3.$$

Since $A + B + C = 180°$,

$$
\begin{aligned}
\tan C &= \tan(180° - A - B) \\
&= -\tan(A + B) \\
&= -\frac{(\tan A + \tan B)}{1 - \tan A\tan B}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ (1 - \tan A\tan B)\tan C &= -\tan A - \tan B \\
\text{i.e. } \tan C - \tan A\tan B\tan C &= -\tan A - \tan B \\
\text{i.e. } \tan A + \tan B + \tan C &= \tan A\tan B\tan C.
\end{aligned}
$$

Now

$$\tan 60° = \tan[(\theta + 60°) - \theta] = \frac{\tan(\theta + 60°) - \tan\theta}{1 + \tan\theta\tan(60° + \theta)}$$

i.e. $\sqrt{3}(1 + \tan\theta\tan\overline{60° + \theta}) = \tan(\theta + 60°) - \tan\theta$

$$\therefore\ 1 + \tan\theta\tan(60° + \theta) = \frac{1}{\sqrt{3}}[\tan(\theta + 60°) - \tan\theta] \dots . \quad (1).$$

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Similarly

$$1 + \tan\theta \tan(\theta - 60°) = -\frac{1}{\sqrt{3}}[\tan(\theta - 60°) - \tan\theta] \ldots\ldots (2).$$

Also

$$\tan 120° = \tan[(\theta + 60°) - (\theta - 60°)] = \frac{\tan(\theta + 60°) - \tan(\theta - 60°)}{1 + \tan(\theta + 60°)\tan(\theta - 60°)}.$$

Now $\tan 120° = -\sqrt{3}$

$$\therefore\ -\sqrt{3}[1 + \tan(\theta + 60°)\tan(\theta - 60°)] = \tan(\theta + 60°) - \tan(\theta - 60°)$$

i.e. $1 + \tan(\theta + 60°)\tan(\theta - 60°)$

$$= -\frac{1}{\sqrt{3}}[\tan(\theta + 60°) - \tan(\theta - 60°)] \ldots\ldots (3)$$

(1) + (2) + (3) gives

$$3 + \tan\theta\tan(\theta + 60°) + \tan\theta\tan(\theta - 60°) + \tan(\theta + 60°)\tan(\theta - 60°) = 0$$

$$\therefore\ \tan\theta\tan(\theta + 60°) + \tan\theta\tan(\theta - 60°) + \tan(\theta + 60°)\tan(\theta - 60°) = -3.$$

EXAMPLE (L.U.).

(i) Prove that

$$\sin^2 \tfrac{1}{2}(A + B)\cos^2 \tfrac{1}{2}(A - B) - \cos^2 \tfrac{1}{2}(A + B)\sin^2 \tfrac{1}{2}(A - B) = \sin A \sin B.$$

(ii) If the angles are positive and acute, prove that

$$\cos^{-1} 3/\sqrt{10} + \cos^{-1} 2/\sqrt{5} = \tfrac{1}{4}\pi$$

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(i) $\sin^2 \tfrac{1}{2}(A + B)\cos^2 \tfrac{1}{2}(A - B) - \cos^2 \tfrac{1}{2}(A + B)\sin^2 \tfrac{1}{2}(A - B)$

$$= [\sin \tfrac{1}{2}(A + B)\cos \tfrac{1}{2}(A - B) + \cos \tfrac{1}{2}(A + B)\sin \tfrac{1}{2}(A - B)] \times [\sin \tfrac{1}{2}(A + B)\cos \tfrac{1}{2}(A - B) - \cos \tfrac{1}{2}(A + B)\sin \tfrac{1}{2}(A - B)]$$

$$= \sin[\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B)]\sin[\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B)]$$

(using $\sin(x + y) = \sin x \cos y + \cos x \sin y$ and $\sin(x - y) = \sin x \cos y - \cos x \sin y$)

$= \sin A \sin B.$

(ii) Let $\theta_1 = \cos^{-1} 3/\sqrt{10}$ and $\theta_2 = \cos^{-1} 2/\sqrt{5}$, where θ_1 and θ_2 are positive and acute. Therefore $\cos\theta_1 = 3/\sqrt{10}$, $\cos\theta_2 = 2/\sqrt{5}$, and from these

$$\sin\theta_1 = \sqrt{(1 - \cos^2\theta_1)} = \sqrt{(1 - 9/10)} = 1/\sqrt{10},$$
$$\sin\theta_2 = \sqrt{(1 - \cos^2\theta_2)} = \sqrt{(1 - 4/5)} = 1/\sqrt{5}$$

($\sin\theta_1$ and $\sin\theta_2$ are positive since θ_1 and θ_2 are acute).

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$= \frac{3}{\sqrt{10}} \times \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{10}} \times \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{50}}$$
$$= 1/\sqrt{2}.$$

Therefore since θ_1 and θ_2 are both acute, $\theta_1 + \theta_2 = \tfrac{1}{4}\pi$.

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Trigonometric Functions of Multiple and Submultiple Angles. It has been proved that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \ldots\ldots\ldots (1)$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \ldots\ldots\ldots (2)$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let $x = y = \theta$ in each of these identities, then,

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\left.\begin{aligned} &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}\right\} \text{ (using } \sin^2\theta + \cos^2\theta = 1)$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

The last two results for cos 2θ should also be memorised in the following forms

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta).$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta).$$

If the angle θ be replaced by $\frac{1}{2}\varphi$ the following results are obtained:

$$\left.\begin{aligned}
\sin\varphi &= 2\sin\tfrac{1}{2}\varphi\cos\tfrac{1}{2}\varphi \\
\cos\varphi &= \cos^2\tfrac{1}{2}\varphi - \sin^2\tfrac{1}{2}\varphi \\
&= 2\cos^2\tfrac{1}{2}\varphi - 1 \\
&= 1 - 2\sin^2\tfrac{1}{2}\varphi \\
\tan\varphi &= \frac{2\tan\frac{1}{2}\varphi}{1 - \tan^2\frac{1}{2}\varphi} \\
\cos^2\tfrac{1}{2}\varphi &= \tfrac{1}{2}(1 + \cos\varphi) \\
\sin^2\tfrac{1}{2}\varphi &= \tfrac{1}{2}(1 - \cos\varphi)
\end{aligned}\right\}$$

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NOTE. The quadrant in which the angle θ lies will determine the sign *to be used* for cos θ and sin θ when *using* the formulae

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta) \text{ and } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta).$$

EXAMPLE. *Without using tables*, find the sine, cosine and tangent of $22\frac{1}{2}°$.

Using the previous formulae and the fact that sin $22\frac{1}{2}°$, cos $22\frac{1}{2}°$, tan $22\frac{1}{2}°$ are all positive,

$$\sin^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 - \cos 45°) = \tfrac{1}{2}(1 - 1/\sqrt{2})$$
$$= (1/2\sqrt{2})(\sqrt{2} - 1) = \tfrac{1}{4}(2 - \sqrt{2})$$
$$\therefore \sin 22\tfrac{1}{2}° = \sqrt{\tfrac{1}{4}(2 - \sqrt{2})} = \tfrac{1}{2}\sqrt{(2 - \sqrt{2})}$$
$$\cos^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 + \cos 45°) = \tfrac{1}{2}(1 + 1/\sqrt{2})$$
$$= \tfrac{1}{4}(2 + \sqrt{2})$$
$$\therefore \cos 22\tfrac{1}{2}° = \tfrac{1}{2}\sqrt{(2 + \sqrt{2})}.$$
$$\tan 22\tfrac{1}{2}° = \frac{\sin 22\frac{1}{2}°}{\cos 22\frac{1}{2}°} = \frac{\frac{1}{2}\sqrt{(2 - \sqrt{2})}}{\frac{1}{2}\sqrt{(2 + \sqrt{2})}} = \sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$$
$$= \sqrt{\frac{(2 - \sqrt{2})^2}{(2 + \sqrt{2})(2 - \sqrt{2})}} = \frac{2 - \sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1.$$

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Theorem. *To find sin 2θ and cos 2θ in terms of t = tan θ.*

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{1}{\cos^2\theta}} = \frac{2\tan\theta}{\sec^2\theta} = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$= \frac{2t}{1+t^2}.$$

$$\cos 2\theta = \frac{\cos^2\theta - \sin^2\theta}{1} = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta} \quad (\cos^2\theta + \sin^2\theta = 1)$$

$$= \frac{\dfrac{\cos^2\theta}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta}} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-t^2}{1+t^2}.$$

NOTE. If $t_1 = \tan \frac{1}{2}x$, by similar reasoning,

$$\sin x = \frac{2t_1}{1+t_1^2}, \quad \cos x = \frac{1-t_1^2}{1+t_1^2}.$$

Theorem. *To find sin 3x in terms of sin x, and cos 3x in terms of cos x.*

$$\begin{aligned} \sin 3x &= \sin(2x+x) = \sin 2x\cos x + \cos 2x \sin x \\ &= 2\sin x\cos x \times \cos x + (1 - 2\sin^2 x)\sin x \\ &= 2\sin x\cos^2 x + \sin x - 2\sin^3 x \\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x. \end{aligned}$$

$$\begin{aligned} \cos 3x &= \cos(2x+x) = \cos 2x\cos x - \sin 2x\sin x \\ &= (2\cos^2 x - 1)\cos x - 2\sin x\cos x \times \sin x \\ &= 2\cos^3 x - \cos x - 2\cos x\sin^2 x \\ &= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) \\ &= 4\cos^3 x - 3\cos x. \end{aligned}$$

EXAMPLE. Show that sin 3θ = cos 2θ when θ = 18°, and use the result to find sin 18° without using tables.

When θ = 18°, sin 3θ = sin 54° = cos 36° = cos 2θ.

Now
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
therefore when θ = 18°
$$\sin 3\theta = \cos 2\theta,$$
$$\text{i.e. } 3\sin\theta - 4\sin^3\theta = 1 - 2\sin^2\theta,$$
$$\text{i.e. } 4\sin^3\theta - 2\sin^2\theta - 3\sin\theta + 1 = 0 \quad \ldots\ldots(1).$$

Now sin θ = 1 satisfies this equation therefore (sin θ − 1) is a factor of the L.H.S. of (1) and dividing the L.H.S. by sin θ − 1 the remaining factor is

$$4\sin^2\theta + 2\sin\theta - 1 = 0 \quad \ldots\ldots(2)$$

Clearly, since θ = 18°, sin θ ≠ 1, therefore θ satisfies (2).

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Solving (2) as a quadratic in $\sin\theta$,

$$\sin\theta = \frac{-1 \pm \sqrt{1+4}}{4} = \frac{-1 \pm \sqrt{5}}{4}.$$

But $\sin 18^\circ$ is positive, therefore only the positive sign can be used

$$\text{i.e. } \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

EXAMPLE (L.U.). Prove that:

(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$,

(ii) $\dfrac{\cos 3\theta}{\cos\theta} - \dfrac{\cos 6\theta}{\cos 2\theta} = 2(\cos 2\theta - \cos 4\theta)$

(iii) $\tan\alpha + \tan(60^\circ + \alpha) + \tan(120^\circ + \alpha) = 3\tan 3\alpha$.

(i) This has been proved as a theorem.

(ii) From (i), $\dfrac{\cos 3\theta}{\cos\theta} = 4\cos^2\theta - 3$ ....(1),

and $\dfrac{\cos 6\theta}{\cos 2\theta} = 4\cos^2 2\theta - 3$ (replacing θ by 2θ in (1)),

$$\therefore \frac{\cos 3\theta}{\cos\theta} - \frac{\cos 6\theta}{\cos 2\theta} = (4\cos^2\theta - 3) - (4\cos^2 2\theta - 3)$$

$$= 4\cos^2\theta - 4\cos^2 2\theta$$

$$= 2(1 + \cos 2\theta) - 2(1 + \cos 4\theta)$$

(using $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$)

$$= 2(\cos 2\theta - \cos 4\theta).$$

(iii) $\tan\alpha + \tan(60^\circ + \alpha) + \tan(120^\circ + \alpha)$

$$= \tan\alpha + \frac{\tan 60^\circ + \tan\alpha}{1 - \tan 60^\circ \tan\alpha} + \frac{\tan 120^\circ + \tan\alpha}{1 - \tan 120^\circ \tan\alpha}$$

$$= \tan\alpha + \frac{\tan\alpha + \sqrt{3}}{1 - \sqrt{3}\tan\alpha} + \frac{\tan\alpha - \sqrt{3}}{1 + \sqrt{3}\tan\alpha}$$

($\tan 60^\circ = \sqrt{3} = -\tan 120^\circ$)

$$= \frac{\tan\alpha(1 - 3\tan^2\alpha) + (\tan\alpha + \sqrt{3})(1 + \sqrt{3}\tan\alpha) + (\tan\alpha - \sqrt{3})(1 - \sqrt{3}\tan\alpha)}{1 - 3\tan^2\alpha}$$

$$= \frac{\tan\alpha - 3\tan^3\alpha + (4\tan\alpha + \sqrt{3}\tan^2\alpha + \sqrt{3}) + (4\tan\alpha - \sqrt{3}\tan^2\alpha - \sqrt{3})}{1 - 3\tan^2\alpha}$$

$$= \frac{9\tan\alpha - 3\tan^3\alpha}{1 - 3\tan^2\alpha}$$

Now $\tan 3\alpha = \dfrac{\tan 2\alpha + \tan\alpha}{1 - \tan\alpha\tan 2\alpha} = \dfrac{(2\tan\alpha)/(1 - \tan^2\alpha) + \tan\alpha}{1 - \tan\alpha\,.\,(2\tan\alpha)/(1 - \tan^2\alpha)}$

$$= \frac{2\tan\alpha + \tan\alpha(1 - \tan^2\alpha)}{(1 - \tan^2\alpha) - 2\tan^2\alpha} = \frac{3\tan\alpha - \tan^3\alpha}{1 - 3\tan^2\alpha}$$

$$\therefore \tan\alpha + \tan(60^\circ + \alpha) + \tan(120^\circ + \alpha) = 3\tan 3\alpha.$$

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EXAMPLE. Solve the following equations:

$$\text{(i)}\ \cos^2 x + \cos 2x = 2\cos x$$
$$\text{(ii)}\ \tan 2\varphi + 6\cot\varphi = 0$$

(i) Using $\cos 2x = 2\cos^2 x - 1$ the equation becomes

$$\cos^2 x + 2\cos^2 x - 1 = 2\cos x$$
$$\text{i.e.}\ 3\cos^2 x - 2\cos x - 1 = 0$$
$$\therefore (3\cos x + 1)(\cos x - 1) = 0$$
$$\therefore \cos x = -\tfrac{1}{3} \text{ or } 1.$$

When $\cos x = -0{\cdot}33333$, $x = m\ .\ 360° \pm 109°\ 28'$.
When $\cos x = 1$, $x = n\ .\ 360°$. (m, n are integers.)

(ii) Using $\tan 2\varphi = \dfrac{2\tan\varphi}{1 - \tan^2\varphi}$ the equation becomes

$$\frac{2\tan\varphi}{1-\tan^2\varphi} + \frac{6}{\tan\varphi} = 0,$$
$$\text{i.e. } 2\tan^2\varphi + 6(1 - \tan^2\varphi) = 0.$$
$$\text{i.e. } 4\tan^2\varphi = 6,\ \therefore \tan\varphi = \pm\sqrt{1{\cdot}5} = \pm 1{\cdot}2247,$$
$$\therefore \varphi = n\ .\ 180° \pm 50°\ 46' \quad (n \text{ any integer}).$$

Theorem. *To find the values of R and α when $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$, and R is positive with $0 \leqslant \alpha \leqslant 360°$.*

$$a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$$
$$\equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

Using $\theta = 0°$ $\quad a = R\cos\alpha$ .... (1)

$\theta = 90°$ $\quad b = R\sin\alpha$ .... (2)

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$(1)^2 + (2)^2$ gives, $\quad a^2 + b^2 = R^2(\cos^2\alpha + \sin^2\alpha) = R^2$

$$\therefore R = \sqrt{a^2 + b^2}$$

(2) ÷ (1) gives, $\quad \tan\alpha = b/a$

NOTE. The quadrant in which α lies is determined by the equations (1) and (2) with R positive.

By using a similar method the values of R and α can be determined in the following cases:

$$\text{(i)}\ a\cos\theta + b\sin\theta \equiv R\cos(\theta + \alpha)$$
$$\text{(ii)}\ a\cos\theta + b\sin\theta \equiv R\sin(\theta + \alpha)$$
$$\text{(iii)}\ a\cos\theta + b\sin\theta \equiv R\sin(\theta - \alpha).$$

In all these cases $R = \sqrt{a^2 + b^2}$, but the value of α will vary.

Since $a\cos\theta + b\sin\theta \equiv \sqrt{a^2 + b^2}\cos(\theta - \alpha)$, and the greatest value of $\cos x$ is unity and its least value is -1, it follows that the greatest value of $a\cos\theta + b\sin\theta$ is $\sqrt{(a^2 + b^2)}$ and its least value is $-\sqrt{(a^2 + b^2)}$.

If it is required to trace the graph of $y = a\sin\theta + b\cos\theta$, it is advisable to convert the expression into the form $\sqrt{(a^2 + b^2)}\ .\ \cos(\theta - \alpha)$, where α must be determined, and the graph is then seen to be an ordinary cosine curve with the origin moved along the θ axis through a distance α.

Theorem. *To solve the equation $a\cos x + b\sin x = c$, where a, b, c are constants.*

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Method (*i*). Let $a\cos x + b\sin x \equiv R\cos(x - \alpha)$, where R is positive and $0 \leqslant \alpha \leqslant 360°$. (Could use $R\sin(x + \alpha)$, etc.)

$$\therefore a\cos x + b\sin x \equiv R\cos x\cos\alpha + R\sin x\sin\alpha.$$

Using $x = 0°$, $\quad a = R\cos\alpha \ldots\ldots\ldots\ldots\ldots\ldots (1).$

Using $x = 90°$, $\quad b = R\sin\alpha \ldots\ldots\ldots\ldots\ldots\ldots (2).$

$(1)^2 + (2)^2$ gives, $\quad a^2 + b^2 = R^2(\cos^2\alpha + \sin^2\alpha) = R^2$

$$\therefore R = \sqrt{(a^2 + b^2)}.$$

$(2) \div (1)$ gives $\tan\alpha = b/a$, and the quadrant in which α lies is determined from (1) and (2).

With these values of R and α the given equation becomes,

$$R\cos(x - \alpha) = c,$$

$$\text{i.e. } \cos(x - \alpha) = \frac{c}{\sqrt{(a^2 + b^2)}}.$$

If $$\cos^{-1}\frac{c}{\sqrt{(a^2 + b^2)}} = \beta,$$

$$\text{then } x - \alpha = n \,.\, 360° \pm \beta,$$

$$\therefore x = n \,.\, 360° \pm \beta + \alpha,$$

where n is any integer.

Method (*ii*). It has been shown that

$$\sin x = \frac{2t}{1 + t^2}, \quad \cos x = \frac{1 - t^2}{1 + t^2}, \text{ where } t = \tan\tfrac{1}{2}x.$$

Substituting these in the given equation it becomes

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$$a\frac{(1 - t^2)}{1 + t^2} + \frac{2bt}{1 + t^2} = c,$$

$$\text{i.e. } a(1 - t^2) + 2bt = c(1 + t^2),$$

$$\text{i.e. } (a + c)t^2 - 2bt + (c - a) = 0.$$

This is a quadratic equation in t, and from this can be obtained two values of t each giving rise to a set of values for x.

EXAMPLE (L.U.). If $t = \tan\frac{1}{2}\theta$, show that $\cos\theta = (1 - t^2)/(1 + t^2)$ and $\sin\theta = 2t/(1 + t^2)$.

Hence, and otherwise, find all the angles between 0° and 360° which satisfy the equation $52\cos\theta + 39\sin\theta = 60$.

The first part of the question has been proved as a theorem.

Method (*i*). Using the values of $\cos\theta$ and $\sin\theta$ in terms of t, the equation becomes

$$\frac{52(1 - t^2)}{1 + t^2} + 39 \times \frac{2t}{1 + t^2} = 60$$

$$\text{i.e. } 52(1 - t^2) + 78t = 60(1 + t^2)$$

$$\therefore 112t^2 - 78t + 8 = 0$$

$$\text{i.e. } 56t^2 - 39t + 4 = 0$$

$$\therefore (7t - 4)(8t - 1) = 0$$

$$\therefore t = \tan\tfrac{1}{2}\theta = 4/7 = 0{\cdot}57143 \ldots\ldots\ldots\ldots (1)$$

$$\text{or } t = \tan\tfrac{1}{2}\theta = \tfrac{1}{8} = 0{\cdot}125 \ldots\ldots\ldots\ldots (2)$$

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From (1), $\frac{1}{2}\theta = n \,.\, 180^\circ + 29^\circ\, 45'$ $\therefore$ $\theta = n \,.\, 360^\circ + 59^\circ\, 30'$.

From (2), $\frac{1}{2}\theta = m \,.\, 180^\circ + 7^\circ\, 7\frac{1}{2}'$ $\therefore$ $\theta = m \,.\, 360^\circ + 14^\circ\, 15'$. ($m$ and n are integers.)

Clearly, values of θ between 0° and 360° are only obtained for $m = 0$ and $n = 0$, and these are $\theta = 14^\circ\, 15'$ or $59^\circ\, 30'$.

Method (*ii*). Let $52 \cos\theta + 39 \sin\theta \equiv R \sin(\theta + \alpha)$, where R is positive and $0 \leqslant \theta \leqslant 360^\circ$.

Therefore $52 \cos\theta + 39 \sin\theta \equiv R \sin\theta \cos\alpha + R \cos\theta \sin\alpha$.

When $\theta = 0$, $\quad 52 = R \sin\alpha$ ....(1).

When $\theta = 90^\circ$, $\quad 39 = R \cos\alpha$ ....(2).

$(1)^2 + (2)^2$ gives, $R^2 = 52^2 + 39^2 = 13^2(4^2 + 3^2) = 13^2 \times 25$,

$$\therefore\ R = 65.$$

From (1) and (2), α is in the first quadrant and

$$\tan\alpha = \frac{52}{39} = \frac{4}{3} = 1{\cdot}33333,$$

$$\therefore\ \alpha = 53^\circ\, 8'.$$

With these values the given equation can be written

$$65 \sin(\theta + 53^\circ\, 8') = 60$$

$$\therefore\ \sin(\theta + 53^\circ\, 8') = \frac{60}{65} = \frac{12}{13} = 0{\cdot}92308$$

$$\therefore\ \theta + 53^\circ\, 8' = n \,.\, 180^\circ + (-1)^n \,.\, 67^\circ\, 23' \quad (n \text{ any integer})$$

$$\therefore\ \theta = n \,.\, 180^\circ + (-1)^n\, 67^\circ\, 23' - 53^\circ\, 8'.$$

The only values of θ between 0° and 360° are given by $n = 0$ and $n = 1$.

When $n = 0$, $\quad \theta = 67^\circ\, 23' - 53^\circ\, 8' = 14^\circ\, 15'$,

and when $n = 1$, $\theta = 180^\circ - 67^\circ\, 23' - 53^\circ\, 8' = 59^\circ\, 29'$.

NOTE. The slight discrepancy in the results is due to the use of tables.

EXAMPLE (L.U.). (i) If $\tan\theta = 4/3$, and if $0^\circ < \theta < 360^\circ$, find, without tables, the possible values of $\tan\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$. (ii) Solve the equation $10 \sin^2\frac{1}{2}\theta - 5 \sin\theta = 4$, giving values of θ between 0° and 360°.

(i)
$$\tan\theta = \frac{2 \tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta} = \frac{4}{3},$$

$$\therefore\ 6 \tan\tfrac{1}{2}\theta = 4 - 4 \tan^2\tfrac{1}{2}\theta,$$

$$\text{i.e. } 2 \tan^2\tfrac{1}{2}\theta + 3 \tan\tfrac{1}{2}\theta - 2 = 0,$$

$$\therefore\ (2 \tan\tfrac{1}{2}\theta - 1)(\tan\tfrac{1}{2}\theta + 2) = 0, \ \therefore\ \tan\tfrac{1}{2}\theta = \tfrac{1}{2} \text{ or } -2.$$

Both values are permissible since $\frac{1}{2}\theta$ lies between 0° and 180°, and hence $\tan\frac{1}{2}\theta$ can be positive or negative.

Now $\operatorname{cosec}^2\frac{1}{2}\theta = 1 + \cot^2\frac{1}{2}\theta$, i.e. $\dfrac{1}{\sin^2\frac{1}{2}\theta} = 1 + \dfrac{1}{\tan^2\frac{1}{2}\theta}$

When $\tan\frac{1}{2}\theta = \frac{1}{2}$, $\dfrac{1}{\sin^2\frac{1}{2}\theta} = 1 + 4 = 5$ $\therefore$ $\sin^2\frac{1}{2}\theta = \dfrac{1}{5}$

$$\therefore\ \sin\tfrac{1}{2}\theta = \pm\frac{1}{\sqrt{5}} = \pm\frac{\sqrt{5}}{5}$$

Now $0^\circ < \frac{1}{2}\theta < 180^\circ$ $\therefore$ $\sin\frac{1}{2}\theta$ is positive $\therefore$ $\sin\frac{1}{2}\theta = \sqrt{5}/5$.

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When $\tan \frac{1}{2}\theta = -2$, $\frac{1}{\sin^2 \frac{1}{2}\theta} = 1 + \frac{1}{4} = \frac{5}{4}$

$\therefore \sin^2 \frac{1}{2}\theta = 4/5$, and as before, $\sin \frac{1}{2}\theta = 2\sqrt{5}/5$.

(ii) $\sin^2 \frac{1}{2}\theta = \frac{1}{2}(1 - \cos\theta)$ therefore the equation can be written

$$5(1 - \cos\theta) - 5\sin\theta = 4$$
$$\text{i.e. } 5\cos\theta + 5\sin\theta = 1$$
$$\therefore \cos\theta + \sin\theta = \tfrac{1}{5} \quad \ldots\ldots (1)$$

Let $\cos\theta + \sin\theta \equiv R\cos(\theta - \alpha)$, where $0 < \alpha < 360°$ and R is positive. Therefore $\cos\theta + \sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$.

Using $\theta = 0$, $\quad 1 = R\cos\alpha \ldots\ldots (2)$

and $\theta = 90°$, $\quad 1 = R\sin\alpha \ldots\ldots (3)$.

$(2)^2 + (3)^2$ gives, $\quad 2 = R^2, \therefore R = \sqrt{2}$.

From (2) and (3), α lies in the first quadrant and $\tan\alpha = 1$,

$$\therefore \alpha = 45°.$$

Using these in (1) the equation becomes

$$\sqrt{2}\cos(\theta - 45°) = \tfrac{1}{5}, \; \therefore \cos(\theta - 45°) = \sqrt{2}/10 = 0{\cdot}14142.$$

$\therefore \theta - 45° = n \,.\, 360° \pm 81°\,52'$, $\therefore \theta = n \,.\, 360° \pm 81°\,52' + 45°$.

Tan values of θ between 0° and 360° are given by

$n = 0$ with + sign, i.e. $\theta = 81°\,52' + 45° = 126°\,52'$

$n = 1$ with − ve sign, i.e. $\theta = 360° - 81°\,52' + 45° = 323°\,8'$.

EXAMPLE (L.U.). If $\tan 2x + \tan 2y = 0$, prove that $x + y$ is an integral multiple of $\frac{1}{2}\pi$.

Find all pairs of angles x, y which lie between 0° and 180°, and satisfy the simultaneous equations

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$$\tan x + \tan y + 3 = 0, \quad \tan 2x + \tan 2y = 0.$$

$$\tan 2x + \tan 2y = 0, \; \therefore \frac{\sin 2x}{\cos 2x} + \frac{\sin 2y}{\cos 2y} = 0,$$

$\therefore \sin 2x\cos 2y + \cos 2x\sin 2y = 0$, (multiply through by $\cos 2x\cos 2y$)

i.e. $\sin(2x + 2y) = 0$,

$\therefore 2x + 2y = n\pi$, where n is any integer,

$\therefore x + y = n \,.\, \frac{1}{2}\pi$.

Since $\tan 2x + \tan 2y = 0$, by part (i) $x + y = \frac{1}{2}n\pi$. But, both x and y lie between 0° and 180°

$$\therefore 0 < (x + y) < 360°$$

$$\therefore x + y = 1 \,.\, \tfrac{1}{2}\pi, \; 2 \,.\, \tfrac{1}{2}\pi \text{ or } 3 \,.\, \tfrac{1}{2}\pi = 90°, 180°, \text{ or } 270°.$$

Now $\quad \tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

Since $\quad \tan x + \tan y = -3 \ldots\ldots (1)$

$$\tan(x + y) = \frac{-3}{1 - \tan x\tan y} \ldots\ldots (2)$$

Also since $\tan x + \tan y$ is negative, $x + y \neq 90°$.

When $x + y = 270°$, $\tan(x + y) = \infty \quad \therefore \quad 1 - \tan x\tan y = 0$ (from (2))

i.e. $\tan x\tan y = 1 \ldots\ldots (3)$.

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When $x + y = 180°$, $\tan(x + y) = 0$, which is impossible unless $1 - \tan x \tan y = \infty$ giving $\tan x$ and $\tan y$ each infinite and therefore $x = y = 90°$ which does not satisfy (1), therefore $x + y \neq 180°$.

Substituting from (3) in (1) for $\tan y$,

$$\tan x + \frac{1}{\tan x} = -3$$

$$\begin{aligned} \text{i.e. } \tan^2 x + 3\tan x + 1 &= 0 \\ \therefore \tan x &= \tfrac{1}{2}(-3 \pm \sqrt{5}) = \tfrac{1}{2}(-3 \pm 2{\cdot}2361) \\ &= -0{\cdot}3820 \text{ or } -2{\cdot}6181 \\ \therefore x &= 180° - 20°\,54' = 159°\,6' \\ \text{or } x &= 180° - 69°\,6' = 110°\,54'. \end{aligned}$$

Since $x + y = 270°$ the corresponding values of y are $110°\,54'$ and $159°\,6'$, therefore required solutions are $\left.\begin{array}{l} x = 159°\,6' \\ y = 110°\,54' \end{array}\right\}$, $\left.\begin{array}{l} 110°\,54' \\ 159°\,6' \end{array}\right\}$

The Addition and Subtraction Theorems. These deal with the sum and difference of two sines and also of two cosines.

It has been shown that:

$$\begin{aligned} \sin(A + B) &= \sin A \cos B + \cos A \sin B \ldots\ldots\ldots(1), \\ \sin(A - B) &= \sin A \cos B - \cos A \sin B \ldots\ldots\ldots(2), \\ \cos(A + B) &= \cos A \cos B - \sin A \sin B \ldots\ldots\ldots(3), \\ \cos(A - B) &= \cos A \cos B + \sin A \sin B \ldots\ldots\ldots(4). \end{aligned}$$

(1) + (2) gives, $\sin(A + B) + \sin(A - B) = 2\sin A \cos B \ldots(5)$.
(1) − (2) gives, $\sin(A + B) - \sin(A - B) = 2\cos A \sin B \ldots(6)$.
(3) + (4) gives, $\cos(A + B) + \cos(A - B) = 2\cos A \cos B \ldots(7)$.
(4) − (3) gives, $\cos(A - B) - \cos(A + B) = 2\sin A \sin B \ldots(8)$.

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Let $A + B = x$, $A - B = y$, $\therefore A = \tfrac{1}{2}(x + y)$, $B = \tfrac{1}{2}(x - y)$, and the results (5), (6), (7), (8) become,

$$\left.\begin{aligned} \sin x + \sin y &= 2\sin\tfrac{1}{2}(x + y)\cos\tfrac{1}{2}(x - y). \\ \sin x - \sin y &= 2\cos\tfrac{1}{2}(x + y)\sin\tfrac{1}{2}(x - y). \\ \cos x + \cos y &= 2\cos\tfrac{1}{2}(x + y)\cos\tfrac{1}{2}(x - y). \\ \cos y - \cos x &= 2\sin\tfrac{1}{2}(x + y)\sin\tfrac{1}{2}(x - y). \end{aligned}\right\}.$$

It is extremely important that these results should be memorised and the peculiarity in the result $\cos y - \cos x$ be noted, and in addition the results (5), (6), (7) and (8), which are also very important, should be memorised in the following reversed forms:

$$\left.\begin{aligned} 2\sin A \cos B &= \sin(A + B) + \sin(A - B) \\ 2\cos A \sin B &= \sin(A + B) - \sin(A - B) \\ 2\cos A \cos B &= \cos(A + B) + \cos(A - B) \\ 2\sin A \sin B &= \cos(A - B) - \cos(A + B) \end{aligned}\right\}.$$

EXAMPLE. Find in the product form:

(i) $\cos 2\theta + \cos 3\theta$, (ii) $\cos\theta - \cos 4\theta$,
(iii) $\sin n\phi + \sin n\theta$, (iv) $\sin 2\phi - \sin 4\phi$,
(v) $\sin(\alpha + \beta) + \sin(\alpha - \beta)$, (vi) $\cos(x + 2y) - \cos y$.

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(i) $\cos 2\theta + \cos 3\theta = 2\cos\frac{2\theta + 3\theta}{2}\cos\frac{3\theta - 2\theta}{2} = 2\cos\frac{5\theta}{2}\cos\frac{\theta}{2}.$

(ii) $\cos\theta - \cos 4\theta = 2\sin\frac{\theta + 4\theta}{2}\sin\frac{4\theta - \theta}{2} = 2\sin\frac{5\theta}{2}\sin\frac{3\theta}{2}.$

(iii) $\sin n\varphi + \sin n\theta = 2\sin\frac{n\varphi + n\theta}{2}\cos\frac{n\varphi - n\theta}{2}$

$$= 2\sin\frac{n}{2}(\theta + \varphi)\cos\frac{n}{2}(\varphi - \theta).$$

(iv) $\sin 2\varphi - \sin 4\varphi = -(\sin 4\varphi - \sin 2\varphi)$

$$= -2\cos\frac{4\varphi + 2\varphi}{2}\sin\frac{4\varphi - 2\varphi}{2} = -2\cos 3\varphi\sin\varphi.$$

(v) $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

$$= 2\sin\frac{(\alpha + \beta) + (\alpha - \beta)}{2}\cos\frac{(\alpha + \beta) - (\alpha - \beta)}{2} = 2\sin\alpha\cos\beta.$$

(vi) $\cos(x + 2y) - \cos y = -[\cos y - \cos(x + 2y)]$

(largest angle in first cosine, ∴ take out −ve sign)

$$= -2\sin\frac{y + x + 2y}{2}\sin\frac{(x + 2y) - y}{2}$$

$$= -2\sin\frac{x + 3y}{2}\sin\frac{x + y}{2}.$$

EXAMPLE. Express the following products as the sum or difference of two sines or two cosines:

(i) $2\sin\theta\cos 3\theta$, (ii) $2\cos x\cos 3x$,
(iii) $2\sin 4\theta\cos\theta$, (iv) $\sin(A + B)\sin B$,
(v) $\cos(n\theta + \alpha)\cos(n\theta - \alpha)$, (vi) $\cos(x + 2y)\sin y$.

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(i) $2\sin\theta\cos 3\theta = 2\cos 3\theta\sin\theta = \sin(3\theta + \theta) - \sin(3\theta - \theta)$
$= \sin 4\theta - \sin 2\theta.$

(ii) $2\cos x\cos 3x = 2\cos 3x\cos x = \cos(3x + x) + \cos(3x - x)$
$= \cos 4x + \cos 2x.$

(iii) $2\sin 4\theta\cos\theta = \sin(4\theta + \theta) + \sin(4\theta - \theta) = \sin 5\theta + \sin 3\theta.$

(iv) $\sin(A + B)\sin B = \frac{1}{2}\{\cos[(A + B) - B] - \cos[(A + B) + B]\}$
$= \frac{1}{2}[\cos A - \cos(A + 2B)].$

(v) $\cos(n\theta + \alpha)\cos(n\theta - \alpha) = \frac{1}{2}\{\cos[(n\theta + \alpha) + (n\theta - \alpha)]$
$+ \cos[(n\theta + \alpha) - (n\theta - \alpha)]\}$
$= \frac{1}{2}[\cos 2n\theta + \cos 2\alpha].$

(vi) $\cos(x + 2y)\sin y = \frac{1}{2}\{\sin[(x + 2y) + y] - \sin[(x + 2y) - y]\}$
$= \frac{1}{2}[\sin(x + 3y) - \sin(x + y)].$

EXAMPLE. Prove that

$$\frac{\sin 7x - \sin 3x - \sin 5x + \sin x}{\cos 7x + \cos 3x - \cos 5x - \cos x} = \tan 2x.$$

$$\text{L.H.S.} = \frac{(\sin 7x - \sin 3x) - (\sin 5x - \sin x)}{(\cos 7x + \cos 3x) - (\cos 5x + \cos x)}$$

$$= \frac{2\cos 5x\sin 2x - 2\cos 3x\sin 2x}{2\cos 5x\cos 2x - 2\cos 3x\cos 2x}$$

$$= \frac{2\sin 2x(\cos 5x - \cos 3x)}{2\cos 2x(\cos 5x - \cos 3x)} = \frac{\sin 2x}{\cos 2x} = \tan 2x.$$

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Angles of a Triangle. In problems on the angles of a triangle it is known that the sum of the angles is 180°, and this is made use of together with the following trigonometric identities:

(i) $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$,
(ii) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$,
(iii) $\sin x + \sin y = 2\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y)$,
(iv) $\sin x - \sin y = 2\cos\frac{1}{2}(x + y)\sin\frac{1}{2}(x - y)$,
(v) $\cos x + \cos y = 2\cos\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y)$,
(vi) $\cos x - \cos y = 2\sin\frac{1}{2}(x + y)\sin\frac{1}{2}(y - x)$.

EXAMPLE (L.U.). (i) Prove that, if $A + B + C = 180°$ then

$$\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

(ii) Show that, if $(1 + \sin\theta)\cot\theta = 4a$, $(1 - \sin\theta)\cot\theta = 4b$, then

$$ab = (a^2 - b^2)^2.$$

(i)
$$\frac{(\sin 2A + \sin 2B) + \sin 2C}{(\sin A + \sin B) + \sin C}$$

$$= \frac{2\sin(A + B)\cos(A - B) + \sin 2C}{2\sin\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B) + \sin C}$$

$$= \frac{2\sin(180° - C)\cos(A - B) + 2\sin C\cos C}{2\sin(90° - \frac{1}{2}C)\cos\frac{1}{2}(A - B) + 2\sin\frac{1}{2}C\cos\frac{1}{2}C}$$

$$= \frac{\sin C\cos(A - B) + \sin C\cos(180° - A - B)}{\cos\frac{1}{2}C\cos\frac{1}{2}(A - B) + \cos\frac{1}{2}C\sin[90° - \frac{1}{2}(A + B)]}$$

$$= \frac{\sin C[\cos(A - B) - \cos(A + B)]}{\cos\frac{1}{2}C[\cos\frac{1}{2}(A - B) + \cos\frac{1}{2}(A + B)]}$$

$$= \frac{\sin C \times 2\sin A\sin B}{\cos\frac{1}{2}C \times 2\cos\frac{1}{2}A\cos\frac{1}{2}B}$$

$$= \frac{2\sin\frac{1}{2}C\cos\frac{1}{2}C \times 2\sin\frac{1}{2}A\cos\frac{1}{2}A \times 2\sin\frac{1}{2}B\cos\frac{1}{2}B}{\cos\frac{1}{2}C\cos\frac{1}{2}A\cos\frac{1}{2}B}$$

$$= 8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

(ii)
$$a = \tfrac{1}{4}(1 + \sin\theta)\cot\theta, \quad b = \tfrac{1}{4}(1 - \sin\theta)\cot\theta$$

$$\therefore ab = \tfrac{1}{16}(1 - \sin^2\theta)\cot^2\theta = \tfrac{1}{16}\cos^2\theta \times \cot^2\theta$$

$$= \tfrac{1}{16}\cos^4\theta/\sin^2\theta.$$

$$a^2 - b^2 = \tfrac{1}{16}(1 + \sin\theta)^2\cot^2\theta - \tfrac{1}{16}(1 - \sin\theta)^2\cot^2\theta$$

$$= \frac{\cot^2\theta}{16}[(1 + 2\sin\theta + \sin^2\theta) - (1 - 2\sin\theta + \sin^2\theta)]$$

$$= \frac{4\cot^2\theta\sin\theta}{16} = \frac{1}{4}\cdot\frac{\cos^2\theta}{\sin\theta}.$$

$$\therefore (a^2 - b^2)^2 = \frac{1}{16}\cdot\frac{\cos^4\theta}{\sin^2\theta}$$

$$\therefore ab = (a^2 - b^2)^2.$$

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EXAMPLE (L.U.). Prove that,

(i) $\tan\alpha - 2\tan(\alpha + \frac{1}{4}\pi) + \tan(\alpha + \frac{1}{2}\pi) = \dfrac{(\tan\alpha + 1)(\tan^2\alpha + 1)}{\tan\alpha(\tan\alpha - 1)}$

(ii) In any triangle ABC, $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$.

(i) $\tan\alpha - 2\tan(\alpha + \frac{1}{4}\pi) + \tan(\alpha + \frac{1}{2}\pi)$

$$= \tan\alpha - \frac{2(\tan\alpha + \tan\frac{1}{4}\pi)}{1 - \tan\alpha\tan\frac{1}{4}\pi} - \cot\alpha$$

$$= \tan\alpha - \frac{2(\tan\alpha + 1)}{1 - \tan\alpha} - \frac{1}{\tan\alpha}$$

$$= \frac{\tan^2\alpha(1 - \tan\alpha) - 2\tan\alpha(\tan\alpha + 1) - (1 - \tan\alpha)}{\tan\alpha(1 - \tan\alpha)}$$

$$= \frac{\tan^2\alpha - \tan^3\alpha - 2\tan^2\alpha - 2\tan\alpha - 1 + \tan\alpha}{\tan\alpha(1 - \tan\alpha)}$$

$$= -\frac{(\tan^3\alpha + \tan^2\alpha + \tan\alpha + 1)}{\tan\alpha(1 - \tan\alpha)}$$

$$= \frac{[\tan^2\alpha(1 + \tan\alpha) + (1 + \tan\alpha)]}{\tan\alpha(\tan\alpha - 1)} = \frac{(\tan\alpha + 1)(\tan^2\alpha + 1)}{\tan\alpha(\tan\alpha - 1)}.$$

(ii) $\sin^2 A + \sin^2 B - \sin^2 C$

$$\begin{aligned}
&= \tfrac{1}{2}(1 - \cos 2A) + \tfrac{1}{2}(1 - \cos 2B) - \sin^2 C \\
&= 1 - \tfrac{1}{2}\cos 2A - \tfrac{1}{2}\cos 2B - \sin^2 C \\
&= (1 - \sin^2 C) - \tfrac{1}{2}(\cos 2A + \cos 2B) \\
&= \cos^2 C - \cos(A + B)\cos(A - B) \\
&= \cos C\cos(180° - A - B) - \cos(180° - C)\cos(A - B) \\
&= -\cos C\cos(A + B) + \cos C\cos(A - B) \\
&= \cos C[\cos(A - B) - \cos(A + B)] \\
&= \cos C \times 2\sin A\sin B = 2\sin A\sin B\cos C.
\end{aligned}$$

NOTE The addition and subtraction theorems can also be used to find the solutions of trigonometric equations and proving identities involving three or more sines or cosines, as shown in the following example. The method is usually to group the sines or cosines in pairs, usually one pair consisting of those with the largest and smallest angles. ($1 = \cos 0°$.)

EXAMPLE (L.U.). Prove that $\cos A + \cos B = 2\cos\frac{1}{2}(A + B)\cos\frac{1}{2}(A - B)$. Find all the values of θ between 0 and 2π which satisfy the equation $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.

The first part of the question is bookwork.

The given equation can be written

$$\begin{aligned}
(\cos 6\theta + 1) + (\cos 4\theta + \cos 2\theta) &= 0 \\
\text{i.e. } 2\cos^2 3\theta + 2\cos 3\theta\cos\theta &= 0 \\
\text{i.e. } 2\cos 3\theta(\cos 3\theta + \cos\theta) &= 0 \\
\therefore\ 2\cos 3\theta \times 2\cos 2\theta\cos\theta &= 0
\end{aligned}$$

$$\therefore\ \cos\theta = 0,\quad \cos 2\theta = 0,\quad \text{or } \cos 3\theta = 0.$$

From $\cos\theta = 0$, $\theta = 2m\pi \pm \frac{1}{2}\pi$.
From $\cos 2\theta = 0$, $2\theta = 2n\pi \pm \frac{1}{2}\pi$ $\therefore\ \theta = n\pi \pm \frac{1}{4}\pi$.
From $\cos 3\theta = 0$, $3\theta = 2p\pi \pm \frac{1}{2}\pi$ $\therefore\ \theta = \frac{2}{3}p\pi \pm \frac{1}{6}\pi$.
(m, n, p are integers.)

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Using $m = 0, 1$; $n = 0, 1, 2$; $p = 0, 1, 2, 3$; the following values of θ between 0 and 2π are obtained:

$$m = 0, 1, \qquad \theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$n = 0, 1, 2, \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

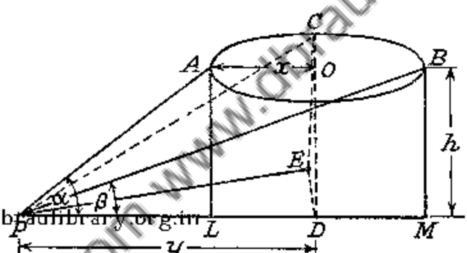
$$p = 0, 1, 2, 3, \quad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

Hence, required values of θ are

$$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \frac{11\pi}{6}.$$

We conclude the chapter with examples involving compound angles.

EXAMPLE (L.U.). An aeroplane is describing a horizontal circle of radius x at a height h; a man is standing on the ground at a distance y from the projection of the centre of the circular path on the ground, where $y > x$;



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and A, B denote the points of the path which are nearest to and furthest from the man. The man notices that the angles of elevation of the aeroplane when it is at A and at B are α, β respectively. Prove that

$$x : y : h = \sin(\alpha - \beta) : \sin(\alpha + \beta) : 2 \sin\alpha \sin\beta.$$

Taking $\alpha = 30°$, $\beta = 25°$, find the angle of elevation of the aeroplane when it is at a point on its path equidistant from A and B.

O is the centre of the circle and D its projection on the level ground. C is a point on the circle midway between A and B. L and M are the projections of A and B on the level ground and will lie on PL and PD produced respectively, where P is the man's position.

$$PL = PD - LD = y - x; \quad PM = PD + DM = y + x$$

From the diagram,

$$h = AL = PL \tan\alpha, \text{ i.e. } h = (y - x)\tan\alpha \ldots\ldots\ldots\ldots (1).$$
$$h = BM = PM \tan\beta, \text{ i.e. } h = (y + x)\tan\beta \ldots\ldots\ldots (2).$$

From (1) and (2), $(y - x)\tan\alpha = (y + x)\tan\beta$,

$$\therefore y(\tan\alpha - \tan\beta) = x(\tan\alpha + \tan\beta),$$

$$\text{i.e. } x\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}\right) = y\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}\right),$$

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i.e. $x(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = y(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$,

i.e. $x\sin(\alpha+\beta) = y\sin(\alpha-\beta)$ ....(3),

$\therefore\ x : y = \sin(\alpha-\beta) : \sin(\alpha+\beta)$.

Using (3) in (1),

$$h = \left\{\frac{x\sin(\alpha+\beta)}{\sin(\alpha-\beta)} - x\right\}\tan\alpha$$

$$= x\left\{\frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{\sin(\alpha-\beta)}\right\}\tan\alpha$$

$$= 2x\cos\frac{(\alpha+\beta)+(\alpha-\beta)}{2}\sin\frac{(\alpha+\beta)-(\alpha-\beta)}{2}\,\frac{\tan\alpha}{\sin(\alpha-\beta)}$$

$$= 2x\cos\alpha\sin\beta\,\frac{\tan\alpha}{\sin(\alpha-\beta)}$$

$$= 2x\frac{\sin\alpha\sin\beta}{\sin(\alpha-\beta)},$$

$$\therefore\ x : h = \sin(\alpha-\beta) : 2\sin\alpha\sin\beta.$$

Hence, $x : y : h = \sin(\alpha-\beta) : \sin(\alpha+\beta) : 2\sin\alpha\sin\beta$.

If E be the projection of C on the ground, $DE = x$, and $\angle PDE = 90°$.

Using Pythagoras' theorem

$$PE^2 = PD^2 + ED^2 = y^2 + x^2 \text{ .... (4)}$$

If θ be the angle of elevation of C, $PE = EC\cot\theta = h\cot\theta$.

Using this in (4)

$$h^2\cot^2\theta = y^2 + x^2$$

$$\therefore\ \cot^2\theta = \frac{y^2}{h^2} + \frac{x^2}{h^2}$$

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$$= \frac{\sin^2(\alpha+\beta)}{4\sin^2\alpha\sin^2\beta} + \frac{\sin^2(\alpha-\beta)}{4\sin^2\alpha\sin^2\beta}$$

(by previous result).

Using $\alpha = 30°$, $\beta = 25°$ in this,

$$\cot^2\theta = \frac{\sin^2 55° + \sin^2 5°}{\sin^2 25°} \qquad (\sin 30° = \tfrac{1}{2})$$

$$= \frac{0{\cdot}67100 + 0{\cdot}00760}{\sin^2 25°} = 0{\cdot}6786/\sin^2 25° \qquad \text{(using tables)}$$

$$\therefore\ \log(\cot\theta) = \tfrac{1}{2}\log 0{\cdot}6786 - \log(\sin 25°)$$
$$= \tfrac{1}{2}(\bar{1}{\cdot}83161) - \bar{1}{\cdot}62595$$
$$= 0{\cdot}28986,$$
$$\therefore\ \theta = 27°\ 10'.$$

EXAMPLE (L.U.). A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled outwards from the wall through a distance a, causing its top to fall a distance b down the wall, while its inclination to the horizontal decreases to β.

Show that $a = b\tan\frac{1}{2}(\alpha+\beta)$.

Let AB be the initial position of the ladder and A_1B_1 its final position as shown, with O the intersection of the wall and horizontal in the plane of the ladder.

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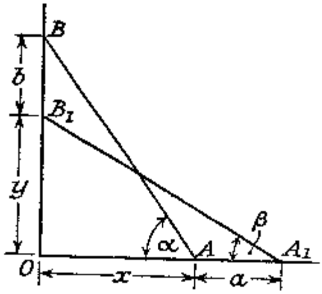

Let $OA = x$, and $OB_1 = y$, and the length of the ladder be

$$l = AB = A_1B_1.$$

From the right-angled triangles OAB, OA_1B_1,

$$x = l\cos\alpha \qquad (1),$$
$$y + b = l\sin\alpha \qquad (2),$$
$$x + a = l\cos\beta \qquad (3),$$
$$y = l\sin\beta \qquad (4).$$

(3) − (1) gives, $$a = l(\cos\beta - \cos\alpha) \qquad (5).$$

(2) − (4) gives, $$b = l(\sin\alpha - \sin\beta) \qquad (6).$$

(5) ÷ (6) gives, $$\frac{a}{b} = \frac{\cos\beta - \cos\alpha}{\sin\alpha - \sin\beta}$$

$$= \frac{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}$$

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$$= \tan\tfrac{1}{2}(\alpha+\beta),$$

$$\therefore\ a = b\tan\tfrac{1}{2}(\alpha+\beta).$$

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EXAMPLES VI

NOTE. A few of the following examples will require the use of the sine rule which is given in the next chapter.

1. Explain how the inclination of a straight line to a plane is measured. An isosceles triangle ABC in which $AB = AC = 2 \cdot BC = 2a$ lying in a horizontal plane π is rotated about the base BC until A is a vertical height a above the plane π. Calculate the angle through which the triangle ABC is rotated and the inclination of AC to the horizontal in its final position.

2. A, B, C, D are four landmarks in the same horizontal level. B is four miles N. 31° E. from A; C is six miles S. 10° 15′ E. from B; D is three miles E. from C. Calculate the distance and bearing of D from A.

3. An aeroplane is observed at the same instant from three stations A, B, C in a horizontal straight line, but not in a vertical plane through the aeroplane. If $AB = BC = c$, and the angles of elevation from A, B, C are respectively α, β, γ, prove that the height of the aeroplane is

$$c\sqrt{2}/(\cot^2\alpha + \cot^2\gamma - 2\cot^2\beta)^{\frac{1}{2}}.$$

(Hint. If O be the foot of the perpendicular on the horizontal plane from aeroplane, then $OA^2 + OC^2 = 2OB^2 + 2AB^2$.)

4. The three edges of a tetrahedron $OABC$ meeting at a vertex O have the same length a and make equal angles θ with each other. Prove the following results:

(i) $AB = BC = CA = 2a \sin \frac{1}{2}\theta$.

(ii) If p be the perpendicular from O to the plane ABC, then

$$3p^2 = (1 + 2\cos\theta)a^2.$$

(iii) The volume of the tetrahedron is $a^3(1 - \cos\theta)(1 + 2\cos\theta)^{\frac{1}{2}}$.

5. The elevation of the top Q of a flagstaff PQ from three distant points A, B, C, which are in a horizontal line with P, are θ, 2θ, 3θ respectively. Prove that $AB = 3BC$ approximately. (Use $\tan\theta \simeq \theta$.)

6. A pyramid has a square base $ABCD$ of side a, all its lateral edges are equal, and the distance of the apex E from the base is $2a$. Determine the angle between the planes AEB and $ABCD$, and the angle which AE makes with the plane $ABCD$.

7. The ridges of two roofs meet at right angles and the roofs are inclined at angles of α and β to the horizon. Prove that, if φ be the inclination to the horizon of the line of intersection of the roofs, $\cot^2\varphi = \cot^2\alpha + \cot^2\beta$.

8. A pyramid stands on a rectangular base the lengths of whose sides are a and b. The faces terminating in the edges of length a are inclined at an angle θ to the horizontal and the vertex is vertically above the mid-point of the base. Find the volume of the pyramid and prove that its surface area is

$$\tfrac{1}{2}b[a \sec\theta + (a^2 + b^2 \tan^2\theta)^{\frac{1}{2}}].$$

9. AB is a horizontal line 12 feet long drawn on a plane inclined at 30° to the horizontal. From B a length BC of 10 feet is measured on the plane in a direction making 60° with AB produced; and at C a vertical pole CD, 10 feet long, is erected. Find the distance AD, and the inclination of AD to the plane.

10. If θ be measured in radians, prove that

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$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1$$

(See work on Limits.)

C is a point due north of A on level ground. From A a base line AB of length 320 feet is set off in a direction 42° S. of E. From B the bearing of C is found to be 19° 18′ W. of N. Calculate AC and find what error would be made in AC if the bearing of C from B were wrongly taken with an error of 15′.

11. From a point on the top of a tree the angle of depression of a small plant on horizontal ground is 24° 15′. From a point on a wall 10 feet above the ground, 50 feet from the tree and due north of it, the angle of depression of the plant is 11° 17′ and its bearing is 63° E. of N. Find the height above the ground of the point on the tree.

12. $ABCD$ is a trapezium; AB, DC are the parallel sides, BC is perpendicular to them, and θ denotes the angle ADB. If $BC = p$, $CD = q$, obtain an expression for the length of AD, and show that the length of AB is

$$\frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}.$$

13. The angles of elevation of a tower from two places A, B on level ground through its base are α, β respectively. A is due south of the tower, and B is due east of A. If $AB = a$, show that the height of the tower is

$$\frac{a \sin\alpha \sin\beta}{[\sin(\alpha + \beta)\sin(\alpha - \beta)]^{\frac{1}{2}}}.$$

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14. Explain how the angle between (i) a plane and a straight line, (ii) two planes, is measured.

A pyramid $DABC$ stands on an equilateral triangle as base, the vertex D being equidistant from A, B, C and at a height above the base equal to treble the length of a side.

Find (i) the inclination of a lateral edge to the base, and (ii) the mutual inclination of two lateral faces.

15. PN is a line perpendicular to a plane NAB, A and B being points in this plane such that $AB = 4{\cdot}13$, $\angle NAB = 63° 30'$, $\angle NBA = 41° 45'$, $\angle PAN = 27° 12'$.

Calculate the length of the perpendicular PN and the angle PBN.

16. OA, OB, and OC are three mutually perpendicular lines and $OA = a$, $OB = b$, $OC = c$. Show that the angle between the planes OBC and ABC is $\tan^{-1} a(b^2 + c^2)^{\frac{1}{2}}/bc$.

Calculate the length of the perpendicular from O to the plane ABC when $OA = 6$ cm., $OB = 3$ cm., $OC = 4$ cm.

17. A tower is built on the top of a hill, the side of the latter being inclined at an angle θ to the horizontal. P and Q are points on the hillside, on a line of greatest slope through the foot of the tower. From P the elevation of the top of the tower is α and from Q it is β. If P is higher than Q and $PQ = c$, prove that the height of the tower is

$$\frac{c \sin(\alpha - \theta) \sin(\beta - \theta)}{\sin(\alpha - \beta) \cos \theta}$$

18. Two lighthouses, the heights of whose lanterns above sea level are 150 feet and 250 feet respectively, are at such a distance apart that the light of each is just visible from the lantern of the other. Find this distance, given that the radius of the earth is 3,959 miles.

Also find the dip of the horizon from each lighthouse.

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19. A building has a hemispherical roof, and a man stands near it with his eyes on the same level as the centre of the sphere. The elevation of the roof is α. The man then walks a distance a towards the centre, and the elevation of the roof becomes β. Prove that the radius of the sphere is

$$\frac{\frac{1}{2}a \sin \alpha \sin \beta}{\sin \frac{1}{2}(\beta - \alpha) \cos \frac{1}{2}(\beta + \alpha)}$$

Calculate the radius, given that $a = 16$ feet, $\alpha = 30°$, $\beta = 60°$.

20. A motorist's danger signal is in the form of an equilateral triangle on a horizontal base facing due south. The sun's bearing is S.W., and its altitude is 60°. Prove that the shadow is a right-angled triangle, and calculate the angles of this triangle.

21. C and D are two points on a horizontal plane, which lie in the vertical plane through the line of flight of an aeroplane, and E is a point on the line CD produced in the direction of D. A and B are two positions of the aeroplane. The following observations are made: $\angle ACD = 64°$, $\angle ADE = 82°$, $\angle BCD = 18°$, $\angle BDE = 25°$, $CD = 100$ feet. Determine the inclination of the line of flight AB to the horizontal.

22. From two points A and B, 100 yards apart, the angles BAC, ABC to the foot of a tower at C are 40° and 120° respectively, and at A the elevation of the top of the tower is 9°. Find the height of the tower if A, B, C all lie in the same horizontal plane.

23. A flagstaff, 60 feet long, is 10° out of the vertical towards the north, in a vertical plane running north and south.

When the sun is due west and at an elevation above the horizon of 23° 15',

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find the direction in which the shadow of the flagstaff is pointing, and its length correct to the nearest tenth of a foot.

24. A, B, and C are three points on a level base line with $AB = BC = a$. The angles of elevation of the top of a tower from A, B, and C are respectively α, β, γ. Find the height of the tower.

If $a = 950$ yards, $\tan \alpha = 1/50$, $\tan \beta = 1/52$, $\tan \gamma = 1/170$, find the height of the tower.

25. Explain how the inclination of a straight line to a plane is measured.

An equilateral triangle ABC lying in a horizontal plane is rotated through an angle of 30° about the side AB. Find the inclination of AC to its original position and to the horizontal plane.

26. Two vertical columns of equal height stand on level ground, and a man on the ground midway between them observes the angle of elevation of the top of either of them to be α. On walking d feet directly towards one of them, its elevation increases to β. Prove that the elevation of the other has decreased to γ, where $\cot \gamma = 2 \cot \alpha - \cot \beta$, and that the height of either column above the eye of the observer is $d \sin \alpha \sin \beta / \sin (\beta - \alpha)$ feet.

27. AB is a vertical flagstaff with the end A on level ground, and C is its mid-point. The portions AC, CB subtend angles α, β respectively at a point P on the ground such that $AP = n \cdot AB$. Show that $\tan \beta = n/(2n^2 + 1)$.

Hence, show that the greatest possible value of β, whatever n may be, is $\cot^{-1} 2\sqrt{2}$, and that β has this value when $n = \frac{1}{2}\sqrt{2}$. (Use calculus.)

28. A, B, C are three points in a horizontal plane, C being the foot of a vertical tower. B is 80 yards due north of A, and C is due east of B. From A the elevation of the tower is 15° 25′, and from B it is 23° 30′. Calculate the height of the tower and the distance BC.

29. $ABCD$ is a rectangle. Through AB a plane is drawn making an angle α with the plane of the rectangle and through AD a plane making an angle β. If the line of intersection of the two planes drawn makes an angle θ with the plane $ABCD$, prove that $\cot^2 \theta = \cot^2 \alpha + \cot^2 \beta$.

Calculate θ when $\alpha = 43° 14'$ and $\beta = 37° 15'$.

30. (i) Express $5 \cos x - 12 \sin x$ in the form $r \cos (x + \alpha)$, and hence find the smallest positive angle which makes the expression greatest.

(ii) Prove the identity $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

31. (i) Find the pairs of angles between 0° and 180° which satisfy the equations $\sin (x + y) = 0{\cdot}5$, $\sin (x - y) = -0{\cdot}5$.

(ii) If $\sin \theta + \sin 2\theta = a$, and $\cos \theta + \cos 2\theta = b$, prove that

$$(a^2 + b^2)(a^2 + b^2 - 3) = 2b.$$

32. (i) Find values of A and B between 0° and 180° that satisfy the equations

$$A - B = 12° 18', \quad \cos (A + B) = 0{\cdot}44568.$$

(ii) Find the values of x between 0° and 360° satisfying the equation

$$10 \sin^2 x + 10 \sin x \cos x - \cos^2 x = 2.$$

33. Prove that

(i) $2 \cot \frac{1}{2}A + \tan A = \tan A \cot^2 \frac{1}{2}A$;

(ii) $\sin \theta + \sin (\theta + \alpha) + \sin (\theta + 2\alpha) + \sin (\theta + 3\alpha)$

$$= 4 \sin\left(\theta + \frac{3}{2}\alpha\right)\cos \alpha \cos \tfrac{1}{2}\alpha.$$

34. (i) Prove that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

(ii) Prove the identity

$$\sin^3 A + \sin^3 (120° + A) + \sin^3 (240° + A) = -\tfrac{3}{4}\sin 3A.$$

(iii) Find the values of $\cos 3{,}360°$, $\operatorname{cosec} (-3{,}840°)$.

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35. Find all the values of θ satisfying the equations,

$$(a)\ 2\tan\theta + 3\sec\theta = 4\cos\theta,$$
$$(b)\ \cos p\theta + \cos(p+2)\theta = \cos\theta.$$

36. Prove that $\cos B - \cos A = 2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.
If $S = \sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta$, prove that $2S\sin\theta = 1 - \cos 8\theta$. Solve the equation $S = 0$.

37. Draw a graph of $3\tan 2\theta - 2$ for all values of θ from 0 to π.
Hence, find the solutions of the equation $3\tan 2\theta - 2 + \theta = 0$ between 0 and π.

38. By drawing the graphs of $\sin 3x$ and $\cos 2x$ for values of x between $0°$ and $360°$, find five solutions of the equation $\sin 3x = \cos 2x$.
Verify your solutions by writing this equation in the form

$$\cos(90° - 3x) = \cos 2x,$$

and obtaining the general solution.

39. Prove that

(i) $\tan\theta + \sec\theta = \tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$,

(ii) $\cos A + \cos B - \cos C - \cos(A+B+C)$
$$= 4\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(C+A).$$

40. Express $a\cos\theta + b\sin\theta$ in the form $r\cos(\theta - \alpha)$, finding r and α in terms of a and b.
Find the value of θ between $0°$ and $180°$ which satisfies the equation $12\cos\theta + 5\sin\theta = 9$.

41. Prove that $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$, $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$.
Find the values of θ between 0 and 2π which satisfy the equation

$$55\cos\theta - 48\sin\theta = 51.$$

42. (i) If A, B, C are the angles of a triangle, prove that
$$(\sin B - \cos B)^2 + (\sin C - \cos C)^2 - (\sin A - \cos A)^2 = 1 - 4\sin A\cos B\cos C.$$

(ii) Solve completely the equation $\sin 3\theta\cos 3\theta - \cos^2 2\theta + \frac{1}{2} = 0$, where θ is measured in degrees.

43. Show that $a\cos\theta + b\sin\theta$ may be written in the form $r\cos(\theta - \alpha)$ and determine r and α.
Hence, or otherwise, solve the equation $8\cos\theta + 6\sin\theta = -7$, giving the solutions between $0°$ and $360°$ and the general solution.

44. Prove that

(i) $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

(ii) $\tan\frac{1}{2}(A+B) = \dfrac{\sin A + \sin B}{\cos A + \cos B}$.

(iii) If $A + B + C = 180°$,
$$\cos A + \cos B + \cos C = 1 + \sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C.$$

45. Show that all the angles given by the expression $n \cdot 360° \pm \theta$ have the same cosine as the angle θ, n denoting any integer.
Solve completely the equation $10\sin\theta + 13\cot\theta = 14\operatorname{cosec}\theta$.

46. (i) Without using tables, prove that $\cos 165° + \sin 165° = \cos 135°$.
(ii) Find the general value of θ, in degrees, which satisfies simultaneously the equations $\tan\theta = \sqrt{3}$, $\sec\theta = -2$.

47. (i) Prove that, if $t = \tan\frac{1}{2}\theta$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$.

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Hence, express the square root of $\dfrac{1+\sin\theta}{(3\sin\theta+4\cos\theta+5)}$ in terms of t.

(ii) Prove that $4\sin(60^\circ-\theta)\sin\theta\sin(60^\circ+\theta)=\sin 3\theta$.

48. If $\sin\alpha = 8/17$, $\sin\beta = 28/53$, find, without the use of tables, $\sin(\alpha-\beta)$ and $\cos(\alpha+\beta)$, α and β being acute angles.

Prove that

(i) $$\frac{\sin(\theta+\varphi)+\cos(\theta-\varphi)}{\sin(\theta-\varphi)+\cos(\theta+\varphi)}=\frac{\cos\varphi+\sin\varphi}{\cos\varphi-\sin\varphi}.$$

(ii) If $$\alpha+\beta+\gamma+\delta=2\pi$$
$$\cos\alpha-\cos\beta+\cos\gamma-\cos\delta=4\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\gamma)\cos\tfrac{1}{2}(\alpha+\delta).$$

49. Show that

(i) $\cos 2\theta = 2\cos^2\theta - 1$.

(ii) $a\cos\theta + b\sin\theta$ can be expressed in the form $r\cos(\theta-\alpha)$.

Hence, or otherwise, find the maximum and minimum values of
$$5\cos^2\theta+2\sin^2\theta+4\cos\theta\sin\theta+1.$$

50. Prove that, for all values of the angle θ,

(i) $$\sin\theta\sin(\theta+120^\circ)+\sin\theta\sin(\theta-120^\circ)+\sin(\theta+120^\circ)\sin(\theta-120^\circ)=-\tfrac{3}{4}.$$

(ii) $\sin^2\theta+\sin^2(\theta+120^\circ)+\sin^2(\theta-120^\circ)=3/2$.

51. For all values of A and B, prove that
$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$
If A, B, C are the angles of a triangle, prove that
$$\sin^2 A+\cos A\sin B\sin C=1+\cos A\cos B\cos C.$$

52. If k be a positive number and α be an angle between 0 and 2π, show that the angles θ, φ determined by the equations $\sin\theta-\sin\varphi=k$, $\theta-\varphi=\alpha$, are real provided that $k \leqslant 2\sin\frac{1}{2}\alpha$.

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If $k=1$, $\alpha=\frac{1}{3}\pi$, determine a pair of values of θ and φ which satisfy the equations.

53. Prove that

(i) $\cos\alpha+\cos(\alpha+120^\circ)+\cos(\alpha+240^\circ)=0$.

(ii) $$(\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha)=\tan 4\alpha(\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cos 7\alpha).$$

54. Prove that $\cos A+\cos B=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

If θ and φ vary so that $(\theta+\varphi)$ remains constant and equal to K, prove that the greatest value of $\cos\theta+\cos\varphi$ is $2\cos K/2$, and find the least value of $\cos\theta\cos\varphi$.

55. If $\sin\theta+\sin\theta\cos\theta=a$, and $\cos\theta+\cos^2\theta=b$, prove that
$$(a^2+b^2)(a^2+b^2-2b)=a^2.$$

56. Express $a\cos^2\theta+b\sin\theta\cos\theta+c\sin^2\theta$ in the form
$$A\cos(2\theta+B)+C,$$
where A, B, and C are each independent of θ.

Use the result to find the greatest and least values of the expression
$$8\cos^2\theta+9\sin\theta\cos\theta-4\sin^2\theta,$$
and find also an acute angle θ for which this expression is equal to zero.

57. If $a\cos\theta+b\sin\theta\equiv A\cos(\theta-\alpha)$, find the values of A and α.

Solve completely the equation $\cos x-2\sin x+1=0$.

Draw the graph of $y=\cos x-2\sin x$ for $-2\pi<x<2\pi$, and from it find the solutions of the above equation which exist in this range.

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58. (i) Prove geometrically that $\sin 2\theta = 2 \sin \theta \cos \theta$.

(ii) Prove that

$$\sin 4\theta = -8 \sin \theta \sin \left(\theta - \frac{\pi}{4}\right) \sin \left(\theta - \frac{\pi}{2}\right) \sin \left(\theta - \frac{3\pi}{4}\right).$$

59. If A, B, and $(A + B)$ are acute angles, prove geometrically that

$$\cos (A + B) = \cos A \cos B - \sin A \sin B.$$

Show that $\cos 5\theta = \cos \theta(16 \cos^4 \theta - 20 \cos^2 \theta + 5)$.

Deduce that the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are

$$\cos \frac{\pi}{10}, \cos \frac{3\pi}{10}, \cos \frac{7\pi}{10}, \cos \frac{9\pi}{10},$$

and find the values of these cosines in surd form.

60. Find all the solutions of $2 \cos x - \sin x = 1$.

Draw the graph of $2 \cos x - \sin x$ from $x = -2\pi$ to $x = 2\pi$. From your graph find the values of x which satisfy the equations (i) $2 \cos x - \sin x = 1$, (ii) $2 \cos x - \sin x = x$.

61. Express $\sin 2\theta$ and $\cos 3\theta$ in terms of $\sin \theta$ and $\cos \theta$. Use the fact that, when $\theta = \frac{1}{10}\pi$, $\sin 2\theta = \cos 3\theta$, to prove that $\sin \frac{1}{10}\pi = \frac{1}{4}(\sqrt{5} - 1)$. Deduce the value of $\sin \frac{1}{5}\pi$.

62. Prove that $\cos^2 \theta + \sin^2 \theta = 1$, whatever the magnitude of θ. If $\cos (\alpha + \beta) = \frac{1}{2}$, and $\cos (\alpha - \beta) = \frac{3}{4}$, find general expressions for all the values of α and β satisfying these equations.

63. Express $\sin \frac{1}{2}\theta$, $\cos \frac{1}{2}\theta$ in terms of $\cos \theta$, and explain briefly how the ambiguities of sign which occur are resolved for any particular value of θ.

If e be a positive number < 1, θ, ϕ are positive angles each less than π, and if $(1 + e \cos \theta)(1 - e \cos \phi) = 1 - e^2$, prove that:

$$\text{(i) } \sin \frac{\phi}{2} = \sqrt{\left(\frac{1 - e}{1 + e \cos \theta}\right)} \sin \frac{\theta}{2};$$

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$$\text{(ii) } \cos \frac{\phi}{2} = \sqrt{\frac{1 + e}{1 + e \cos \theta}} \cdot \cos \frac{\theta}{2};$$

$$\text{(iii) } \tan \frac{\phi}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\theta}{2}.$$

64. If $2A$ be an acute angle, prove the formula

$$\tan 2A = 2 \tan A/(1 - \tan^2 A),$$

and deduce a formula for $\tan 3A$ in terms of $\tan A$.

$ABCD$ is a straight line, and P a point on the perpendicular to the line through A, such that the segments AB, BC, CD subtend equal angles at P. If $AB = a$, $AP = h$, find in terms of a and h expressions for the lengths of BC and CD.

If $BC = \frac{2}{3}CD$, show that $h = \sqrt{11}a$.

65. Find an expression for all angles which have their tangents equal to the tangent of a given angle.

Find the values of θ between $0°$ and $360°$ which satisfy the equation $3 \cos^2 \theta - 4 \cos \theta \sin \theta + \sin^2 \theta = 2$.

66. Prove that $\cos \theta = (1 - t^2)/(1 + t^2)$, $\sin \theta = (2t)/(1 + t^2)$, where $t = \tan \frac{1}{2}\theta$. Show how the equation $a \cos \theta + b \sin \theta = c$ may be solved by the above substitution.

If $\theta = \alpha$, $\theta = \beta$ are two solutions of the equation, show that

$$\tan \tfrac{1}{2}\alpha \cdot \tan \tfrac{1}{2}\beta = (c - a)/(c + a)$$

and deduce that $\cos \frac{1}{2}(\beta - \alpha)/\cos \frac{1}{2}(\beta + \alpha) = c/a$.

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67. Write down formulae expressing $\sin A \pm \sin B$ and $\cos A \pm \cos B$ as products.

Prove that (i) $4 \cos \theta \cos \left(\theta + \frac{2\pi}{3}\right) \cos \left(\theta + \frac{4\pi}{3}\right) = \cos 3\theta.$

(ii) $\cos^2 x + \cos^2 (x + \frac{1}{3}\pi) + \cos^2 (x - \frac{1}{3}\pi) = 1{\cdot}5.$

68. (i) Show that $\sin \theta + \sin 3\theta + \sin 5\theta + \sin 7\theta = 16 \sin \theta \cos^2 \theta \cos^2 2\theta.$

(ii) If $\tan \alpha = 2$, obtain all the solutions of the equation

$$\tan (\theta - 2\alpha) + 11 \tan (\theta + 2\alpha) = 0$$

that lie between $0°$ and $360°$.

69. Obtain the general solutions of the equations

$$\text{(i) } 5 \sin \theta - 3 \cos \theta = 2.$$
$$\text{(ii) } \cot 2\theta - 2 \tan \theta = 2.$$

70. If $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) = 0$, prove that

$$\beta = (2k + 1)\pi \pm \tfrac{1}{3}\pi,$$

where k is an integer or zero, unless $(\alpha + \beta)$ is a multiple of π.

71. Prove that $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

If $\theta = 18°$, prove that $\cos 3\theta = \sin 2\theta$, and hence show that $\sin 18°$ is a root of the equation $4x^2 + 2x - 1 = 0$.

Deduce the values of $\sin 18°$ and $\cos 18°$ to five decimal places.

72. Draw the graph of $\cos (\theta + 60°)$ for values of θ between $-180°$ and $+180°$. From your graph obtain three solutions of the equation

$$\cos \theta - \sqrt{3} \sin \theta = -\theta/60°,$$

where θ is measured in degrees.

73. Draw the graph of

$$y = \sin \left(2x + \frac{2\pi}{3}\right)$$

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from $x = 0$ to $x = 2\pi$. Use your graph to find the positive values which satisfy the equation

$$x = 5 \sin \left(2x + \frac{2\pi}{3}\right),$$

and to find the values between which k must lie in order that

$$kx = \sin \left(2x + \frac{2\pi}{3}\right)$$

may have at least two positive roots.

74. Define a radian and find, to the nearest integer, the number of seconds in a radian, taking the value of π to be $3{\cdot}141593$.

Angles being measured in radians, use the curve $y = \sin x$ and a suitable straight line to solve approximately the equation $x - \sin x = 1{\cdot}5$. Express the value of x so obtained in degrees.

75. A taut belt passes round two pulleys of radii 6 cm. and 2 cm. respectively. The straight portions of the belt are direct common tangents to the pulleys and are inclined to each other at an angle of 2α radians.

If the total length of the belt is 44 cm., show that $\pi + \alpha + \cot \alpha = 5{\cdot}5$. Draw the graphs of $\cot \alpha$ and $5{\cdot}5 - \pi - \alpha$ and hence find α.

76. Measuring x in radians, draw the graph of the function $\sin (x + \frac{1}{6}\pi)$ for value of x from 0 to 2π.

Use your graph to find, as accurately as you can, the solutions of the equation $\pi \sin (x + \frac{1}{6}\pi) = \pi - x$, which lie between 0 and 2π.

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77. Draw the graph of $1 + \cos 2x$, where x is measured in radians, for values of x from 0 to 2π.

Use your graph to determine approximately the least value of x which satisfies the equation $4\cos^2 x = x$.

78. An observer whose eye is 5 feet above the ground finds that, when standing at a point X on the horizontal line between the bases L, M of two vertical poles, the angular elevations of the tops of the poles are both 24°. On moving to a point 15 feet nearer M, he finds that the angular elevation of the top of one pole has increased by as much as the other has diminished. It is known that the length of one pole is twice that of the other. Find their lengths.

79. A cylindrical tower of radius r is surmounted by a hemispherical dome of the same radius. From points A and B on the level ground on which the tower stands the elevations of the highest visible points of the dome are α and β respectively. Prove that, if B is a distance d nearer the axis of the tower than A, the height to the top of the dome is

$$\frac{d \sin \alpha \sin \beta}{\sin (\alpha - \beta)} + \frac{2r \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta}{\cos \frac{1}{2}(\beta - \alpha)}.$$

80. A, B are two points on one bank of a straight river, and C, D are two points on the other bank, the direction from A to B along this river being the same as that from C to D.

If $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, prove that

$$CD = a \sin \alpha \sin \gamma / \sin \beta \sin (\alpha + \beta + \gamma).$$

Calculate CD and the width of the river when $a = 100$ feet, $\alpha = 36°$, $\beta = 24°$, $\gamma = 46°$.

81. An aeroplane is flying in a straight line in a direction making an angle θ with the horizontal, and so that its height is diminishing. The line of flight passes vertically above an object A on the ground, which is horizontal. When the aeroplane is at P, PA makes an angle α with the line of flight and when it has travelled a distance l to a point Q, QA makes an angle β with the line of flight.

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Prove that the height of the aeroplane when vertically over A is

$$\frac{l \sin \alpha \sin \beta}{\sin (\beta - \alpha) \cos \theta},$$

and that the distance travelled by the aeroplane from P before it reaches the ground is

$$\frac{l \sin (\alpha + \theta) \sin \beta}{\sin (\beta - \alpha) \sin \theta}.$$

82. Prove geometrically that $\sec^2 \theta - \tan^2 \theta = 1$.

Given $\sec \theta + \tan \theta = u$, express $\tan \theta$ and $\tan \frac{1}{2}\theta$ in terms of u.

Find all values between 0° and 360° which satisfy the equation

$$\sec^2 \tfrac{1}{2}\theta = 2\sqrt{2} \tan \tfrac{1}{2}\theta.$$

83. If α, β are both acute angles, prove that

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

If $\tan \alpha = \cos 2i \tan \beta$, prove that

$$\tan (\beta - \alpha) = \sin 2\beta / (\cot^2 i + \cos 2\beta).$$

84. (i) Prove that $\tan \frac{1}{2}(A + B) = (\sin A + \sin B)/(\cos A + \cos B)$.

(ii) If $x \tan \theta = y \tan \varphi$ and $(x + y) \cos (\theta - \varphi) + (x - y) \cos (\theta + \varphi) = 2a$, prove that $x^2 \cos^2 \varphi + y^2 \sin^2 \varphi = a^2$.

85. Prove that the value of

$$\frac{\cot A}{1 + \cot A} \times \frac{\cot (45° - A)}{1 + \cot (45° - A)}$$

the same for all values of the angle A.

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86. Prove that $\cos 2\theta = (1 - t^2)/(1 + t^2)$, where $t = \tan\theta$.

Use this result to deduce that $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$, and find similarly the value of $\tan 15^\circ$.

87. From the equation $11\cos\theta + 7\sin\theta = 13$ derive the value of $\tan\frac{1}{2}\theta$, and hence all the values of θ between 0° and 360°.

88. Prove that $\cos 3\theta = \cos\theta\,(2\cos 2\theta - 1)$. Show that

$$\frac{\cos 3A \sin 3B}{\cos A \sin B} = 2\cos 2(A+B) + 2\cos 2(A-B) + 2\cos 2A - 2\cos 2B - 1.$$

89. Express $a\cos\theta + b\sin\theta$ in the form $A\cos(\theta - \alpha)$, where A and α are independent of θ.

Hence, or otherwise, show that $5\cos\theta + 3\cos(\theta + 60^\circ)$ cannot be greater than 7 or less than -7.

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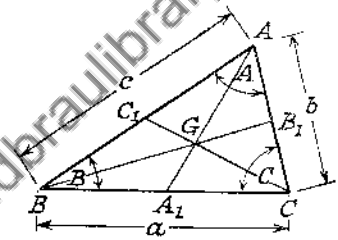
CHAPTER VII

Trigonometry

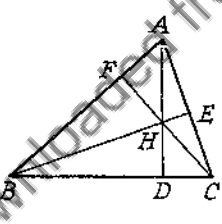
Properties of a Triangle

Standard Notation for a Triangle *ABC*. In discussing a triangle *ABC*, the following notation is most usual. (All theorems employed require geometrical proof.)



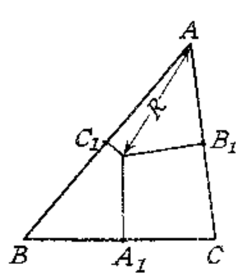
$BC = a$, $CA = b$, $AB = c$, $\angle CAB = \angle A$, $\angle ABC = \angle B$, $\angle BCA = \angle C$; A_1, B_1, C_1 are the mid-points of the sides BC, CA, AB respectively; the lines AA_1, BB_1, CC_1 are the three *medians* of the triangle and are concurrent, meeting at a point G, known as the *centroid* of the triangle; area of triangle $ABC = \triangle$, semiperimeter

$$= s = \tfrac{1}{2}(a + b + c).$$



The points D, E, F are the feet of the perpendiculars from the vertices A, B, C respectively on the opposite sides. The lines AD, BE, CF are known as the *altitudes* of the triangle and are concurrent, meeting at the point H known as the *orthocentre* of the triangle. The triangle DEF is the *pedal* triangle of triangle ABC.

The perpendicular bisectors of the sides of the triangle ABC are concurrent and meet at a point O which is known as the *circumcentre* of the triangle ABC. The circle with O as centre and OA as radius will pass through B and C and is known as the *circumscribed circle* (or *circumcircle*) of the triangle and its radius is denoted by R.



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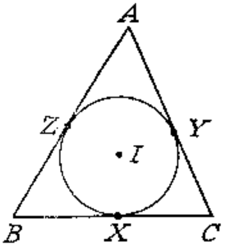
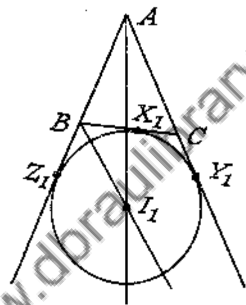


The internal bisectors of the angles A, B, and C are concurrent and meet at a point I known as the *incentre* of the triangle. A circle, with I as centre, touching one side of the triangle, will also touch the other two sides, and is known as the *inscribed circle* of triangle ABC. Its radius is denoted by r, and its points of contact with BC, CA, AB respectively are X, Y, Z.

The external bisectors of the angles B and C, and the internal bisector of $\angle A$ are also concurrent and meet at a point denoted by I_1 known as the first *ecentre* of triangle ABC. The circle with I_1 as centre touching BC internally at X_1 will touch AC and AB externally at Y_1 and Z_1 respectively. This circle is known as the *first escribed circle* (ecircle) of the triangle ABC, and its radius is denoted by r_1.



A similar circle drawn in the space opposite B is known as the *second escribed circle*, and has its centre at I_2, radius r_2, and touches BC, CA, AB at X_2, Y_2, Z_2 respectively. (CA internally and BC, BA externally.)

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Similarly the *third escribed circle* will be in the space opposite C, and has centre I_3, radius r_3, and points of contact with CB, CA, AB will be X_3, Y_3, Z_3 respectively. (Z_3 internal division, and X_3, Y_3 external division.)

Theorem. *The sine rule. To prove that, in any triangle ABC,*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

When the proof is required for *any* triangle the following three cases must always be considered:

(i) an acute-angled triangle,
(ii) an obtuse-angled triangle,
(iii) a right-angled triangle.

Case (i) (diagram (i) on next page).

Draw AD and BE the altitudes of triangle ABC.

Then, $$AD = AB \sin B = AC \sin C$$

$$\therefore\ c \sin B = b \sin C, \text{ i.e. } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

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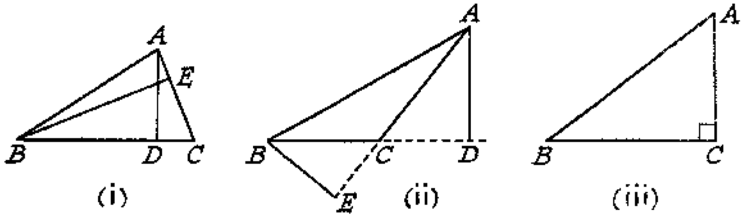

(i) (ii) (iii)

Also $\qquad BE = BC \sin C = AB \sin A.$

$$\therefore\ a \sin C = c \sin A, \text{ i.e. } \frac{a}{\sin A} = \frac{c}{\sin C}.$$

$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Case (*ii*) (diagram (ii)), $\angle C$ obtuse.

Draw AD the perpendicular on BC produced, and BE the perpendicular on AC produced.

$$\therefore\ AD = AB \sin B = AC \sin (180° - C).$$

$$\therefore\ c \sin B = b \sin (180° - C) = b \sin C.$$

$$\text{i.e. } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by using the altitude BE,

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

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$$\therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Case (*iii*) (diagram (iii)), $\widehat{C} = 90°$.

$$AC = AB \sin B, \text{ i.e. } b = c \sin B$$

Since $\qquad \sin 90° = 1,\ b \sin 90° = c \sin B$

$$\therefore\ \frac{b}{\sin B} = \frac{c}{\sin 90°} = \frac{c}{\sin C}.$$

Similarly,

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \therefore\ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Hence, for any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Theorem. *The cosine rule. To prove that, in any triangle ABC,*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

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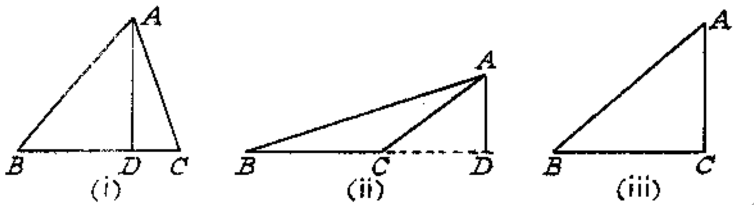
There are the three cases to be considered as shown by the three diagrams, viz. (i) an acute-angled triangle, (ii) an obtuse-angled triangle, (iii) a right-angled triangle.



Case (*i*). Draw AD an altitude of the triangle.

Using Pythagoras' theorem,

$$\begin{aligned} AB^2 &= AD^2 + BD^2 \\ &= (AC^2 - CD^2) + BD^2 \\ &= AC^2 - CD^2 + (BC - CD)^2 \\ &= AC^2 - CD^2 + BC^2 - 2BC \,.\, CD + CD^2 \\ &= BC^2 + AC^2 - 2BC \,.\, CD \\ &= a^2 + b^2 - 2a \,.\, b \cos C \\ \therefore\ c^2 &= a^2 + b^2 - 2ab \cos C, \\ \text{or } \cos C &= (a^2 + b^2 - c^2)/2ab. \end{aligned}$$

Case (*ii*). C obtuse.

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Draw the perpendicular from A on BC produced.
Using Pythagoras,

$$\begin{aligned} AB^2 &= AD^2 + BD^2 \\ &= (AC^2 - CD^2) + BD^2 \\ &= AC^2 - CD^2 + (BC + CD)^2 \\ &= AC^2 - CD^2 + BC^2 + 2BC \,.\, CD + CD^2 \\ &= BC^2 + CA^2 + 2BC \,.\, CD \\ \text{i.e. } c^2 &= a^2 + b^2 + 2a \,.\, b \cos (180^\circ - C) \\ &= a^2 + b^2 - 2ab \cos C, \end{aligned}$$

$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Case (*iii*). $\angle C = 90^\circ$.

Using Pythagoras,

$$\begin{aligned} AB^2 &= BC^2 + CA^2, \\ \text{i.e. } c^2 &= a^2 + b^2 = a^2 + b^2 - 2ab \cos 90^\circ \qquad (\cos 90^\circ = 0) \\ &= a^2 + b^2 - 2ab \cos C, \end{aligned}$$

$$\text{or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

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Thus, in all cases,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \quad \text{or } c^2 = a^2 + b^2 - 2ab \cos C.$$

The results for $\cos B$ and $\cos A$ can be proved in a similar manner.

EXAMPLE. The sides of a triangle are x, y, and $\sqrt{(x^2 + y^2 + xy)}$. Find the greatest angle.

The greatest angle C, of the triangle ABC, will be opposite the greatest side $c = \sqrt{(x^2 + y^2 + xy)}$.

Using the cosine rule, $\cos C = (a^2 + b^2 - c^2)/2ab$, where $a = x$, $b = y$, $c = \sqrt{(x^2 + y^2 + xy)}$.

$$\therefore \cos C = \frac{x^2 + y^2 - (x^2 + y^2 + xy)}{2xy} = \frac{-xy}{2xy} = -\tfrac{1}{2}$$

$$\therefore \angle C = 120°.$$

Theorem. *In any triangle ABC, to find the values of $\cos A/2$, $\sin A/2$, $\tan A/2$, etc., in terms of the sides a, b, c.*

In any triangle ABC,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\therefore 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc}$$

$$\text{i.e. } 2\cos^2 A/2 = \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc}$$

$$= \frac{2s[(b + c + a) - 2a]}{2bc} = \frac{2s(2s - 2a)}{2bc}$$

$$\therefore \cos^2 A/2 = \frac{s(s - a)}{bc}.$$

But $A/2$ is acute, therefore $\cos A/2$ is positive.

Hence, $$\cos A/2 = \sqrt{\frac{s(s - a)}{bc}},$$

Similarly, $$\cos B/2 = \sqrt{\frac{s(s - b)}{ca}},$$

$$\cos C/2 = \sqrt{\frac{s(s - c)}{ab}}.$$

$$\text{Also } 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b - c)^2}{2bc} = \frac{(a + b - c)(a - b + c)}{2bc}$$

$$\therefore 2\sin^2 A/2 = \frac{[(a + b + c) - 2c][(a + b + c) - 2b]}{2bc}$$

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$$= \frac{(2s-2c)(2s-2b)}{2bc} = \frac{2(s-b)(s-c)}{bc}$$

$\therefore \sin^2 A/2 = (s-b)(s-c)/bc.$

$A/2$ is acute, therefore $\sin A/2$ is positive.

Hence, $\sin A/2 = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$

Similarly, $\sin B/2 = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$

$\sin C/2 = \sqrt{\dfrac{(s-a)(s-b)}{ab}}.$

From these results,

$$\tan A/2 = \frac{\sin A/2}{\cos A/2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \div \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Also, $\tan B/2 = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}},$

$\tan C/2 = \sqrt{\dfrac{(s-a)(s-b)}{s(s-a)}}.$

$$\sin A = 2 \sin A/2 \cos A/2 = 2\sqrt{\frac{(s-b)(s-c)}{bc}} \times \sqrt{\frac{s(s-a)}{bc}}$$

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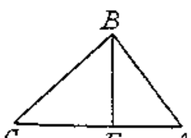
$$= \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Also, $\sin B = \dfrac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)},$

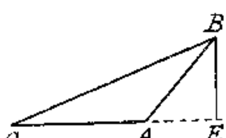
$\sin C = \dfrac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}.$

Theorem. *To prove that the area $\triangle$ of the triangle ABC is given by* $\triangle = \frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}.$

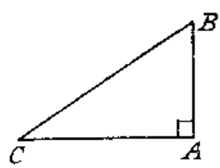
Considering the three cases shown by the diagrams (i) acute-angled triangle, (ii) $\angle A$ obtuse, (iii) $\angle A = 90°$.



(i)



(ii)



(iii)

Case (i). BE is an altitude.

Area of triangle $ABC = \frac{1}{2}CA.BE = \frac{1}{2}bc \sin A.$

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By drawing the other altitudes,

$$\triangle = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

$$\therefore \triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

Case (*ii*). Draw perpendicular BE from B on CA produced.

$$\triangle = \tfrac{1}{2}CA.BE = \tfrac{1}{2}bc \sin(180° - A) = \tfrac{1}{2}bc \sin A.$$

By drawing in the other altitudes,

$$\triangle = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

$$\therefore \triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

Case (*iii*). $\triangle = \tfrac{1}{2}CA.BA = \tfrac{1}{2}bc = \tfrac{1}{2}bc \sin 90° = \tfrac{1}{2}bc \sin A.$

Also, $\triangle = \tfrac{1}{2}bc = \tfrac{1}{2}b.a \sin C \qquad (c = a \sin C,\ b = a \sin B.)$

$= \tfrac{1}{2}c.a \sin B.$

$$\therefore \triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C.$$

Using $\sin A = (2/bc)\sqrt{s(s-a)(s-b)(s-c)}$, and $\triangle = \tfrac{1}{2}bc \sin A$,

$$\triangle = \tfrac{1}{2}bc \times (2/bc)\sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

Thus, for any triangle ABC,

$$\triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C = \sqrt{s(s-a)(s-b)(s-c)}.$$

Theorem. *To prove that, in any triangle ABC,* $a = b \cos C + c \cos B$, $b = c \cos A + a \cos C$, $c = a \cos B + b \cos A$.

Using the diagrams shown, with (i) $\angle A$, $\angle B$, $\angle C$ acute, (ii) $\angle C$ obtuse, (iii) $C = 90°$.

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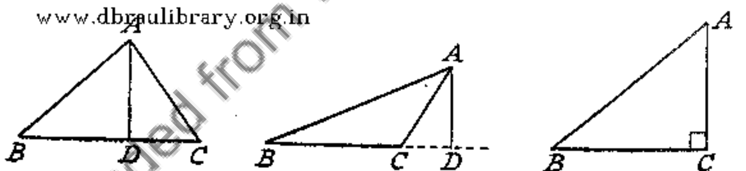

(i) (ii) (iii)

AD is an altitude in cases (i) and (ii).

Case (*i*). $a = BC = BD + CD$

$= b \cos C + c \cos B.$

Case (*ii*). $a = BC = BD - CD = c \cos B - b \cos(180° - C)$

$= c \cos B + b \cos C = b \cos C + c \cos B.$

Case (*iii*). $a = BC = AB \cos B = c \cos B$

$= c \cos B + b \cos 90° \qquad (\cos 90° = 0)$

$= c \cos B + b \cos C = b \cos C + c \cos B.$

By drawing in the other altitudes it can be proved, in a similar manner in each case, that

$$b = c \cos A + a \cos C, \text{ and } c = a \cos B + b \cos A.$$

Hence, in any triangle ABC,

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$$\left.\begin{aligned} a &= b\cos C + c\cos B \\ b &= c\cos A + a\cos B \\ c &= a\cos B + b\cos A \end{aligned}\right\}.$$

Theorem. *To prove that, in any triangle ABC,*

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}, \quad \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2},$$

$$\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}.$$

Using the sine rule, in any triangle ABC,

$$\frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)},$$

$$\therefore\ b = k\sin B,\ c = k\sin C.$$

$$\frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\cos\tfrac{1}{2}(B+C)\sin\tfrac{1}{2}(B-C)}{2\sin\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(B-C)}$$

$$= \frac{1}{\tan\tfrac{1}{2}(B+C)} \times \tan\frac{B-C}{2}$$

$$= \frac{1}{\tan(90^\circ - A/2)} \times \tan\frac{B-C}{2}$$

$(A+B+C=180^\circ)$

$$= \frac{1}{\cot A/2} \times \tan\frac{B-C}{2}$$

$(\tan(90^\circ - \theta) = \cot\theta)$

$$\therefore\ \tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot A/2.$$

Similarly,

$$\left.\begin{aligned} \tan\frac{C-A}{2} &= \frac{c-a}{c+a}\cot B/2, \\ \tan\frac{A-B}{2} &= \frac{a-b}{a+b}\cot C/2. \end{aligned}\right\}$$

EXAMPLE. Prove that $a\cos\tfrac{1}{2}(B-C) = (b+c)\sin A/2$.

By the sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say),

$$\therefore\ a = k\sin A,\quad b = k\sin B,\quad c = k\sin C,$$

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$$\therefore \frac{b+c}{a} = \frac{k\sin B + k\sin C}{k\sin A} = \frac{\sin B + \sin C}{\sin A}$$

$$= \frac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\sin A/2\cos A/2}$$

$$= \frac{\sin(90^\circ - \frac{1}{2}A)\cos\frac{1}{2}(B-C)}{\sin A/2\cos A/2}$$

$$= \frac{\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}{\sin A/2\cos A/2} = \frac{\cos\frac{1}{2}(B-C)}{\sin A/2},$$

$$\therefore \cos\tfrac{1}{2}(B-C) = \frac{b+c}{a}\sin A/2,$$

i.e. $a\cos\frac{1}{2}(B-C) = (b+c)\sin A/2$.

EXAMPLE (L.U.). Prove that, in any triangle ABC, $b\cos A + a\cos B = c$ From this, and two similar identities, deduce the following:

(i) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} = \dfrac{c}{ab}$;

(ii) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$;

(iii) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$.

The first part of the question has been proved in an earlier theorem.

(i) The identity in the first part and the two similar ones are

$$b\cos A + a\cos B = c \quad \ldots\ldots (1)$$

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$$c\cos A + a\cos C = b \quad \ldots\ldots (2)$$

$$c\cos B + b\cos C = a \quad \ldots\ldots (3)$$

Dividing (1) by ab, (2) by ca, (3) by bc,

$$\frac{\cos A}{a} + \frac{\cos B}{b} = \frac{c}{ab} \quad \ldots\ldots (4)$$

$$\frac{\cos A}{a} + \frac{\cos C}{c} = \frac{b}{ca} \quad \ldots\ldots (5)$$

$$\frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a}{bc} \quad \ldots\ldots (6)$$

Adding (4), (5), (6) in (i),

$$2\left(\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}\right) = \frac{c}{ab} + \frac{b}{ca} + \frac{a}{bc}$$

$$= \frac{c^2+b^2+a^2}{abc}$$

$$\therefore \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc} \quad \ldots\ldots (7)$$

(iii) (7) − (4) gives,

$$\frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc} - \frac{c}{ab} = \frac{a^2+b^2+c^2-2c^2}{2abc} = \frac{a^2+b^2-c^2}{2abc}.$$

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$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Solution of any Triangle *ABC*. An *element* of a triangle means either a side or an angle of the triangle, and thus there are six elements to any triangle.

To *solve* a given triangle means: given three *independent* elements to find the other three.

NOTE. The three angles of a triangle are not independent since their sum is equal to two right angles.

The case of the right-angled triangle was considered when dealing with heights and distances, and the remaining cases are classified as follows:

Case (*i*). *Three sides given*, i.e. a, b, c.

Method (*a*). Use any set of formulae for $\sin A/2$, $\cos A/2$, $\tan A/2$, or $\sin A$, etc., to find A, B, C, where

$$\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos A/2 = \sqrt{\frac{s(s-a)}{bc}},$$

$$\tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \quad \sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

Method (*b*). Use the cosine rule, i.e. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, etc., to find A, B, C.

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NOTE. The second method is only used when a, b, c are small whole numbers or easy fractions, or when a certain angle is required accurately. (Cannot be used with logarithms.)

The first method is to be used in all other cases since it lends itself to the use of logarithms.

When using the first method it is advisable to find s, $(s-a)$, $(s-b)$, $(s-c)$ initially.

EXAMPLE. Given $a = 5{\cdot}6$ inches, $b = 7{\cdot}6$ inches, $c = 10{\cdot}8$ inches, solve the triangle.

$s = \frac{1}{2}(5{\cdot}6 + 7{\cdot}6 + 10{\cdot}8) = 12$, $(s-a) = 6{\cdot}4$, $(s-b) = 4{\cdot}4$, $(s-c) = 1{\cdot}2$.

$$\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{4{\cdot}4 \times 1{\cdot}2}{7{\cdot}6 \times 10{\cdot}8}} = \sqrt{\frac{11}{171}}$$

$$\begin{aligned} \therefore \log(\sin A/2) &= \tfrac{1}{2}(\log 11 - \log 171) = \tfrac{1}{2}(1{\cdot}04139 - 2{\cdot}23300) \\ &= \tfrac{1}{2}(\bar{2}{\cdot}80839) = \bar{1}{\cdot}40420, \\ \therefore A/2 &= 14^\circ\, 41\tfrac{1}{2}' \quad \therefore A = 29^\circ\, 23'. \end{aligned}$$

$$\sin B/2 = \sqrt{\frac{(s-a)(s-c)}{ac}} = \sqrt{\frac{6{\cdot}4 \times 1{\cdot}2}{5{\cdot}6 \times 10{\cdot}8}} = \sqrt{\frac{8}{63}}$$

$$\begin{aligned} \therefore \log(\sin B/2) &= \tfrac{1}{2}(\log 8 - \log 63) = \tfrac{1}{2}(0{\cdot}90309 - 1{\cdot}79934) \\ &= \tfrac{1}{2}(\bar{1}{\cdot}10375) = \bar{1}{\cdot}55188, \\ \therefore B/2 &= 20^\circ\, 52\tfrac{1}{2}' \quad \therefore B = 41^\circ\, 45'. \end{aligned}$$

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$$\sin C/2 = \sqrt{\frac{(s-a)(s-b)}{ab}} = \sqrt{\frac{6{\cdot}4 \times 4{\cdot}4}{5{\cdot}6 \times 7{\cdot}6}} = \sqrt{\frac{2 \times 44}{7 \times 19}} = \sqrt{\frac{88}{133}},$$

$$\therefore \log(\sin C/2) = \tfrac{1}{2}(\log 88 - \log 133) = \tfrac{1}{2}(1{\cdot}94448 - 2{\cdot}12385)$$
$$= \tfrac{1}{2}(\bar{1}{\cdot}82063) = \bar{1}{\cdot}91032,$$
$$\therefore C/2 = 54^\circ\,26', \quad \therefore C = 108^\circ\,52'.$$

Check. $A + B + C = 29^\circ\,23' + 41^\circ\,45' + 108^\circ\,52' = 180^\circ$.

EXAMPLE. Solve the triangle, $a = 6$ inches, $b = 5$ inches, $c = 4$ inches.

In this case the lengths of the sides are small numbers and it is quicker to use the cosine rule.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{5^2 + 4^2 - 6^2}{2 \times 5 \times 6} = \frac{5}{40} = \frac{1}{8} = 0{\cdot}125$$
$$\therefore \angle A = 82^\circ\,49'.$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2bc} = \frac{4^2 + 6^2 - 5^2}{2 \times 6 \times 4} = \frac{27}{48} = \frac{9}{16} = 0{\cdot}5625,$$
$$\therefore \angle B = 55^\circ\,46'.$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{6^2 + 5^2 - 4^2}{2 \times 6 \times 5} = \frac{45}{60} = \frac{3}{4} = 0{\cdot}75,$$
$$\therefore \angle C = 41^\circ\,25'.$$

Check. $A + B + C = 82^\circ\,49' + 55^\circ\,46' + 41^\circ\,25' = 180^\circ$.

Case (*ii*). *Two sides and included angle given.* (Use b, c, and A.)

Method (*a*). Use $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot A/2$ to find $(B - C)$.

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Now $B + C = 180^\circ - A$ = known quantity, therefore B and C can be found, and a is then found by using the sine rule.

Method (*b*). Use the cosine formula $a^2 = b^2 + c^2 - 2bc\cos A$ in order to find a and then the sine rule to find B and C.

NOTE. Method (*b*) is only suitable when b and c are small numbers, and when $\angle A = 60^\circ$ or 120°, whilst method (*a*) can be adapted for use with logarithms.

EXAMPLE (L.U.). Prove that in any triangle ABC

(i) $\tan\tfrac{1}{2}(B - C) = \dfrac{b-c}{b+c}\cot A/2$;

(ii) $a = (b + c)\cos\varphi$, where $\sin\varphi = \dfrac{2\sqrt{bc}}{b+c}\cos\tfrac{1}{2}A$.

If $b = 321$ feet, $c = 123$ feet, $\angle A = 29^\circ\,16'$, find B, C, and a.

(i) This has already been proved as a theorem.

(ii) Using the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos A = (b^2 + 2bc + c^2) - 2bc - 2bc\cos A$$
$$= (b + c)^2 - 2bc(1 + \cos A) = (b + c)^2 - 4bc\cos^2\tfrac{1}{2}A$$
$$= (b + c)^2 - (b + c)^2\sin^2\varphi \qquad (\sin\varphi = [2\sqrt{bc}/(b + c)]\cos\tfrac{1}{2}A)$$
$$= (b + c)^2(1 - \sin^2\varphi) = (b + c)^2\cos^2\varphi,$$
$$\therefore a = (b + c)\cos\varphi.$$

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When $b = 321$ feet, $c = 123$ feet, $A = 29° 16'$, using

$$\tan \tfrac{1}{2}(B - C) = \frac{b-c}{b+c} \cot A/2,$$

$$\tan \tfrac{1}{2}(B - C) = \frac{198}{444} \cot 14° 38' = \frac{33}{74} \cot 14° 38'$$

$$\begin{aligned} \therefore \log [\tan \tfrac{1}{2}(B - C)] &= \log 33 + \log(\cot 14° 38') - \log 74 \\ &= 1{\cdot}51851 + 0{\cdot}58318 - 1{\cdot}86923 \\ &= 0{\cdot}23246 \end{aligned}$$

$$\therefore \tfrac{1}{2}(B - C) = 59° 39' \quad \therefore B - C = 119° 18' \quad \ldots\ldots\ldots(1).$$

Now $\quad B + C = 180° - A = 150° 44' \ldots\ldots\ldots\ldots\ldots (2).$

From (1) and (2), $\quad 2B = 270° 2', \quad \therefore B = 135° 1',$
and $C = 15° 43'$.

Using the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \therefore a = \frac{b \sin A}{\sin B} = \frac{321 \sin 29° 16'}{\sin 135° 1'},$$

$$\begin{aligned} \therefore \log a &= \log 321 + \log(\sin 29° 16') - \log(\sin 135° 1') \\ &= 2{\cdot}50651 + \bar{1}{\cdot}68920 - \bar{1}{\cdot}84936 \\ &= 2{\cdot}34635, \\ \therefore a &= 222{\cdot}0. \end{aligned}$$

Case (*iii*). *Given one side and two angles* (e.g. a, $\angle B$, $\angle C$).

Since $A + B + C = 180°$, $A = 180° - B - C$. The sides b, c can now be found by using the sine rule.

EXAMPLE. Solve the triangle in which $a = 10$ inches, $\angle B = 41° 24'$, $C = 35° 18'$.

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$$\angle A = 180° - B - C = 180° - 41° 24' - 35° 18' = 103° 18'.$$

Using the sine rule,

$$b = \frac{a \sin B}{\sin A} = \frac{10 \sin 41° 24'}{\sin 103° 18'},$$

$$\begin{aligned} \therefore \log b &= 1 + \log(\sin 41° 24') - \log(\sin 103° 18') \\ &= 1 + \bar{1}{\cdot}82041 - \bar{1}{\cdot}98819 = 0{\cdot}83222, \\ \therefore b &= 6{\cdot}796 \text{ inches.} \end{aligned}$$

$$c = \frac{a \sin C}{\sin A} = \frac{10 \sin 35° 18'}{\sin 103° 18'},$$

$$\begin{aligned} \therefore \log c &= 1 + \log(\sin 35° 18') - \log(\sin 103° 18') \\ &= 1 + \bar{1}{\cdot}76182 - \bar{1}{\cdot}98819 = 0{\cdot}77363, \\ \therefore c &= 5{\cdot}938 \text{ inches.} \end{aligned}$$

Case (*iv*). *Given two sides and the angle opposite one of them* (e.g. given b, c, and $\angle B$).

This case is known as the *ambiguous* case, as it is possible to have 0, 1, or 2 solutions, dependent on the values of b, c, and $\angle B$.

$\angle C$ is obtained by using the sine rule, viz.

$$\sin C = \frac{c}{b} \sin B.$$

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(*A*) *Angle B acute.*

(*a*) If $b < c \sin B$, then $\sin C > 1$, which is impossible, and there will be no solution.

(*b*) If $b = c \sin b$, then $\sin C = 1$ and $\angle C = 90°$, giving only one solution.

(*c*) If $b > c \sin b$ and $b \geqslant c$, then $\angle B \geqslant \angle C$, and $\angle C$ must be acute, and there is only one solution.

(*d*) If $b > c \sin b$ and $b < c$, there are two solutions for C, one being acute and the other obtuse.

(*B*) *Angle B obtuse*

(*a*) There will be no solution if $b \leqslant c$, since $\angle B$ being obtuse is greater than $\angle C$, and there can only be one obtuse angle in a triangle.

(*b*) If $b > c$, there is only one solution, since $\angle C$ must be acute.

Once the value or values of $\angle C$ have been determined, the angle A is found by using $A + B + C = 180°$, and the side a is then determined by using the sine rule.

EXAMPLE (L.U.). Show that there may be two triangles or possibly no triangles at all, having assigned values for a, b, $\angle A$.

When $a = 25$, $b = 30$, $\angle A = 50°$, determine how many such triangles exist and complete their solutions.

The first part of the question has already been proved.

By the sine rule,

$$\sin \angle B = \frac{b}{a} \sin \angle A = \frac{30}{25} \sin 50° = \frac{6}{5} \times 0{\cdot}76604$$

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$$= 0{\cdot}91925 \ldots\ldots\ldots\ldots \quad (1)$$

Since $b > a$, and $b \sin A < a$, there will be two solutions, denoted by the suffixes 1 and 2.

From (1), $\angle B_1 = 66° 49'$, and $\angle B_2 = 113° 11'$

$$\therefore \angle C_1 = 180° - 50° - 66° 49' = 63° 11'.$$

Also
$$c_1 = \frac{a \sin \angle C_1}{\sin \angle A} = \frac{25 \sin 63° 11'}{\sin 50°}$$

$$\therefore \log c_1 = \log 25 + \log (\sin 63° 11') - \log (\sin 50°)$$
$$= 1{\cdot}39794 + \bar{1}{\cdot}95059 - \bar{1}{\cdot}88425 = 1{\cdot}46428.$$
$$\therefore c_1 = 29{\cdot}13.$$
$$\angle C_2 = 180° - 50° - 113° 11' = 16° 49'.$$
$$c_2 = \frac{a \sin \angle C_2}{\sin \angle A} = \frac{25 \sin 16° 49'}{\sin 50°}$$
$$\log c_2 = \log 25 + \log (\sin 16° 49') - \log (\sin 50°)$$
$$= 1{\cdot}39794 + \bar{1}{\cdot}46137 - \bar{1}{\cdot}88425 = 0{\cdot}97506$$
$$\therefore c_2 = 9{\cdot}442$$

$\therefore$ solutions are,
$$\left.\begin{array}{lll} \angle B_1 = 66° 49', & \angle C_1 = 63° 11', & c_1 = 29{\cdot}13, \\ \angle B_2 = 113° 11', & \angle C_2 = 16° 49', & c_2 = 9{\cdot}442. \end{array}\right\}$$

EXAMPLE (L.U.). Prove that, in any triangle whose sides are a, b, c, and corresponding angles A, B, C,

$$\sin A / a = \sin B / b = \sin C / c.$$

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In an obtuse-angled triangle ABC, the angle B is $48° 24'$, and the lengths of AB, AC are respectively 5·38 and 4·29. Solve the triangle completely.

The sine rule has already been proved as a theorem.

Now $\angle B = 48° 24'$, $c = 5{\cdot}38$, $b = 4{\cdot}29$.

Using the sine rule,

$$\sin \angle C = \frac{c \sin B}{b} = \frac{5{\cdot}38}{4{\cdot}29} \sin 48° 24',$$

$$\begin{aligned} \therefore \log(\sin \angle C) &= \log 5{\cdot}38 + \log(\sin 48° 24') - \log 4{\cdot}29 \\ &= 0{\cdot}73078 + \bar{1}{\cdot}87378 - 0{\cdot}63246 = \bar{1}{\cdot}97210, \\ \therefore \angle C &= 69° 41' \text{ or } 180° - 69° 41' = 110° 19'. \end{aligned}$$

But the triangle is obtuse-angled, therefore $\angle C = 110° 19'$ ($\angle A$ cannot be obtuse since $\angle B = 48° 24'$ and $\angle C$ must at least be $69° 41'$).

Hence, $\angle A = 180° - 48° 24' - 110° 19' = 21° 17'$.

From the sine rule,

$$a = \frac{b \sin A}{\sin B} = 4{\cdot}29 \frac{\sin 21° 17'}{\sin 48° 24'},$$

$$\begin{aligned} \therefore \log a &= \log 4{\cdot}29 + \log(\sin 21° 17') - \log(\sin 48° 24') \\ &= 0{\cdot}63246 + \bar{1}{\cdot}55989 - \bar{1}{\cdot}87378 = 0{\cdot}31857, \\ \therefore a &= 2{\cdot}083, \end{aligned}$$

therefore solution is, $a = 2{\cdot}083$, $\angle A = 21° 17'$, $\angle C = 110° 19'$.

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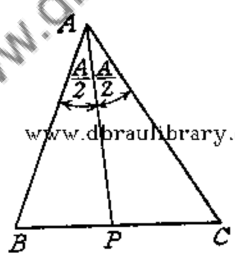
Theorem. *To find the length of the internal bisector of the angle A of the triangle ABC.*



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Let AP be the required bisector.

$$\begin{aligned} \text{Area of triangle } ABC &= \text{area } \triangle ABP + \text{area } \triangle APC. \\ \text{i.e. } \tfrac{1}{2}bc \sin A &= \tfrac{1}{2}c \,.\, AP \sin A/2 + \tfrac{1}{2}b \,.\, AP \sin A/2 \\ \therefore bc \sin A &= AP(b + c) \sin A/2 \\ \text{i.e. } 2bc \sin A/2 \cos A/2 &= AP(b + c) \sin A/2 \end{aligned}$$

$$\therefore AP = \frac{2bc}{b + c} \cos \frac{A}{2}.$$

NOTE. The lengths of the other internal and external bisectors of the angles can be found in a similar manner.

EXAMPLE (L.U.). If the bisector of the angle A of the triangle ABC meets BC in D, prove that,

(i) $AD(b + c) = 2bc \cos A/2$,

(ii) $a = (b + c)\left(1 - \dfrac{AD^2}{bc}\right)^{\frac{1}{2}}$

If $AB = 9$, $AC = 5$, $AD = 6$, find $\angle A$ and a.

(i) This was proved in the previous theorem.

(ii) Using the cosine rule,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A = b^2 + c^2 - 2bc(2\cos^2 A/2 - 1) \\ &= b^2 + 2bc + c^2 - 4bc\cos^2 A/2 \\ &= (b + c)^2 - 4bc \cdot \frac{AD^2(b + c)^2}{4b^2c^2} \qquad \text{(using (i))} \\ &= (b + c)^2\left(1 - \frac{AD^2}{bc}\right), \end{aligned}$$

$$\therefore\ a = (b + c)\left(1 - \frac{AD^2}{bc}\right)^{\frac{1}{2}},$$

$c = 9,\ b = 5,\ AD = 6$

Using (i), $6 \times 14 = 2 \times 5 \times 9\cos A/2,$

$$\therefore\ \cos A/2 = 14/15 = 0{\cdot}93333,\ \therefore\ A/2 = 21^\circ\,2\tfrac{1}{2}',$$

$$\therefore\ A = 42^\circ\,5'.$$

From (ii), $a = 14\left(1 - \frac{36}{45}\right)^{\frac{1}{2}} = 14\left(\frac{1}{5}\right)^{\frac{1}{2}} = \frac{14}{\sqrt{5}} = \frac{14\sqrt{5}}{5}.$

Circles connected with the triangle ABC

Theorem. *To prove that, in any triangle ABC, using standard notation,*

$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\,\Delta}.$$

There are three cases to be considered as shown in the following diagrams.

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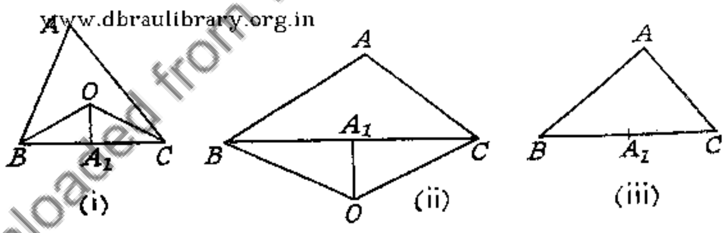

In (i) $\angle A$ is acute; in (ii) $\angle A$ is obtuse; and in (iii) $\angle A = 90^\circ$.

In each case A_1 is the mid-point of BC and in the first two cases O is the circumcentre of triangle ABC, and hence, by geometry, OA_1 is perpendicular to BC and bisects $\angle BOC$.

Also in diagram (i) $\angle BOC = 2\angle A\ \therefore\ \angle BOA_1 = \angle A$.

In (ii), reflex $\angle BOC = 2\angle A\ \therefore\ \angle BOC = 360^\circ - 2A$,

and $\angle BOA_1 = 180^\circ - A$.

From diagram (i), $BA_1 = OB\sin\angle BOA$

$$\text{i.e. } \frac{a}{2} = R\sin A,\ \therefore\ R = \frac{a}{2\sin A}.$$

From diagram (ii), $BA_1 = OB\sin\angle BOA_1$,

$$\therefore\ \tfrac{1}{2}a = R\sin(180^\circ - A) = R\sin A,$$

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$$\text{i.e. } R = \frac{a}{2 \sin A}.$$

From diagram (iii), where A_1 is the circumcentre,

$$BA_1 = R \quad \therefore \tfrac{1}{2}a = R = R \sin 90° = R \sin A,$$
$$\therefore R = a/2 \sin A,$$

$\therefore$ in all cases, $$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A}.$$

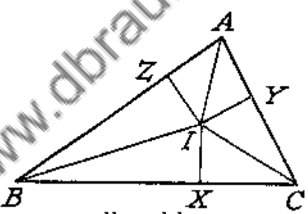
But $$\triangle = \tfrac{1}{2}bc \sin A, \quad \therefore R = \frac{a}{2 \sin A} = \frac{abc}{4\,\Delta}.$$

By the sine rule, $$\frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C},$$

$\therefore$ for any triangle ABC,

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}.$$

Theorem. *With standard notation, to find BX, CX, and AY, and to prove that $r = \triangle/s$ for any triangle ABC.*



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I is the incentre, and X, Y, Z are the points of contact of the inscribed circle with BC, CA, AB respectively. Therefore

$$IX = IY = IZ = r.$$

Area $\triangle ABC$ = area $\triangle IBC$ + area $\triangle ICA$ + area $\triangle IAB$,

$$\text{i.e. } \triangle = \tfrac{1}{2}r \,.\, BC + \tfrac{1}{2}r \,.\, CA + \tfrac{1}{2}r \,.\, AB$$
$$= \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc = (\tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2}c)r = sr,$$
$$\therefore r = \triangle/s.$$

Since the tangents from a point to a circle are equal

$$AZ = AY; \; BZ = BX; \; CX = CY.$$

$$\therefore AZ + BZ + CX = AY + BX + CY$$
$$= \tfrac{1}{2}[(AZ + BZ + CX) + (AY + BX + CY)]$$
$$= \tfrac{1}{2} \times \text{perimeter of } \triangle ABC = s,$$
$$\text{i.e. } AB + CX = AC + BX = s,$$
$$\therefore c + CX = b + BX = s,$$
$$\therefore CX = CY = s - c, \text{ and } BX = BZ = s - b.$$

Similarly, $$AZ = AY = s - a.$$

$$\therefore \left.\begin{aligned} AY &= AZ = s - a \\ BX &= BZ = s - b \\ CX &= CY = s - c \end{aligned}\right\}$$

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Theorem. *To prove that*

$$r = (s-a)\tan A/2 = (s-b)\tan B/2 = (s-c)\tan C/2,$$

and $$r = 4R\sin A/2\sin B/2\sin C/2.$$

Using the diagram and results of the previous theorem:

from triangle IAZ, $IZ = AZ\tan\angle IAZ$, i.e. $r = (s-a)\tan A/2$,
from triangle IBX, $IX = BX\tan\angle IBX$, i.e. $r = (s-b)\tan B/2$,
from triangle ICX, $IX = CX\tan\angle ICX$, i.e. $r = (s-c)\tan C/2$,
i.e. $r = (s-a)\tan A/2 = (s-b)\tan B/2 = (s-c)\tan C/2$.

Now $$BC = BX + XC = r\cot B/2 + r\cot C/2$$

$$\text{i.e. } a = r\left(\frac{\cos B/2}{\sin B/2} + \frac{\cos C/2}{\sin C/2}\right)$$

$$= r\left(\frac{\cos B/2\sin C/2 + \sin B/2\cos C/2}{\sin B/2\sin C/2}\right)$$

$$= r\frac{\sin(B+C)/2}{\sin B/2\sin C/2} = \frac{r\sin(90^\circ - A/2)}{\sin B/2\sin C/2}$$

$$= \frac{r\cos A/2}{\sin B/2\sin C/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

But $$a = 2R\sin A, \text{ since } R = \frac{a}{2\sin A}, \text{ etc.,}$$

$$\therefore\ a = 4R\sin A/2\cos A/2.$$

Using this in (1),

$$4R\sin A/2\cos A/2 = (r\cos A/2)/(\sin B/2\sin C/2),$$

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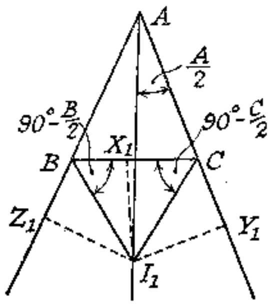
$$\therefore\ r = 4R\sin A/2\sin B/2\sin C/2.$$

Theorem. *To prove that, with standard notation,*

$$r_1 = \frac{\Delta}{s-a} = s\tan A/2 = (s-b)\cot C/2 = (s-c)\cot B/2$$

$$= 4R\sin A/2\cos B/2\cos C/2.$$



I_1 is the first ecentre and X_1, Y_1, Z_1 the points of contact of the first escribed circle with BC, CA, AB respectively as shown in the diagram.

Since I_1 is the first ecentre, $\angle I_1AC = A/2$, $\angle I_1BC = 90^\circ - B/2$, $\angle I_1CB = 90^\circ - C/2$.

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Now, area of $\triangle ABC$ = area $\triangle I_1AB$ + area $\triangle I_1CA$ − area $\triangle I_1BC$,

$$\text{i.e. } \triangle = \tfrac{1}{2}I_1Z_1 \cdot AB + \tfrac{1}{2}I_1Y_1 \cdot CA - \tfrac{1}{2}I_1X_1 \cdot BC$$
$$= \tfrac{1}{2}r_1c + \tfrac{1}{2}r_1b - \tfrac{1}{2}r_1a = r_1\frac{(c+b-a)}{2}$$
$$= r_1\frac{(a+b+c-2a)}{2} = r_1\frac{(2s-2a)}{2}$$
$$= r_1(s-a),$$
$$\therefore \triangle = r_1(s-a), \text{ i.e. } r_1 = \triangle/(s-a).$$

Since the tangents from a point to a circle are equal, $AZ_1 = AY_1$,

$$\text{i.e. } AB + BZ_1 = AC + CY_1 \ldots\ldots\ldots\ldots(1).$$

Also, $BZ_1 = BX_1$ and $CY_1 = CX_1$,

$$\therefore \text{ from (1), } AB + BX_1 = AC + CX_1$$
$$= \tfrac{1}{2}[(AB + BX_1) + (AC + CX_1)]$$
$$= \tfrac{1}{2}(AB + BC + CA) = \tfrac{1}{2} \times 2s = s.$$
$$\text{i.e. } AZ_1 = AY_1 = AB + BX_1 = AC + CX_1 = s$$
$$\therefore BX_1 = s - AB = s - c,\ CX_1 = s - AC = s - b.$$

From triangle I_1AZ_1, $I_1Z_1 = AZ_1 \tan A/2 = s \tan A/2$,

i.e. $r_1 = s \tan A/2$.

From triangle I_1BX_1, $I_1X_1 = BX_1 \tan (90^\circ - B/2) = (s - c) \cot B/2$,

i.e. $r_1 = (s - c) \cot B/2$.

From triangle I_1CX_1, $I_1X_1 = CX_1 \tan (90^\circ - C/2) = (s - b) \cot C/2$

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$\therefore r_1 = (s - b) \cot C/2$.

Thus, $r_1 = s \tan A/2 = (s - c) \cot B/2 = (s - b) \cot C/2$.

Now,

$$a = BC = BX_1 + CX_1 = I_1X_1 \tan B/2 + I_1X_1 \tan C/2$$
$$= r_1 \tan B/2 + r_1 \tan C/2$$
$$= r_1\left\{\frac{\sin B/2}{\cos B/2} + \frac{\sin C/2}{\cos C/2}\right\}$$
$$= r_1\left\{\frac{\sin B/2 \cos C/2 + \cos B/2 \sin C/2}{\cos B/2 \cos C/2}\right\}$$
$$= r_1\frac{\sin (B + C)/2}{\cos B/2 \cos C/2} = r_1\frac{\sin (90^\circ - A/2)}{\cos B/2 \cos C/2}$$
$$= r_1\frac{\cos A/2}{\cos B/2 \cos C/2}.$$

But $a = 2R \sin A = 4R \sin A/2 \cos A/2$,

$$\therefore 4R \sin A/2 \cos A/2 = r_1\frac{\cos A/2}{\cos B/2 \cos C/2},$$
$$\therefore r_1 = 4R \sin A/2 \cos B/2 \cos C/2,$$

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NOTE. The following results for r_2 and r_3 are obtained by a similar process.

$$r_2 = \frac{\Delta}{s-b} = s\tan B/2 = (s-a)\cot C/2 = (s-c)\cot A/2$$
$$= 4R\cos A/2\sin B/2\cos C/2.$$

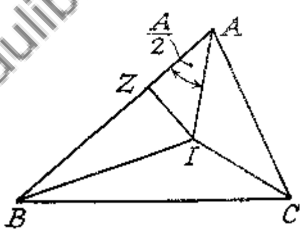
$$r_3 = \frac{\Delta}{s-c} = s\tan C/2 = (s-a)\cot B/2 = (s-b)\cot A/2$$
$$= 4R\cos A/2\cos B/2\sin C/2.$$

Also,

$BZ_2 = BX_2 = s, \quad CX_2 = CY_2 = (s-a), \quad AZ_2 = AY_2 = (s-c);$
$CX_3 = CY_3 = s, \quad BX_3 = BZ_3 = (s-a), \quad AZ_3 = AY_3 = (s-b).$

EXAMPLE (L.U.). Express in terms of the sides of the triangle the distances of its vertices from the centre of the inscribed circle, and calculate the distance from A, when $a = 50$, $b = 37$, $c = 33$.

I is the incentre and Z the point of contact of the inscribed circle with AB.



$$\angle IAZ = \tfrac{1}{2}A$$
$$\therefore AI = IZ/\sin\tfrac{1}{2}A$$
$$= r/\sin\tfrac{1}{2}A.$$

But, with standard notation,

$$= \frac{\Delta}{s} = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)},$$

and $\sin A/2 = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

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$$\therefore AI = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)} \div \sqrt{\frac{(s-b)(s-c)}{bc}}$$
$$= \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}} \times \sqrt{\frac{bc}{(s-b)(s-c)}}$$
$$= \sqrt{\frac{(s-a)bc}{s}}.$$

NOTE. This could also have been obtained by using

$AI = AZ/\cos\tfrac{1}{2}A$, with $AZ = (s-a)$, and $\cos\tfrac{1}{2}A = \sqrt{\dfrac{s(s-a)}{bc}}$.

Similarly, $BI = \sqrt{\dfrac{(s-b)ca}{s}}$, and $CI = \sqrt{\dfrac{(s-c)ab}{s}}$.

When $a = 50$, $b = 37$, $c = 33$, $2s = a + b + c = 120$, $\therefore s = 60$

$$\therefore s - a = 10.$$

Hence, $$AI = \sqrt{\frac{10 \times 37 \times 33}{60}} = \sqrt{\frac{407}{2}} = \sqrt{203{\cdot}5}$$
$$= 14{\cdot}27 \text{ to 4 significant figures.}$$

EXAMPLE (L.U.). Prove, with the usual notation, the formula for the radius of the circumcircle of a triangle:

$$R = \frac{a}{2\sin A}.$$

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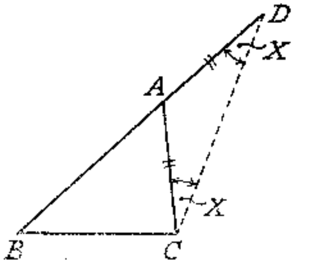
The side BA of a triangle ABC is produced through A to D, where $AD = AC$. If the radii of the circles ABC, BCD, ADC are in the ratio 5 : 8 : 7, calculate the magnitudes of the angles of the triangle.



The first part has been proved as a theorem.

Since $AD = AC$, $\angle ADC = \angle ACD$.

But $\qquad \angle A = \angle ADC + \angle ACD \quad$ (ext. $\angle$ of Δ)

$$\therefore \angle ADC = \angle ACD = A/2.$$

Let R_1, R_2, R_3 be the circum-radii of triangles ABC, BCD, ADC respectively.

$$\therefore R_1 = \frac{BC}{2\sin\angle A} = \frac{AC}{2\sin\angle B} = \frac{AB}{2\sin\angle C}$$

$$R_2 = \frac{BC}{2\sin\angle ADC} = \text{etc.}$$

$$R_3 = \frac{AC}{2\sin\angle ADC} = \text{etc.}$$

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$$\therefore \frac{R_1}{R_2} = \frac{\sin\angle ADC}{\sin\angle A} = \frac{\sin A/2}{2\sin A/2\cos A/2} = \frac{1}{2\cos A/2}$$

$$\therefore \frac{5}{8} = \frac{1}{2\cos A/2}$$

$\therefore$ 10 cos $A/2 = 8$, cos $A/2 = 0{\cdot}8$,

$\therefore A/2 = 36°\ 52'$ $\therefore A = 73°\ 44'$.

Since cos $A/2 = \frac{4}{5}$, sin $A/2 = \frac{3}{5}$ (using right-angled Δ with $\angle A$ acute).

Using $R_1 = \dfrac{AC}{2\sin B}$, $R_3 = \dfrac{AC}{2\sin A/2}$, $\therefore \dfrac{R_1}{R_3} = \dfrac{\sin A/2}{\sin B}$,

$\therefore \dfrac{5}{7} = \dfrac{3/5}{\sin B}$, $\therefore \sin B = \dfrac{21}{25} = 0{\cdot}84$,

$\therefore B = 57°\ 8\frac{1}{2}'$.

Therefore $C = 180° - A - B = 180° - 130°\ 52\frac{1}{2}' = 49°\ 7\frac{1}{2}'$.

EXAMPLE (L.U.). (i) In any triangle ABC, if R and r be the radii of the circumcircle and inscribed circle respectively, prove that,

$$Rr = abc/2(a + b + c)$$

(ii) In an isosceles triangle ABC, BC being equal to CA, the length of AB is $3\sqrt{7}$ inches, and the radius of the circumcircle is 16 inches. Find the length of BC and the radius of the inscribed circle.

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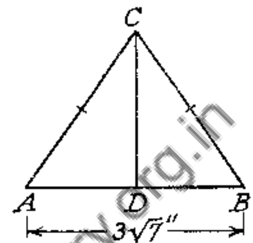
(i) In any triangle ABC, $R = \dfrac{abc}{4\triangle}$, $r = \dfrac{\triangle}{s}$

$$\therefore\ Rr = \frac{abc}{4\triangle} \times \frac{\triangle}{s} = \frac{abc}{4s} = \frac{abc}{2(a+b+c)}.$$

(ii) With standard notation, $R = \dfrac{c}{2\sin C}$.



In this case $R = 16$ inches, $c = 3\sqrt{7}$

$$\therefore\ 16 = \frac{3\sqrt{7}}{2\sin C}$$

$$\therefore\ \sin C = \frac{3\sqrt{7}}{32}.$$

$$\therefore\ \cos C = \pm\sqrt{1-\sin^2 C}$$
$$= \pm\sqrt{1-63/32^2}$$
$$= \pm\frac{1}{32}\sqrt{32^2-63} = \pm\frac{\sqrt{961}}{32} = \pm\frac{31}{32}$$

$$\therefore\ 1-2\sin^2 C/2 = \pm\frac{31}{32},\quad \therefore\ 2\sin^2 C/2 = \frac{1}{32} \text{ or } \frac{63}{32}$$

$$\therefore\ \sin^2 C/2 = \frac{1}{64} \text{ or } \frac{63}{64}.$$

Since $C/2$ must be acute, $\sin C/2$ is positive, and

$$\sin C/2 = \frac{1}{8} \text{ or } \frac{\sqrt{63}}{8}$$

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$$\therefore\ \cos C/2 = \frac{\sqrt{63}}{8} \text{ or } \frac{1}{8}.$$

(Using $\cos^2 x + \sin^2 x = 1$ with $\cos C/2$ +ve.)

Since $AC = AB = a$, from the properties of an isosceles triangle,

$$AB = 2AD = 2a\sin C/2 \quad (D \text{ mid-point of } AB),$$

i.e. $3\sqrt{7} = 2a \times \frac{1}{8}$, or $3\sqrt{7} = 2a\sqrt{63}/8$,

$$\therefore\ a = 12\sqrt{7} \text{ or } 4.$$

When $a = b = 12\sqrt{7}$,

$$s = \tfrac{1}{2}(24\sqrt{7} + 3\sqrt{7}) = \tfrac{27}{2}\sqrt{7},\quad (s-a) = (s-b) = \tfrac{3}{2}\sqrt{7},$$
and $(s-c) = \frac{21}{2}\sqrt{7}$.

$$\therefore\ \triangle = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(\tfrac{27}{2}\sqrt{7})(\tfrac{3}{2}\sqrt{7})(\tfrac{3}{2}\sqrt{7})(\tfrac{21}{2}\sqrt{7})}$$

$$= \tfrac{7}{4}\sqrt{(27\times 3\times 3\times 21)} = \frac{27\times 7}{4}\sqrt{7}.$$

$$\therefore\ r = \frac{\triangle}{s} = \frac{27\times 7}{4}\sqrt{7} \div \frac{27}{2}\sqrt{7} = \frac{7}{2}.$$

When $a = b = 4$

$$s = \tfrac{1}{2}(8+3\sqrt{7}),\ (s-a) = (s-b) = \tfrac{3}{2}\sqrt{7},\ \text{and } (s-c) = \tfrac{1}{2}(8-3\sqrt{7})$$
$$\therefore\ \triangle = \sqrt{\tfrac{1}{2}(8+3\sqrt{7}) \times (\tfrac{3}{2}\sqrt{7}) \times (\tfrac{3}{2}\sqrt{7}) \times \tfrac{1}{2}(8-3\sqrt{7})}$$
$$= \tfrac{3}{4}\sqrt{7}\sqrt{(8+3\sqrt{7})(8-3\sqrt{7})} = \tfrac{3}{4}\sqrt{7}\sqrt{(64-63)} = \tfrac{3}{4}\sqrt{7}.$$

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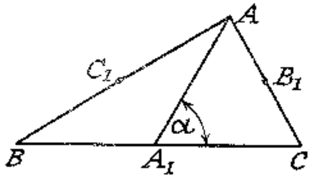
$$\therefore\ r = \frac{\Delta}{s} = \frac{\frac{3}{4}\sqrt{7}}{\frac{1}{2}(8 + 3\sqrt{7})}$$

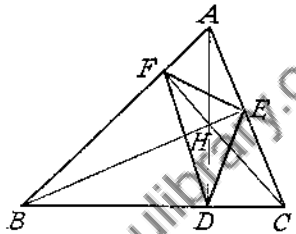
$$= \tfrac{3}{2}(\sqrt{7})\frac{(8 - 3\sqrt{7})}{(8 + 3\sqrt{7})(8 - 3\sqrt{7})} = \tfrac{3}{2}\frac{(\sqrt{7})(8 - 3\sqrt{7})}{64 - 63}$$

$$= \tfrac{3}{2}(8\sqrt{7} - 21).$$

Theorem. *To find the lengths of the medians of a triangle ABC.*

A_1, B_1, C_1 are the mid-points of BC, CA, AB respectively, and AA_1, BB_1, CC_1 will be medians. (Left-hand figure.)





Let $\angle AA_1C = \alpha$, $\therefore\ \angle AA_1B = 180° - \alpha$.

In the triangle AA_1B by the cosine rule,

$$AB^2 = AA_1^2 + BA_1^2 - 2AA_1 \,.\, BA_1 \cos(180° - \alpha)$$
$$\text{i.e. } c^2 = AA_1^2 + \tfrac{1}{4}a^2 + a \,.\, AA_1 \cos\alpha \ldots\ldots\ldots\ldots\ldots\ (1)$$

Using the cosine rule on triangle AA_1C, www.dbraulibrary.org.in

$$b^2 = AA_1^2 + \tfrac{1}{4}a^2 - a \,.\, AA_1 \cos\alpha \ldots\ldots\ldots\ldots (2)$$

(1) + (2) gives, $\quad b^2 + c^2 = 2AA_1^2 + \tfrac{1}{2}a^2$

$$\therefore\ 2AA_1^2 = b^2 + c^2 - \tfrac{1}{2}a^2$$

$$\therefore\ AA_1^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

Similarly,
$$\left.\begin{aligned} \therefore\ AA_1 &= \tfrac{1}{2}\sqrt{(2b^2 + 2c^2 - a^2)} \\ BB_1 &= \tfrac{1}{2}\sqrt{(2c^2 + 2a^2 - b^2)} \\ CC_1 &= \tfrac{1}{2}\sqrt{(2a^2 + 2b^2 - c^2)} \end{aligned}\right\}.$$

NOTE. These results are very important and should be memorised.

EXAMPLE (L.U.). If D, E, F are the feet of the altitudes of an acute-angled triangle ABC, express the sides and angles of the triangle DEF in terms of the sides and angles of the triangle ABC. (Right-hand figure.)

Show that, if R denotes the radius of the circle ABC, the perimeter of the triangle DEF is equal to $4R \sin A \sin B \sin C$.

Let the altitudes meet at H the orthocentre.

Since $\angle HDC = \angle HEC = 90°$, H, D, C, E are concyclic.

$$\therefore\ \angle HDE = \angle HCE \text{ (same segment)}$$
$$= 90° - \angle A.$$

Since $\angle HDB = \angle HFB = 90°$, H, F, B, D are concyclic.

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$$\therefore\ \angle HDF = \angle HBE \text{ (same segment)}$$
$$= 90° - A$$

$$\left.\begin{aligned} \therefore\ \angle FDE = \angle HDE + \angle HDF &= 180° - 2A \\ \text{Similarly,}\qquad \angle DEF &= 180° - 2B \\ \angle EFD &= 180° - 2C \end{aligned}\right\}.$$

NOTE. H is the incentre of the pedal triangle DEF.

$\angle BEC = \angle CFB = 90°$, therefore $FBCE$ is a cyclic quadrilateral.

$\therefore\ \angle AFE = \angle C$ (exterior $\angle$ of cyclic quadrilateral).

Now $AE = AB\cos A = c\cos A$.

Using the sine rule on triangle AFE,

$$\frac{FE}{\sin A} = \frac{AE}{\sin \angle AFE} = \frac{c\cos A}{\sin C}$$
$$= \frac{a}{\sin A}\,.\,\cos A \quad \left(\frac{c}{\sin C} = \frac{a}{\sin A}\right)$$

$$\left.\begin{aligned} \therefore\ FE &= a\cos A \\ \text{Similarly,}\qquad FD &= b\cos B \\ DE &= c\cos C \end{aligned}\right\}.$$

NOTE. Similar triangles AFE, ABC could be used to prove $FE = a\cos A$, for $EF/BC = AE/AB = \cos A$, therefore $EF = BC\cos A = a\cos A$.

Perimeter of triangle $DEF = EF + FD + DE$

$$= a\cos A + b\cos B + c\cos C$$
$$= 2R\sin A\cos A + 2R\sin B\cos B + 2R\sin C\cos C$$

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$$\left(\text{since } R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}\right)$$

$$= R[(\sin 2A + \sin 2B) + \sin 2C]$$
$$= R[2\sin(A + B)\cos(A - B) + 2\sin C\cos C]$$
$$= 2R[\sin C\cos(A - B) + \sin C\cos(180° - A - B)]$$
$$(A + B + C = 180°)$$
$$= 2R\sin C[\cos(A - B) - \cos(A + B)]$$
$$= 2R\sin C \times 2\sin A\sin B = 4R\sin A\sin B\sin C.$$

Quadrilaterals. Certain problems on quadrilaterals can be dealt with by division into two triangles and then by using the properties of triangles as shown in the following examples.

EXAMPLE (L.U.). $ABCD$ is a quadrilateral in which $AB = BC = CD = a$, $\angle ABC = \theta$, $\angle BCD = \phi$. (Left-hand figure on next page.)

Prove that, $AD = a[1 - 8\sin\frac{1}{2}\theta\sin\frac{1}{2}\phi\cos\frac{1}{2}(\theta + \phi)]^{\frac{1}{2}}$, and that the area of the quadrilateral is $2a^2\sin\frac{1}{2}\theta\sin\frac{1}{2}\phi\sin\frac{1}{2}(\theta + \phi)$.

Join AC, then, $\angle BAC = \angle BCA = 90° - \frac{1}{2}\theta$ (isosceles Δ), and $AC = 2a\sin\frac{1}{2}\theta$.

(If BE is the perpendicular on AC then E is the mid-point of AC and BE bisects $\angle ABC$.)

$$\angle ACD = \phi - (90° - \tfrac{1}{2}\theta)$$
$$= \phi + \tfrac{1}{2}\theta - 90°.$$

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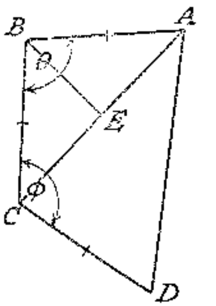
Using the cosine rule on triangle ACD,

$$AD^2 = AC^2 + CD^2 - 2AC \cdot CD \cdot \cos(\varphi + \tfrac{1}{2}\theta - 90^\circ)$$
$$= 4a^2 \sin^2 \tfrac{1}{2}\theta + a^2 - 4a^2 \sin \tfrac{1}{2}\varphi \cos(\varphi + \tfrac{1}{2}\theta - 90^\circ)$$
$$= a^2[1 - 4 \sin \tfrac{1}{2}\theta \cos(90^\circ - \varphi - \tfrac{1}{2}\theta) + 4 \sin^2 \tfrac{1}{2}\theta] \qquad (\cos(-\theta) = \cos\theta)$$
$$= a^2[1 - 4 \sin \tfrac{1}{2}\theta\, [\sin(\varphi + \tfrac{1}{2}\theta) - \sin \tfrac{1}{2}\theta]] \qquad (\cos(90^\circ - x) = \sin x)$$
$$= a^2[1 - 8 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}(\theta + \varphi) \sin \tfrac{1}{2}\varphi]$$
$$\therefore AD = a[1 - 8 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\varphi \cos \tfrac{1}{2}(\theta + \varphi)]^{\frac{1}{2}}.$$

The area of the quadrilateral = area $\triangle ABC$ + area $\triangle ACD$

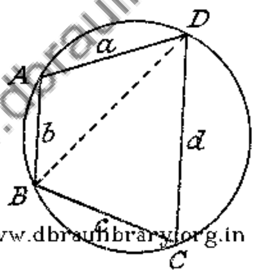
$$= \tfrac{1}{2}a^2 \sin\theta + \tfrac{1}{2}AC \cdot CD \sin \angle ACD$$
$$= \tfrac{1}{2}a^2 \sin\theta + a \sin \tfrac{1}{2}\theta \cdot a \sin(\varphi + \tfrac{1}{2}\theta - 90^\circ)$$
$$= a^2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta - a^2 \sin \tfrac{1}{2}\theta \cos(\varphi + \tfrac{1}{2}\theta) \qquad (\sin(x - 90^\circ) = -\sin(90^\circ - x) = -\cos x)$$
$$= a^2 \sin \tfrac{1}{2}\theta\, [\cos \tfrac{1}{2}\theta - \cos(\varphi + \tfrac{1}{2}\theta)]$$
$$= a^2 \sin \tfrac{1}{2}\theta \times 2 \sin \tfrac{1}{2}\varphi \sin \tfrac{1}{2}(\varphi + \theta)$$
$$= 2a^2 \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\varphi \sin \tfrac{1}{2}(\theta + \varphi).$$



EXAMPLE (L.U.). $ABCD$ is a cyclic quadrilateral in which

$DA = a$, $AB = b$,
$BC = c$, $CD = d$.

The area of the quadrilateral is S. (Right-hand figure.)



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Prove that (i) $S = \tfrac{1}{2}(ab + cd) \sin A$,

(ii) $\cos A = (a^2 + b^2 - c^2 - d^2)/2(ab + cd)$,

(iii) $S^2 = (s - a)(s - b)(s - c)(s - d)$,

where $2s = a + b + c + d$.

(i) Area of quadrilateral $ABCD$

$$= \text{area } \triangle ABD + \text{area } \triangle BCD$$
$$= \tfrac{1}{2}ab \sin A + \tfrac{1}{2}cd \sin C$$
$$= \tfrac{1}{2}ab \sin A + \tfrac{1}{2}cd \sin(180^\circ - A)$$
$$(A + C = 180^\circ \text{ since } A, B, C, D \text{ cyclic})$$
$$= \tfrac{1}{2}ab \sin A + \tfrac{1}{2}cd \sin A$$
$$= \tfrac{1}{2}(ab + cd) \sin A.$$

(ii) Using the cosine rule on

(a) triangle ABD,

$$BD^2 = a^2 + b^2 - 2ab \cos A \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(b) triangle BCD,

$$BD^2 = c^2 + d^2 - 2cd \cos C$$
$$= c^2 + d^2 - 2cd \cos(180^\circ - A)$$
$$= c^2 + d^2 + 2cd \cos A \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

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From (1) and (2),

$$a^2 + b^2 - 2ab\cos A = c^2 + d^2 + 2cd\cos A$$
$$\therefore\ a^2 + b^2 - c^2 - d^2 = 2ab\cos A + 2cd\cos A$$
$$= 2\cos A\,.\,(ab + cd)$$
$$\therefore\ \cos A = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

(iii) From (i),

$$S^2 = \tfrac{1}{4}(ab + cd)^2 \sin^2 A = \tfrac{1}{4}(ab + cd)^2(1 - \cos^2 A)$$
$$= \tfrac{1}{4}(ab + cd)^2\left\{1 - \frac{(a^2 + b^2 - c^2 - d^2)^2}{4(ab + cd)^2}\right\} \quad \text{(using value for } \cos A)$$
$$= \tfrac{1}{16}[(2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2]$$
$$= \tfrac{1}{16}[(2ab + 2cd + a^2 + b^2 - c^2 - d^2)(2ab + 2cd - a^2 - b^2 + c^2 + d^2)]$$
$$= \tfrac{1}{16}[(a + b)^2 - (c - d)^2][(c + d)^2 - (a - b)^2]$$
$$= \tfrac{1}{16}[(a + b + c - d)\,.\,(a + b - c + d)(a - b + c + d)(-a + b + c + d)]$$
$$= \tfrac{1}{16}(2s - 2d)(2s - 2c)(2s - 2b)(2s - 2a)$$
$$= (s - a)(s - b)(s - c)(s - d).$$

Regular Polygons. A *regular polygon* is a polygon with all its sides equal and all its angles equal.

The sum of the angles of any polygon of n sides is $(2n - 4)$ right angles, and therefore the interior angle of a regular polygon of n sides is $(2n - 4)/n$ right angles.

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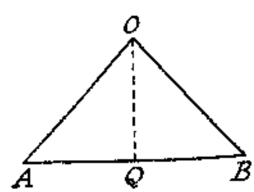
It is sometimes advisable to use the fact that the sum of the exterior angles of any polygon, with its sides produced in order, is equal to four right angles or 360°.

In the case of any regular polygon, it can be proved that the bisectors of the interior angles of the polygon are concurrent, and meet in a point which is equidistant from both the vertices and sides of the regular polygon. This point is therefore the common centre of the circumscribed and inscribed circles of the polygon.

Theorem. *To find the lengths of the radii of the inscribed and circumscribed circles of a regular polygon of n sides in terms of the side a of the polygon.*

Let R and r be the radii of the circumscribed and inscribed circles respectively. AB is a side of the polygon and O the common centre of the two circles. OQ is the perpendicular from O on AB, and, by geometry, OQ bisects $\angle AOB$ and also bisects AB.



$$OA = R,\ OQ = r, \text{ and } AQ = \tfrac{1}{2}a,$$
$$\angle AOB = 2\pi/n\ \therefore\ \angle AOQ = \pi/n.$$

From the right-angled triangle AQO,

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$$\left.\begin{aligned} OA &= \frac{AQ}{\sin \angle AOQ}, \quad \therefore \quad R = \frac{a}{2\sin \pi/n} \\ \text{Also} \quad OQ &= AQ \cot \angle AOQ, \quad \therefore \quad r = \frac{a}{2}\cot\frac{\pi}{n} \end{aligned}\right\}.$$

Theorem. *To find the area of a regular polygon of n sides* (i) *in terms of a side a,* (ii) *in terms of r the radius of the inscribed circle,* (iii) *in terms of R the radius of the circumcircle.*

Using the diagram of the previous theorem,

(i) Area $\triangle AOB = \frac{1}{2}AB \,.\, OQ = \frac{1}{2}a \,.\, \frac{a}{2}\cot\frac{\pi}{n}$

$= \frac{a^2}{4}\cot\frac{\pi}{n}.$

But the area of the polygon $=$ n $\times$ area of triangle AOB

$= n\frac{a^2}{4}\cot\frac{\pi}{n}.$

(ii) Area of the polygon $= n \times$ area $\triangle AOB$
$= n \times OQ \times AQ$
$= n \times r \times r\tan\pi/n$
$= nr^2\tan\pi/n.$

(iii) Area of the polygon $= n \times$ area $\triangle AOB = n \,.\, OQ \,.\, AQ$
$= n \,.\, R\cos\pi/n \times R\sin\pi/n$
$= nR^2\sin\pi/n\cos\pi/n$

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$= n\frac{R^2}{2}\sin\frac{2\pi}{n}.$

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EXAMPLE (L.U.). Prove that, if C be the area of a regular hexagon circumscribed about a circle, and I the area of a similar inscribed hexagon, the value of $\sqrt[3]{(C^2I)}$ differs from that of the area of the circle by about 0·2 per cent.

Let R be the radius of the circle.
By the previous theorem,

$$C = nR^2\tan\pi/n, \text{ where } n = 6$$
$$= 6R^2\tan 30° = 6R^2 \times 1/\sqrt{3} = 2\sqrt{3}R^2.$$

Also, $$I = \frac{nR^2}{2}\sin\frac{2\pi}{n}, \text{ where } n = 6$$

$$= 3R^2\sin 60° = \frac{3\sqrt{3}}{2}R^2.$$

$$\therefore \sqrt[3]{(C^2I)} = \sqrt[3]{\left\{(2\sqrt{3}R^2)^2 . \frac{3\sqrt{3}}{2}R^2\right\}} = \sqrt[3]{\left(12R^4 \times \frac{3\sqrt{3}}{2}R^2\right(}$$
$$= \sqrt[3]{(18\sqrt{3}R^6)} = (\sqrt[6]{972})R^2.$$

Now $\frac{1}{6}\log 972 = \frac{1}{6} \times 2{\cdot}98767 = 0{\cdot}49795$

$\therefore \sqrt[6]{972} = 3{\cdot}147$ to 4 significant figures

Area of circle $= \pi R^2 = 3{\cdot}142R^2$. Therefore value of $\sqrt[3]{(C^2 I)}$ differs approximately from the area of the circle by

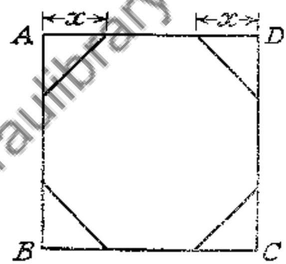
$$\frac{0{\cdot}005R^2}{3{\cdot}142R^2} \times 100 \text{ per cent.}$$

$$= \frac{0{\cdot}5}{3{\cdot}14} \text{ per cent} = 0{\cdot}2 \text{ per cent.}$$

EXAMPLE (L.U.). From the corners of the square *ABCD*, equal isosceles triangles are cut off so as to leave a regular octagon. Prove that the area cut off is $(3 - 2\sqrt{2})$ times the area of *ABCD*.

Compare the perimeter of the octagon with the perimeter of the square *ABCD*.

Let x be a side of one of the isosceles right-angled triangles removed, and a the side of the square. Therefore side of regular octagon $= a - 2x$, and also $= \sqrt{2}x$ (hypotenuse of isosceles triangle).



$$\therefore\ a - 2x = (\sqrt{2})x.$$

$$\therefore\ x(2 + \sqrt{2}) = a.$$

$$\therefore\ x = \frac{a}{2 + \sqrt{2}} = \frac{a(2 - \sqrt{2})}{4 - 2}$$

$$= \frac{a(2 - \sqrt{2})}{2}.$$

Area cut off $= 4 \times \frac{1}{2}x^2 = 2x^2$

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$$= 2\left\{\frac{a(2 - \sqrt{2})}{2}\right\}^2 = \frac{a^2}{2}(4 - 4\sqrt{2} + 2)$$

$$= \frac{a^2}{2}(6 - 4\sqrt{2})$$

$$= a^2(3 - 2\sqrt{2}) = (3 - 2\sqrt{2}) \times \text{area of square } ABCD.$$

Perimeter of octagon $= 8x\sqrt{2} = 4a\,(2 - \sqrt{2})\,\sqrt{2}$.

Perimeter of square $= 4a$.

Therefore ratio of perimeter of octagon to perimeter of square

$$= (2 - \sqrt{2})\sqrt{2} : 1.$$

Further Problems on Heights and Distances. Certain problems on heights and distances could not be dealt with earlier, as they require the use of the sine, cosine rule, etc., and compound angles. Some examples of these will now be illustrated

EXAMPLE (L.U.). An aeroplane is observed simultaneously from two points *A* and *B*, at the same level, *A* being at a distance *c* due north of *B*. From *A* the bearing of the aeroplane is θ east of south at an elevation α, and from *B* the bearing is φ east of north.

Show that the aeroplane is at a height

$$\frac{c \tan \alpha \sin \varphi}{\sin(\theta + \varphi)}$$

and find its elevation from *B*.

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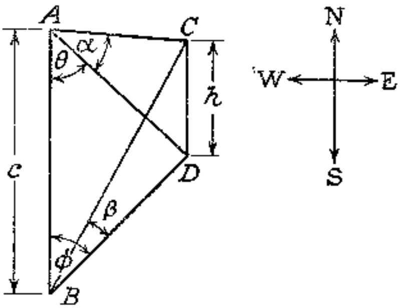
Let C be the position of the aeroplane, and $CD = h$ be its height above the ground, with β its elevation as seen from B.



Now, $AD = h \cot \alpha$, $BD = h \cot \beta$.
Using the sine rule on triangle ABD,

$$\frac{AB}{\sin(180° - \theta - \varphi)} = \frac{AD}{\sin \varphi} = \frac{BD}{\sin \theta},$$

$$\text{i.e. } \frac{c}{\sin(\theta + \varphi)} = \frac{h \cot \alpha}{\sin \varphi} = \frac{h \cot \beta}{\sin \theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\therefore\ h \cot \alpha = \frac{c \sin \varphi}{\sin(\theta + \varphi)}, \quad \therefore\ h = \frac{c \tan \alpha \sin \varphi}{\sin(\theta + \varphi)}$$

Also from (1), $$h \cot \beta = \frac{h \cot \alpha \sin \theta}{\sin \varphi}$$

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$$\therefore\ \cot \beta = \frac{\cot \alpha \sin \theta}{\sin \varphi}$$

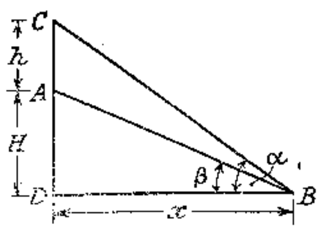
$$\text{i.e. } \tan \beta = \frac{\sin \varphi \tan \alpha}{\sin \theta}$$

$$\therefore\ \beta = \tan^{-1}\left(\frac{\sin \varphi \tan \alpha}{\sin \theta}\right).$$

EXAMPLE (L.U.). A tower of height h stands on the top of a cliff of height H. An observer in a boat finds that the elevation of the top of the tower is α, and that of the top of the cliff is β. Prove that

$$\frac{H}{h} = \frac{\cos \alpha \sin \beta}{\sin(\alpha - \beta)}.$$

If $\alpha = 32°$, $\beta = 23°$, $h = 60$ feet, find H and the distance of the boat from the foot of the cliff.



Let B be the boat, A the top of the cliff, C the top of the tower, and D the point on the cliff on the same level as B.

Let $BD = x$.

Using right-angled triangles,

$$H + h = x \tan \alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$
$$H = x \tan \beta \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

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(1) ÷ (2) gives, $\dfrac{H+h}{H} = \dfrac{\tan\alpha}{\tan\beta}$, $\therefore (H+h)\tan\beta = H\tan\alpha$,

$\therefore H(\tan\alpha - \tan\beta) = h\tan\beta$,

$$\therefore \frac{H}{h} = \frac{\tan\beta}{\tan\alpha - \tan\beta} = \frac{\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}$$

$$= \frac{\sin\beta\cos\alpha}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

(multiplying numerator and denominator by $\cos\alpha\cos\beta$)

$$= \frac{\cos\alpha\sin\beta}{\sin(\alpha - \beta)}.$$

Aliter. From right-angled triangle ADB, $AB = H\operatorname{cosec}\beta$.

Using the sine rule on triangle ABC,

$$\frac{AB}{\sin\angle ACB} = \frac{AC}{\sin\angle ABC},$$

$$\text{i.e. } \frac{H\operatorname{cosec}\beta}{\sin(90^\circ - \alpha)} = \frac{h}{\sin(\alpha-\beta)},$$

$$\therefore \frac{H}{\sin\beta\cos\alpha} = \frac{h}{\sin(\alpha-\beta)},$$

$$\therefore \frac{H}{h} = \frac{\sin\beta\cos\alpha}{\sin(\alpha-\beta)}.$$

If $\alpha = 32^\circ$, $\beta = 23^\circ$, $h = 60$ feet,

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$$\frac{H}{60} = \frac{\cos 32^\circ \sin 23^\circ}{\sin 9^\circ},$$

$$\therefore H = \frac{60\cos 32^\circ \sin 23^\circ}{\sin 9^\circ},$$

$$\therefore \log H = \log 60 + \log(\cos 32^\circ) + \log(\sin 23^\circ) - \log(\sin 9^\circ)$$

$$= 2{\cdot}10412,$$

$$\therefore H = 127{\cdot}1 \text{ feet.}$$

log 60 =	1·77815
log (cos 32°) =	$\bar{1}$·92842
log (sin 23°) =	$\bar{1}$·59188
	1·29845
log (sin 9°) =	$\bar{1}$·19433
	2·10412

From (2), $x = H\cot\beta$.

$$\log x = \log H + \log(\cot 23^\circ) = 2{\cdot}10412 + 0{\cdot}37215$$

$$= 2{\cdot}47627,$$

$$\therefore x = 299{\cdot}4 \text{ feet.}$$

Example (L.U.). A straight pole makes an angle θ with the horizontal ground. From a point a distance a due W. of its foot the elevation of its top is α, and from a point at a distance a due E. of its foot the elevation of its top is β.

Show that the length of the pole is

$$\sqrt{2}a\operatorname{cosec}\theta/(\cot^2\alpha + \cot^2\beta - 2\cot^2\theta)^{\frac{1}{2}}.$$

Let C be the foot of the pole, and D the top of it.

A and B are at distances a from C due W. and E. respectively of C, and E is the foot of the perpendicular from D on the horizontal ground.

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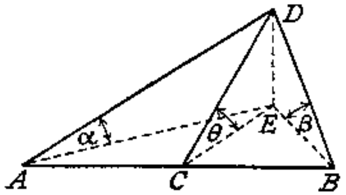
Let $DE = h$, $\therefore$ $AE = h \cot \alpha$, $BE = h \cot \beta$, $CE = h \cot \theta$.



As proved in the theorem on medians of a triangle,

$$2CE^2 = AE^2 + EB^2 - 2AC^2,$$

$$\text{i.e. } 2h^2 \cot^2 \theta = h^2 \cot^2 \alpha + h^2 \cot^2 \beta - 2a^2,$$

$$\therefore\ 2a^2 = h^2(\cot^2 \alpha + \cot^2 \beta - 2 \cot^2 \theta),$$

$$\therefore\ h^2 = \frac{2a^2}{\cot^2 \alpha + \cot^2 \beta - 2 \cot^2 \theta},$$

$$\therefore\ h = \frac{a\sqrt{2}}{(\cot^2 \alpha + \cot^2 \beta - 2 \cot^2 \theta)^{\frac{1}{2}}}.$$

$$CD = \frac{DE}{\sin \theta} = h \operatorname{cosec} \theta$$

$$= \frac{a\sqrt{2} \,.\, \operatorname{cosec} \theta}{(\cot^2 \alpha + \cot^2 \beta - 2 \cot^2 \theta)^{\frac{1}{2}}}.$$

EXAMPLES VII

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1. Prove that, in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

If $a = 8{\cdot}35$, $b = 11{\cdot}65$, $\angle A = 34° \, 36'$, show that there are two possible values of the side c, and find these values.

2. In a triangle ABC, prove that $a^2 = (b - c)^2 + 4\, bc \sin^2 A/2$ and hence that $a = (b - c) \sec \varphi$, where

$$\tan \varphi = \frac{2\sqrt{(bc)} \,.\, \sin A/2}{b - c}.$$

Calculate φ and a when $b = 17{\cdot}5$ cm., $c = 9{\cdot}8$ cm., $\angle A = 68° \, 36'$.

3. In a triangle ABC, prove that $a \cos \frac{1}{2}(B - C) = (b + c) \sin \frac{1}{2}A$.

The sum of two sides of a triangle is 33·7 cm. and the included angle is 56° 24′. Calculate the remaining angles when the third side is 16·3 cm.

4. If E be the middle point of the side CA of triangle ABC, and if S be the area of the triangle, prove that $\cos \angle AEB = (BC^2 - BA^2)/4S$.

Calculate $\angle ABC$, given that $BC = 30{\cdot}2$, $BA = 16{\cdot}8$, $\angle AEB = 42° \, 33'$.

5. Prove that, in any triangle ABC,

(i) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$, (ii) $\dfrac{\sin (A - B)}{\sin (A + B)} = \dfrac{a^2 - b^2}{c^2}$.

6. In a triangle ABC, $a = 18{\cdot}9$ inches, $b = 12{\cdot}2$ inches, $A - B = 30°$. Find the values of A and c.

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7. If in a triangle two sides and the angle opposite one are known, determine the conditions that there may be (i) two triangles, (ii) only one triangle, having the given elements.

If $A = 31°\,42'$, $a = 7$, $b = 5$, show that there is only one triangle and find the remaining side and angles.

8. The sides a, b, and the angle A of a triangle ABC are given. Discuss the conditions that it should be possible to draw one and only one triangle to correspond with these data.

A man walking on a straight road in a direction 25° E. of N. sees due W. of him the top of a hill three miles away at an elevation of 14°. Later he sees it at an elevation of 9° 30′. Find through what distance he has walked during the interval.

9. A chord of a circle of radius R subtends an angle α at a point on the circumference. Show that the length of the chord is $2R \sin \alpha$.

A, B, C, D are four points on a circle and $AB = 3{\cdot}5$ feet. If AB subtends an angle 37° 18′ at C and CD subtends an angle 54° 39′ at B, calculate the length of CD.

10. Prove that, if a, b, c are the sides of a triangle and s the semi-perimeter, the area of the triangle is $\sqrt{s(s - a)(s - b)(s - c)}$.

The lengths of the sides BC, CA, AB of a triangle are 13, 14, 15 feet respectively. If X, Y, Z are the mid-points of BC, CA, AB, find the lengths of AX, BY, CZ.

Determine the area of the triangle ABC and of a triangle whose sides are equal to AX, BY, CZ, and find the ratio of these areas.

11. The lengths of the sides of a triangle are given by $a = (p^2 + q^2)r$, $b = (p^2 + r^2)q$, $c = (p^2 - qr)(q + r)$.

Prove that the area of the triangle is $pqr(p^2 - qr)(q + r)$, and that the radius of the circumcircle is

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$$\frac{1}{4p}(p^2 + q^2)(p^2 + r^2).$$

12. Four equal circles of radii 1 inch have their centres at the corners A, B, C, D of a square of side 2 inches.

Find the radius of the circle whose centre is at O, the centre of the square, touching the other four circles externally.

Find also the area of the curvilinear triangle bounded by the arcs of the circles centres A, B, and O.

13. ABC is a triangle right-angled at A. Give a geometrical construction for an equilateral triangle XYZ having the vertex X on BC, Y on CA, Z on AB, and the side XY parallel to AB.

If the length of each side of the triangle XYZ is x, show that

$$x = bc/(b + c \sin 60°),$$

where $b = CA$, $c = AB$.

14. Two circles of radii 7 inches and 1 inch have their centres 10 inches apart. Show that each external common tangent is of length 8 inches, and makes an angle $\sin^{-1} \frac{3}{5}$ with the line of centres.

Calculate the area bounded by the two external common tangents and by the arcs of the circles that cut the line of centres internally.

15. (i) Prove that

$$\frac{\cos^2 X}{1 - \tan^2 X} + \frac{\sin^2 X}{1 - \cot^2 X} = 1.$$

(ii) If $\sec A - \tan A = x$, prove that $\tan \frac{1}{2}A = \dfrac{1 - x}{1 + x}$

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(iii) Prove, in the usual notation for a triangle, that if

$$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13},$$

then $$\frac{\sin A}{7} = \frac{\sin B}{6} = \frac{\sin C}{5}, \text{ and } \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}.$$

16. Calculate the angles and area of a triangle whose sides are 18·32 cm., 10·15 cm., and 13·47 cm.

17. In a triangle ABC in which $\angle A$ is obtuse, prove that $a/\sin A = b/\sin B$.

Two ships leave port at the same time, the first steaming N. 66° E. at 14 miles per hour and the second N. 47° W. After $1\frac{1}{2}$ hours the first ship reaches A and the second B. If $AB = 34$ miles, find the average speed of the second ship and the bearing of B from A.

18. Prove that, in any triangle, with the usual notation,

(i) $\sin \frac{1}{2}A = [(s-b)(s-c)/bc]^{\frac{1}{2}}$,

(ii) the area $= [s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$.

The sides of a triangle having lengths 13, 14, 15 inches, find the distance of the centre of the inscribed circle from the angular point opposite the shortest side.

19. Prove that, in any triangle ABC,

$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c} \tan \tfrac{1}{2}(B+C).$$

In a triangle ABC, $b = 6·27$, $c = 4·32$, $A = 51°$. Calculate the angles B and C and the length of the internal bisector BD of the angle ABC.

20. A piece of wire is bent into three parts AB, BC, CD, each of the outer parts being at right angles to the plane which contains the other two parts.

If $AB = CD = 2BC = 2a$, calculate (i) the distance from A to D, (ii) the volume of the tetrahedron $ABCD$.

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21. A straight line through the vertex C of a triangle ABC, which cuts AB internally, makes angles θ and φ with the sides CB and CA respectively, and p, q are the lengths of the perpendiculars from A, B respectively on the straight line. Prove that the area of the triangle is $\frac{1}{2}(ap\cos\theta + bq\cos\varphi)$.

Show further that $a^2p^2 + b^2q^2 + 2abpq\cos C = a^2b^2\sin^2 C$.

22. Prove that the area of a triangle of sides a, b, c is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a + b + c$.

Find the lengths of the three perpendiculars from the vertices to the sides of a triangle whose sides are 7, 9, and 12 feet.

23. $ABCDE$ is a pentagon circumscribed to a circle of radius r. The length of the side CD, opposite A, is a; of DE, opposite B, is b; of EA, opposite C, is c; of AB, opposite D, is d; of BC, opposite E, is e.

Prove that $$\tan \tfrac{1}{2}A = \frac{r}{s-b-e}, \quad \tan \tfrac{1}{2}B = \frac{r}{s-c-a},$$

where $2s = a+b+c+d+e$.

Show further that $a \sin \frac{1}{2}C \sin \frac{1}{2}D = r \sin \frac{1}{2}(C+D)$.

24. Prove that, in a triangle ABC, with the usual notation,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

Calculate, to the nearest minute, the angles of the triangle whose sides are 2·73, 3·49, 5·17.

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25. If ABC be a triangle with sides a, b, c, and θ is any angle, prove that $a\cos\theta = b\cos(C+\theta) + c\cos(B-\theta)$.

Prove that (i) $a\cos\frac{1}{2}(B-C) = (b+c)\sin\frac{1}{2}A$.
(ii) $a\sin\frac{1}{2}(B-C) = (b-c)\cos\frac{1}{2}A$.

26. The area of a triangle ABC is 24·67 sq. cm., and the sides b, c are 6·372 cm. and 8·105 cm. respectively. Calculate the angle A.

27. D, E, F are points in the sides BC, CA, AB respectively of the triangle ABC, such that $BD:DC = CE:EA = AF:FB = 12$.

Prove that the ratio of the area of the triangle DEF to the area of the triangle ABC is $1:3$.

28. Prove that, in a triangle of sides a, b, c and angles A, B, C, the radius R of the circumcircle is equal to $a/2\sin A$.

If r be the radius of the incircle, prove that

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = \frac{1}{2Rr}.$$

29. Prove that the area of the triangle whose sides are a, b, c is

$$[s(s-a)(s-b)(s-c)]^{\frac{1}{2}},$$

where $2s = a + b + c$.

If p_1, p_2, p_3 are the perpendiculars from the vertices of a triangle to the opposite sides, and r is the radius of the inscribed circle, prove that

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}.$$

30. Prove that, in any triangle, $\cos A = (b^2 + c^2 - a^2)/2bc$.

A and B are the centres of two circles of radii a and b which meet in C, and θ is the angle ACB. Prove that the length of the common tangent of the two circles is $2\sqrt{(ab)}\sin\frac{1}{2}\theta$.

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31. A, B, C are three points on a straight railway, such that $AB = BC$, and D is a point on one side. The bearings of D are: from A due west, from B 20° south of west, and from C 24° south of west. Find the direction of the railway line.

32. Find the remaining angles and side of each of the triangles in which $A = 40°$, $a = 6·7$ inches, $c = 8·5$ inches.

33. Prove that, with the usual notation for a triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

If the internal bisector of the angle A of a triangle ABC meets BC at D, show that $AD = 2bc\cos\frac{1}{2}A/(b+c)$, and obtain a corresponding expression for the length of the external bisector of $\angle A$.

34. Obtain expressions for $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ in terms of the lengths of the sides of a triangle ABC.

If a triangle be such that $2b = a + c$, prove that

$$2\cot\tfrac{1}{2}B = \cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C.$$

35. Find expressions for the areas of the regular polygons of n sides (a) circumscribed about, and (b) inscribed in a circle of radius r.

Taking $n = 12$ and denoting by A the area of the circumscribing regular polygon, and by a that of the inscribed regular polygon, prove that $\frac{1}{3}(2A + a)$ exceeds the area of the circle by less than 0·07 per cent.

36. Express the radius of the inscribed circle of a triangle in terms of the lengths of the sides.

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The inscribed circle of a triangle ABC touches the sides BC, CA, AB at A', B', C' respectively. Show that the expression

$$BA' \cdot CB' \cdot AC' \cdot (BA' + CB' + AC')$$

is equal to the square of the area of the triangle.

37. Prove that, in a triangle ABC,

$$\tan \tfrac{1}{2}(B - C) = \frac{b - c}{b + c} \cot \tfrac{1}{2}A.$$

Find the remaining side and angles of the triangle ABC, given that $\angle A = 97° 38'$, $b = 10{\cdot}282$ inches, $c = 15{\cdot}737$ inches.

38. Prove that, in any triangle, $a^2 = b^2 + c^2 - 2bc \cos A$.

If OA, OB, OC are three mutually perpendicular lines meeting at O, show that the cotangents of the angles of the triangle ABC are in the ratios of $OA^2 : OB^2 : OC^2$.

39. Prove that, in any triangle, $R = abc/4\triangle$, where R is the radius of the circumcircle and $\triangle$ is the area of the triangle.

The perimeter of a triangle is 21 inches, the length of one of its sides is 6 inches, and the area is $21\sqrt{15}/4$ square inches.

Find the lengths of the other two sides and the radius of the circumcircle.

40. Prove that, in any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Solve both triangles in which $A = 35°$, $a = 14$, $b = 17{\cdot}2$.

41. In an isosceles triangle ABC, the sides AB and AC are equal, the perpendicular from A on BC is 4 inches in length, and the perpendicular from B on AC is 2 inches in length. Find the area of the triangle and the angle ABC.

42. The length of the perimeter of a triangle, in which the sides are in arithmetical progression, is 18 inches, and the area of the triangle is 10 square inches. Find the length of each side to two decimal places.

43. If I_1, I_2, and I_3 are the centres of the escribed circles of a triangle ABC, find the angles of the triangle $I_1I_2I_3$ in terms of $\angle A$, $\angle B$, $\angle C$, and prove that the sides of the triangle $I_1I_2I_3$ are respectively $4R \cos \frac{1}{2}A$, $4R \cos \frac{1}{2}B$, $4R \cos \frac{1}{2}C$.

44. Two triangles ABC, DEF are inscribed in the same circle. The angles of the triangle DEF are respectively $(\pi - 2A)$, $(\pi - 2B)$, $(\pi - 2C)$. Prove that four times the area of $\triangle ABC$ is equal to the product of the perimeter of $\triangle DEF$ and the radius of the circumcircle.

45. In any triangle, prove that $\tan \frac{1}{2}(B - C) = \tan (\frac{1}{4}\pi - \theta) \cot \frac{1}{2}A$, where $\tan \theta = c/b$.

If $A = 65° 18'$, $b = 432$ feet, $c = 357$ feet, find the values of B, C, and a, and the area of the triangle.

46. If p_1, p_2, p_3 are the lengths of the perpendiculars on the opposite sides of a triangle ABC from A, B, C respectively, and R is the radius of the circumcircle, prove that

(i) $p_1 \sin A = p_2 \sin B = p_3 \sin C$.

(ii) $R = \frac{1}{4}\dfrac{a^2 + b^2 + c^2}{p_1 \cos A + p_2 \cos B + p_3 \cos C}$.

47. ABC is an equilateral triangle of which each side is 5 inches long. Points D, E, and F are taken in the sides AB, BC, CA respectively such that $AD = 1$ inch, $BE = 2$ inches, $CF = 2$ inches. Find the angles of the triangle DEF.

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48. $ABCD\ldots$ is a regular n-sided polygon of side a. The sides AB, BC, $CD, \ldots$ are produced in order, through equal distances x, to points P, Q, $R, \ldots$ Write down the relation connecting a, x, and the side l of the regular polygon $PQR\ldots$ so formed.

If the area of this polygon be double that of the original polygon, α denote the angle π/n, and θ be the angle BQP, prove the following results:

(i) $x = \frac{1}{2}a[\sqrt{(1 + \operatorname{cosec}^2 \alpha)} - 1]$.

(ii) $\sqrt{2} \sin \theta = \cos \alpha[\sqrt{(1 + \sin^2 \alpha)} - \sin \alpha]$.

49. In a triangle ABC of area Δ, with the usual notation, prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

If $a - b = kc$, prove that $\sin \frac{1}{2}(A - B) = k \cos \frac{1}{2}C$.

Prove also that $\dfrac{k \sin B}{1 - k \cos B} = \tan \frac{1}{2}(A - B)$.

50. Three circles of radii r_1, r_2, r_3 are placed with their centres at points A, B, C respectively, so that each circle touches the other two externally. Show that the area of the triangle ABC is $\sqrt{r_1 r_2 r_3(r_1 + r_2 + r_3)}$, and that

$$\tan \tfrac{1}{2}A = \sqrt{\frac{r_2 r_3}{r_1(r_1 + r_2 + r_3)}}.$$

If the circles are equal, each of radius r, show that the area inside the triangle ABC bounded by arcs of the three circles is very nearly $0{\cdot}161r^2$.

51. Prove that, in any triangle,

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

and obtain a formula for $\tan \frac{1}{2}A$.

If $a = 237$ feet, $b = 342$ feet, $c = 511$ feet, find the values of A, B, C and the area of the triangle.

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52. Prove that, if S be the area of triangle ABC and if points P and Q be taken in AB and AC so that $PA = QA$, and the area of PAQ is $\frac{1}{2}S$, the length of PQ is $\sqrt{(2S \tan \frac{1}{2}A)}$.

Calculate the lengths of PA and PQ when $BC = 10$, $CA = 17$, $AB = 21$.

53. If A, B, C are the angles of a triangle, prove that

$$1 - \cos^2 A - \cos^2 B - \cos^2 C = 2 \cos A \cos B \cos C.$$

Prove that, if R be the radius of the circumscribing circle of a triangle of sides a, b, c, $a^2 + b^2 + c^2 = 8R^2(1 + \cos A \cos B \cos C)$.

54. Prove that, in any triangle, if

$$\cos \theta = \frac{2\sqrt{bc}}{b + c} \cos \tfrac{1}{2}A,$$

then $a = (b + c) \sin \theta$.

Solve the triangle in which $b = 21{\cdot}3$, $c = 35{\cdot}2$, $\angle A = 51°\,8'$.

55. Show that, if $A = 18°$, $\sin 2A = \cos 3A$, and deduce that

$$\sin 18° = (\sqrt{5} - 1)/4.$$

Alternate sides of a regular pentagon are produced to meet. Show that the points of intersection are the angular points of another regular pentagon, and that the ratio of the area of the pentagon so formed to that of the original pentagon is $(7 + 3\sqrt{5}) : 2$.

56. Prove with the usual notation for a triangle, r_1 being the radius of the escribed circle touching BC internally and R being the radius of the circumcircle, that $r_1 = a \cos \frac{1}{2}B \cos \frac{1}{2}C \sec \frac{1}{2}A = 4R \sin \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

If $r_1 = 2R$ and $B = C$, find the angles of the triangle.

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57. In a triangle ABC, $AB > AC$. The internal and external bisectors of the angle A meet the line BC at D and D' respectively.

If $AD = p$, and $AD' = q$, prove that

$$p = \frac{2bc}{b+c} \cos \tfrac{1}{2}A,$$

and find the corresponding expression for q.

Prove that $\dfrac{\cos \frac{1}{2}A}{p} + \dfrac{\sin \frac{1}{2}A}{q} = \dfrac{1}{b}$; $\dfrac{\cos \frac{1}{2}A}{p} - \dfrac{\sin \frac{1}{2}A}{q} = \dfrac{1}{c}$.

58. Show how to solve a triangle when two sides and the angle opposite one of them are given, explaining carefully the different cases that may arise.

If b, c, B are given, and have such values that two distinct solutions are possible, show that the difference between the third sides of the two triangles is $\sqrt{(b^2 - c^2 \sin^2 B)}$, and that the difference between their areas is

$$c \sin B \sqrt{(b^2 - c^2 \sin^2 B)}.$$

If ABC_1, ABC_2 are the two triangles and the triangle AC_1C_2 is equilateral, show that $2c \sin B = b\sqrt{3}$.

59. $ABCD$ is a convex quadrilateral having $AB = 5$, $BC = 7$, $CD = 6$, $DA = 12$ and the angle DAB is 45°.

Find the angle BCD, writing down any formula you use.

60. If O be the circumcentre of a triangle ABC and H the intersection of the perpendiculars from A, B, C to the opposite sides respectively, prove that $OH^2 = R^2(1 - 8 \cos A \cos B \cos C)$, where R is the radius of the circum-circle.

If $BC = 10$ inches, $\angle ABC = 45°$, $\angle ACB = 75°$, find R and OH correct to two decimal places.

61. Prove that the radius r_1 of the escribed circle of the triangle ABC which touches BC, and touches AB and AC produced, is given by $r_1 = s \tan \frac{1}{2}A$.

If $a = 11$ cm., $b = 8$ cm., $c = 15$ cm., calculate the angle A to the nearest minute, and the radius r_1 correct to three significant figures.

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62. Prove that, in any triangle,

$$\tan \tfrac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \tfrac{1}{2}A.$$

If $b = 11{\cdot}2$ inches, $c = 7{\cdot}5$ inches, $A = 107° 26'$, find a, B, C.

63. If r_1 is the radius of the escribed circle of the triangle ABC opposite the angle A, prove that $r_1 = \Delta/(s - a)$, where Δ is the area of the triangle and $2s = (a + b + c)$.

If r, r_1, r_2, r_3 are the radii of the inscribed and escribed circles respectively, prove that $\Delta abc = r^2(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)$.

64. If R be the radius of the circumcircle of a triangle ABC, whose area is Δ, prove that $R = abc/4\Delta$.

The mid-point of BC is D, and E and F are the feet of the perpendiculars from D to AC and AB respectively, and the area of the triangle DEF is S.

Prove that $R = \frac{1}{4}a\sqrt{(\Delta/S)}$.

65. Prove that, in any triangle ABC,

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

If $a = 14{\cdot}8$ feet, $b = 19{\cdot}4$ feet, $c = 27{\cdot}3$ feet, calculate the angles of the triangle.

66. For a triangle ABC, prove that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \text{ where } R \text{ is the circumradius.}$$

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In a triangle ABC, $AB = 120$, $AC = 65$ and angle $ABC = 26°$. Prove that two triangles can be constructed to satisfy these conditions, and find the difference between the lengths of the remaining side BC in the two triangles.

67. (*a*) Prove that in any triangle, in the usual notation,

$$\text{(i) } a^2 = b^2 + c^2 - 2bc\cos A,$$
$$\text{(ii) } b\cos C - c\cos B = (b^2 - c^2)/a,$$
$$\text{(iii) } b\cos B - c\cos C = (c^2 - b^2)\cos A/a.$$

(*b*) Prove the identity

$$\sin(\theta + \varphi)\sin(\theta - \varphi) = (\sin\theta + \sin\varphi)(\sin\theta - \sin\varphi).$$

68. If R, r are respectively the radii of the circumscribed and inscribed circles of the triangle ABC, and $2s = a + b + c$, prove that

$$a = 2R\sin A \text{ and } r = (s - a)\tan\tfrac{1}{2}A.$$

If $R = 15$, $r = 5$, $s = 35$, show that $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$, $\tan\frac{1}{2}C$ satisfy the cubic equation in t

$$7t^3 - 13t^2 + 7t - 1 = 0.$$

Verify that one of the angles is a right angle and find approximate values of the other two angles.

69. (*a*) Prove, in the usual notation, that the area of any triangle ABC is given by $\Delta = \frac{1}{2}bc\sin A = [s(s - a)(s - b)(s - c)]^{\frac{1}{2}}$, where $s = \frac{1}{2}(a + b + c)$.

(*b*) The median CN of a triangle ABC meets the side AB in the point N. If θ is the angle ANC, prove that

$$\frac{2c\cos\theta}{a^2 - b^2} = \frac{c\sin\theta}{2\Delta} = \frac{1}{CN}.$$

70. Prove that the radius of the escribed circle touching the side BC of the triangle ABC is $a\cos\frac{1}{2}B\cos\frac{1}{2}C\sec\frac{1}{2}A$.

In a triangle ABC the angle B is 36°, the radius of the escribed circle touching BC is 5 inches and the length of either tangent from A to this circle is 12 inches. Calculate (i) the angle A, (ii) the length of BC, (iii) the radius of the inscribed circle of the triangle ABC.

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CHAPTER VIII

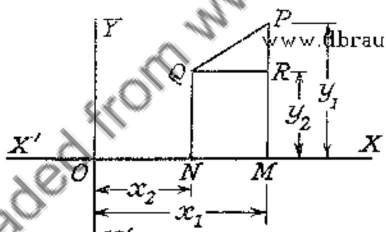
Co-ordinate Geometry—The Straight Line and Circle

There is a basic similarity between graphical work and co-ordinate geometry. The usual axes $X'OX$, $Y'OY$ at right angles are taken, with O as the origin, and the same conventions for sign as in graphs, but the scales of x and y along the two axes are always the *same* for co-ordinate geometry.

The notation $P \equiv (x, y)$ means that the point P has its *abscissa* x and its *ordinate* of value y, and x and y are known as the *coordinates* of the point P with respect to the rectangular axes $X'OX$, $Y'OY$.

It will be noted that, for convenience, all proofs are given for the first quadrant, but the results are valid for all quadrants.

Theorem. *To prove that the distance between the points* $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$ is $\sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}$.



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PM, QN are the ordinates of P and Q respectively, and QR is perpendicular to PM.

From the diagram, $PR = y_1 - y_2$, $QR = x_1 - x_2$.

Using Pythagoras' theorem on triangle PQR,

$$PQ^2 = QR^2 + PR^2$$
$$= (x_1 - x_2)^2 + (y_1 - y_2)^2.$$
$$\therefore\ PQ = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}.$$

Theorem. *To find the point in which the join of PQ is divided internally in the ratio of* $\lambda_2 : \lambda_1$, *where* $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$.

Let $R \equiv (\bar{x}, \bar{y})$ be the required point of division. PM, QN, RS are the ordinates of P, Q, R respectively; QD is the perpendicular from Q on RS and RE the perpendicular from R on PM.

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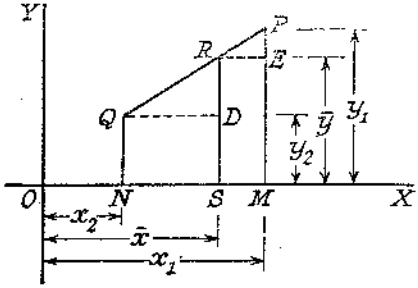

From the diagram it can be seen that

$$QD = \bar{x} - x_2,\ SM = RE = x_1 - \bar{x}.$$

From the similar triangles QRD, PRE, it follows that

$$\frac{QD}{RE} = \frac{QR}{RP}, \text{ i.e. } \frac{\bar{x} - x_2}{x_1 - \bar{x}} = \frac{\lambda_1}{\lambda_2} \left(\frac{PR}{RQ} = \frac{\lambda_2}{\lambda_1} \right)$$

$$\therefore\ \lambda_2\bar{x} - \lambda_2 x_2 = \lambda_1 x_1 - \lambda\bar{x}.$$

$$\therefore\ \bar{x}(\lambda_1 + \lambda_2) = \lambda_1 x_1 + \lambda_2 x_2.$$

$$\therefore\ \bar{x} = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}.$$

Similarly, $$\bar{y} = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2}.$$

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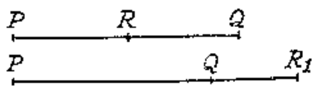
NOTE. The line PQ is divided at R in the ratio $\lambda_2 : \lambda_1$ and not $\lambda_1 : \lambda_2$,

i.e. $\dfrac{PR}{RQ} = \dfrac{\lambda_2}{\lambda_1}$.



The line PQ is said to be divided *internally* at R if R lies between P and Q, and externally at R_1 if R_1 lies on PQ or QP produced. The ratio of external division of PQ will be $PR_1 : R_1Q$, but R_1Q is measured in the opposite direction to PR_1 and the two lengths will have opposite signs. Hence, the ratio for external division will be negative, whilst that for internal division will be positive (PR and RQ are of the same sign). Thus, instead of saying a given line is divided externally in the ratio of 3 : 2, it can be said that it is divided in the ratio of $-3 : 2$.

The formulae proved for a point of internal division can be used for a point of external division providing that it is remembered that the ratio $\lambda_2 : \lambda_1$ is negative, i.e. λ_2 and λ_1 are of opposite signs.

In the case of the mid-point of a line, $\lambda_2 : \lambda_1 = 1 : 1$, and if $(\bar{x}, \bar{y})$ be the mid-point of PQ, where $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, then $\bar{x} = \frac{1}{2}(x_1 + x_2)$, $\bar{y} = \frac{1}{2}(y_1 + y_2)$.

EXAMPLE. (i) Find the distance between the points $(2, -3)$, $(-3, 2)$.

(ii) Find the points dividing the join of $(1, 2)$, $(3, 1)$ in the ratio $1 : 2$, internally and externally.

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(i) Using the result of the first theorem with $x_1 = 2,\ y_1 = -3$; $x_2 = -3,\ y_2 = 2$, the required distance

$$= \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]} = \sqrt{[(2+3)^2 + (-3-2)^2]}$$
$$= \sqrt{(25 + 25)} = \sqrt{50} = 5\sqrt{2}.$$

(ii) Let $(\bar{x}_1, \bar{y}_1)$ be the point of internal division, with $x_1 = 1, y_1 = 2$; $x_2 = 3, y_2 = 1$; $\lambda_2 = 1,\ \lambda_1 = 2$.

$$\therefore\ \bar{x}_1 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} = \frac{2\,.\,1 + 1\,.\,3}{2+1} = \tfrac{5}{3},$$

$$\bar{y}_1 = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2} = \frac{2\,.\,2 + 1\,.\,1}{2+1} = \tfrac{5}{3},$$

therefore required point is $(\frac{5}{3}, \frac{5}{3})$.

Let $(\bar{x}_2, \bar{y}_2)$ be the point of external division with $\lambda_2 = -1,\ \lambda_1 = 2$.

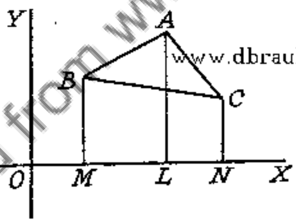
$$\therefore\ \bar{x}_2 = \frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} = \frac{2-3}{2-1} = -1,$$

$$\bar{y}_2 = \frac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2} = \frac{4-1}{2-1} = 3,$$

therefore required point is $(-1, 3)$.

Theorem. *To find the area of the triangle joining the points* (x_1, y_1), (x_2, y_2), (x_3, y_3), *taken in an anti-clockwise order.*

A, B, C are the points (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, and AL, BM, CN the ordinates of A, B, C.



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From the diagram,

$$ML = (x_1 - x_2),\ MN = (x_3 - x_2),\ NL = (x_3 - x_1).$$

Area of $\triangle ABC$

$$= \text{area of trapezium } ABML + \text{area of trapezium } ALNC - \text{area of trapezium } BMNC.$$
$$= \tfrac{1}{2}ML(AL + BM) + \tfrac{1}{2}LN(AL + CN) - \tfrac{1}{2}MN(BM + CN)$$
$$= \tfrac{1}{2}(x_1 - x_2)(y_1 + y_2) + \tfrac{1}{2}(x_3 - x_1)(y_1 + y_3) - \tfrac{1}{2}(x_3 - x_2)(y_2 + y_3)$$
$$= \{\tfrac{1}{2}x_1[(y_1 + y_2) - (y_1 + y_3)] + x_2[(y_2 + y_3) - (y_1 + y_2)] + x_3[(y_1 + y_3) - (y_2 + y_3)]\}$$
$$= \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)].$$

NOTE. When the points are considered (lettered) in a clockwise direction, it is found that the area obtained will be the above result with a negative sign.

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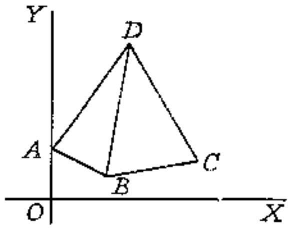
EXAMPLE. Calculate the area of the quadrilateral whose vertices are (0, 3), (3, 1), (10, 2), and (6, 10).

Let the given points be A, B, C, D in that order.

Using the formula obtained,



$$\text{Area } \triangle ABD = \tfrac{1}{2}[0(1-10) + 3(10-3) + 6(3-1)]$$
$$= \tfrac{1}{2} \times 33 = 33/2 \text{ square units}$$
$$\text{Area } \triangle BCD = \tfrac{1}{2}[3(2-10) + 10(10-1) + 6(1-2)]$$
$$= \tfrac{1}{2} \times 60 = 30 \text{ square units}$$

therefore area of quadrilateral $ABCD$

$$= \text{area } \triangle ABD + \text{area } \triangle BCD = 46\tfrac{1}{2} \text{ square units.}$$

Theorem. *To show that a linear equation in x and y always represents a straight line.*

Consider the equation

$$ax + by + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which is the general linear equation in x and y, where a, b, c are constants.

Let $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$ be any two points lying on the curve that is represented by equation (1).

NOTE. The necessary condition that a point shall lie on a curve is that the co-ordinates of the point shall satisfy the equation to the curve.

Since A and B lie on the curve represented by (1)

$$ax_1 + by_1 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$ax_2 + by_2 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let λ_2 and λ_1 be any constants.

$$(2) \times \frac{\lambda_1}{\lambda_1 + \lambda_2} \text{ gives } \frac{a\lambda_1 x_1}{\lambda_1 + \lambda_2} + \frac{b\lambda_1 y_1}{\lambda_1 + \lambda_2} + \frac{c\lambda_1}{\lambda_1 + \lambda_2} = 0 \;..\; (4).$$

$$(3) \times \frac{\lambda_2}{\lambda_1 + \lambda_2} \text{ gives } \frac{a\lambda_2 x_2}{\lambda_1 + \lambda_2} + \frac{b\lambda_2 y_2}{\lambda_1 + \lambda_2} + \frac{c\lambda_2}{\lambda_1 + \lambda_2} = 0 \;..\; (5)$$

(4) + (5) gives

$$\frac{a(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1 + \lambda_2} + \frac{b(\lambda_1 y_1 + \lambda_2 y_2)}{\lambda_1 + \lambda_2} + \frac{c(\lambda_1 + \lambda_2)}{\lambda_1 + \lambda_2} = 0$$

$$\text{i.e. } a\bar{x} + b\bar{y} + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where $\quad \bar{x} = \dfrac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2}, \quad \bar{y} = \dfrac{\lambda_1 y_1 + \lambda_2 y_2}{\lambda_1 + \lambda_2},$

i.e. $(\bar{x}, \bar{y})$ is the point dividing the straight line AB in the ratio $\lambda_2 : \lambda_1$.

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Equation (6) shows that the point $(\bar{x}, \bar{y})$ lies on the curve represented by equation (1). Since $\lambda_2 : \lambda_1$ is *any* ratio, the point $(\bar{x}, \bar{y})$ is any point on the straight line *AB*. Hence, any point on the straight line *AB* must lie on the curve represented by $ax + by + c = 0$. Thus, this equation represents a straight line.

Conversely it can be shown that a straight line is always represented by a linear equation in x and y. The equation $ax + by + c = 0$ is the *general* equation of a straight line.

The *slope* of a straight line is the tangent of the angle that the line makes with the positive direction of *OX*.

If a straight line cut *OX* and *OY* at *A* and *B* respectively, then *OA* and *OB* are known as the *intercepts* on the two axes.

The intercept on *OX* is *positive* if *A* lies to the right of *O* and negative if *A* lies to the left of *O*.

The intercept on *OY* is positive if *B* lies above *O* and negative if it lies below *O*.

NOTE. If $P \equiv (x, y)$ be *any* point on a given curve, and an equation can be obtained (by geometrical or other means) involving x, y and the various *constants* present, then this equation will be the equation representing the curve.

Theorem. *To find the equation of the straight line of slope m, that makes an intercept of c on OY.*

Let $P \equiv (x, y)$ be any point on the straight line, and *B* the point in which the line cuts *OY*. *PM* is the ordinate of *P* and *BL* is perpendicular to *PM*.

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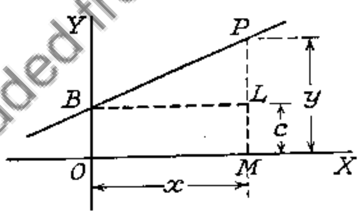

From the diagram, $PL = y - c$, $BL = x$.
Slope of $PB = PL/BL = (y - c)/x$

$$\therefore (y - c)/x = m, \text{ i.e. } y - c = mx, \text{ i.e. } y = mx + c.$$

This is the required equation since *P* is *any* point on the line, and is known as the *slope equation* of the line.

Theorem. *To find the equation of a straight line making intercepts of a and b on OX and OY respectively.*

Let the given line cut *OX* and *OY* respectively at *A* and *B*. $P \equiv (x, y)$ is any point on *AB*, and *PM* is the perpendicular from *P* on *OY*.

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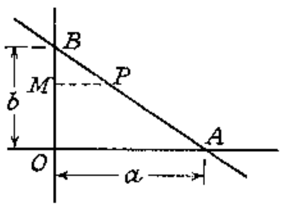
By similar triangles (left-hand figure),

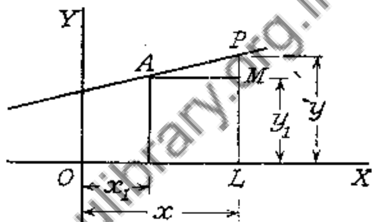
$$\frac{BM}{BO} = \frac{PM}{AO},$$

$$\text{i.e. } \frac{b-y}{b} = \frac{x}{a}, \quad \therefore 1 - \frac{y}{b} = \frac{x}{a},$$

$$\text{i.e. } \frac{x}{a} + \frac{y}{b} = 1,$$

which is the required equation since P is *any* point on the line.





Theorem. *To find the equation of a straight line of slope m which passes through the point* (x_1, y_1). (Right-hand figure).

A is the point (x_1, y_1) and $P \equiv (x, y)$ is any point on the line. PL is the ordinate of P, and AM the perpendicular from A on PL.

From the diagram, $PM = (y - y_1)$, $AM = (x - x_1)$.

The slope of the given line is

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$$\frac{PM}{AM} = \frac{y - y_1}{x - x_1}$$

$$\therefore \frac{y - y_1}{x - x_1} = m, \text{ i.e. } y - y_1 = m(x - x_1),$$

which is the required equation.

Aliter. The equation of line of slope m is

$$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

(c is intercept on OY)

Since A lies on this line,

$$y_1 = mx_1 + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

(1) − (2) gives

$$y - y_1 = mx - mx_1$$
$$= m(x - x_1),$$

which is the required equation, since it contains x, y, and only known constants.

Theorem. *To find the equation of the straight line passing through the points* (x_1, y_1), (x_2, y_2).

A and B are the points (x_1, y_1), (x_2, y_2) respectively. $P \equiv (x, y)$ is any point on AB, and AM is the ordinate of A. PR and BN are perpendiculars on AM from P and B respectively.

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From the diagram (left-hand figure),

$$AR = y_1 - y,\ PR = x_1 - x,$$
$$AN = y_1 - y_2,\ BN = x_1 - x_2.$$

The slope of AB $= \dfrac{AR}{PR} = \dfrac{y_1 - y}{x_1 - x} = \dfrac{y - y_1}{x - x_1}.$

Also, the slope of AB $= \dfrac{AN}{BN} = \dfrac{y_1 - y_2}{x_1 - x_2}.$

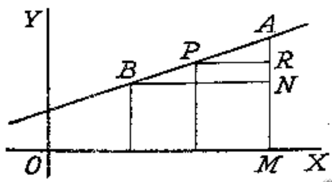
Hence, $\dfrac{y - y_1}{x - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2},$

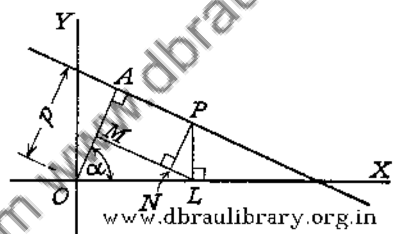
which is the required equation, when written in the form

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}.$$

NOTE. The slope of the line (straight) joining (x_1, y_1), (x_2, y_2) is

$$\frac{y_1 - y_2}{x_1 - x_2}.$$





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Theorem. *To find the equation of the straight line, the perpendicular on which from the origin O makes an angle α with OX and is of length p.*

$P \equiv (x, y)$ is any point on the line and $OA = p$ is the perpendicular from O on this line; PL is the ordinate of P and LM the perpendicular from P on OA; PN is the perpendicular from P on LM. (Right-hand figure.)

By geometry, $\angle PLN = \alpha$, and $MA = PN$,

$$OA = OM + MA = OM + PN = OL \cos\alpha + PL \sin\alpha$$
$$\therefore p = x\cos\alpha + y\sin\alpha.$$

Since P is any point on the line, the equation of the line is

$$x\cos\alpha + y\sin\alpha = p,$$

which is known as the *perpendicular form* of the equation of a straight line.

NOTE. The perpendicular from O on any straight line is always taken to be positive, i.e. p is always positive.

A Summary of the Equations of a Straight Line

(i) General equation $ax + by + c = 0$ (a, b, c constants).

(ii) Slope equation $y = mx + c$ (m = slope, c = intercept on OY).

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(iii) Intercept equation $x/a + y/b = 1$ (a and b intercepts on OX and OY respectively).

(iv) Line slope m through (x_1, y_1) $y - y_1 = m(x - x_1)$.

(v) Line through (x_1, y_1), (x_2, y_2) $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$.

(vi) Perpendicular form. $x \cos \alpha + y \sin \alpha = p$ (perpendicular from O on line makes an angle α with OX and is of length p).

NOTE. The particular form of the equation of a straight line to be used is dependent on the question, and, in general, any form of the equation of a straight line can be converted into any other by algebraic manipulation.

Thus, $ax + by + c = 0$ can be written

$$y = -\frac{a}{b}x - \frac{c}{b} \text{ (slope form)},$$

showing that its slope is $-a/b$ and its intercept on OY is $-c/b$.

Also, this equation can be written $ax + by = -c$

$$\text{i.e. } \frac{x}{-c/a} + \frac{y}{-c/b} = 1 \text{ (intercept form)},$$

showing that its intercepts on OX and OY are $-c/a$ and $-c/b$ respectively.

EXAMPLE. Find the equations of the following lines:

(i) Slope $-\frac{1}{3}$, intercept on OY is $-3/2$.

(ii) Slope $+\frac{1}{2}$, passing through $(-1, 3)$.

(iii) Passing through $(1, -3)$, $(2, 5)$.

(iv) Making intercepts of -2, $-\frac{1}{2}$ on OX and OY respectively.

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(v) Perpendicular from O making an angle $150°$ with OX and of length 2 units.

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NOTE. Fractions must be cleared in the results.

The previous notation is used throughout.

(i) Using $y = mx + c$, the equation of the line is

$$y = -\tfrac{1}{3}x - 3/2, \quad \text{i.e. } 6y = -2x - 9,$$
$$\text{i.e. } 2x + 6y + 9 = 0.$$

(ii) Using $y - y_1 = m(x - x_1)$, the equation is

$$y - 3 = \tfrac{1}{2}(x + 1), \quad \text{i.e. } 2y - 6 = x + 1$$
$$\text{i.e. } 2y = x + 7.$$

(iii) Using $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$, the equation is

$$\frac{y + 3}{-3 - 5} = \frac{x - 1}{1 - 2}, \quad \text{i.e. } \frac{y + 3}{8} = x - 1,$$
$$\therefore y + 3 = 8x - 8, \quad \text{i.e. } y = 8x - 11.$$

(iv) From the equation $x/a + y/b = 1$, the line has as its equation

$$\frac{x}{-2} + \frac{y}{-\frac{1}{2}} = 1,$$
$$\text{i.e. } \tfrac{1}{2}x + 2y = -1,$$
$$\text{i.e. } x + 4y = -2.$$

(v) Using $x \cos \alpha + y \sin \alpha = p$, the required equation is

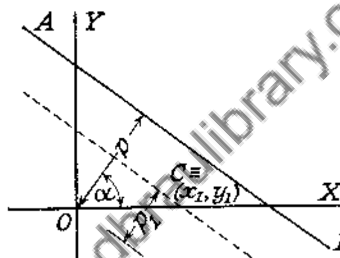
$$x \cos 150° + y \sin 150° = 2,$$

$$\text{i.e. } -x\frac{\sqrt{3}}{2} + \tfrac{1}{2}y = 2,$$

$$\text{i.e. } y = x\sqrt{3} + 4.$$

Theorem. *To find the length of the perpendicular from the point (x_1, y_1) on the straight line $x \cos \alpha + y \sin \alpha = p$.*

AB is the given line, the perpendicular on which from the origin O is of length p and makes an angle α with OX; p_1 is the perpendicular from O on a line through $C \equiv (x_1, y_1)$ parallel to AB.



Since the line through C is parallel to AB, its equation is

$$x \cos \alpha + y \sin \alpha = p_1.$$

Since C lies on this line,

$$x_1 \cos \alpha + y_1 \sin \alpha = p_1.$$

From the diagram it can be seen that the perpendicular from C on AB is $p - p_1$,

i.e. $p - x_1 \cos \alpha - y_1 \sin \alpha$.

Note. In the proof C has been taken on the same side of the line AB as the origin and the result is clearly positive. When C is on the opposite side of AB to the origin $p_1 > p$, and the perpendicular from C in this case on the line AB will be negative.

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Theorem. *To find the values of p and α in order that the equations $ax + by + c = 0$ and $x \cos \alpha + y \sin \alpha = p$, shall represent the same straight line.*

Since the two equations represent the same line, they are equivalent equations, i.e. the coefficients must be in the same ratio.

$$\therefore \frac{\cos \alpha}{a} = \frac{\sin \alpha}{b} = \frac{-p}{c} = k \text{ (say)}$$

$$\therefore \cos \alpha = ak, \ \sin \alpha = bk, \ p = -ck.$$

Hence,

$$\cos^2 \alpha + \sin^2 \alpha = a^2k^2 + b^2k^2$$

$$\text{i.e. } 1 = k^2(a^2 + b^2)$$

$$\therefore k = \pm 1/\sqrt{(a^2 + b^2)}.$$

Thus,

$$\left.\begin{aligned} \cos \alpha &= \pm\frac{a}{\sqrt{(a^2 + b^2)}} \\ \sin \alpha &= \pm\frac{b}{\sqrt{(a^2 + b^2)}} \\ p &= \mp\frac{c}{\sqrt{(a^2 + b^2)}} \end{aligned}\right\}$$

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These results give the values for p and α, and, since p must always be positive, the positive sign for p (and therefore the negative signs for $\cos\alpha$ and $\sin\alpha$) will be taken when c is positive, and the negative sign (positive sign for $\sin\alpha$ and $\cos\alpha$) when c is negative.

Theorem. *To find the length of the perpendicular from (x_1, y_1) on the line $ax + by + c = 0$.*

If $x\cos\alpha + y\sin\alpha = p$ be the perpendicular form of the equation of the given straight line, by the previous theorem,

$$\cos\alpha = \pm\frac{a}{\sqrt{(a^2+b^2)}}, \quad \sin\alpha = \pm\frac{b}{\sqrt{(a^2+b^2)}}, \quad p = \mp\frac{c}{\sqrt{(a^2+b^2)}}.$$

Now the length of the perpendicular from (x_1, y_1) on the line $x\cos\alpha + y\sin\alpha = p$ is $p - x_1\cos\alpha - y_1\sin\alpha$, and using the above values for $\cos\alpha$, etc., the required length is

$$\mp\frac{c}{\sqrt{(a^2+b^2)}} - x_1\left\{\pm\frac{a}{\sqrt{(a^2+b^2)}}\right\} - y_1\left\{\pm\frac{b}{\sqrt{(a^2+b^2)}}\right\}$$
$$= \mp\frac{c}{\sqrt{(a^2+b^2)}} \mp \frac{ax_1}{\sqrt{(a^2+b^2)}} \mp \frac{by_1}{\sqrt{(a^2+b^2)}}$$
$$= \pm\frac{(ax_1+by_1+c)}{\sqrt{(a^2+b^2)}}.$$

NOTE. Since the perpendicular from the origin (0, 0) on any line must be positive, the positive sign will be taken in this result when c is positive, and the negative sign when c is negative.

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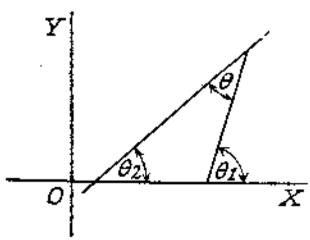
Theorem. *To find the angle θ between two straight lines whose slope are m_1, m_2.*

Let the two given lines make angles θ_1 and θ_2 respectively with OX. Then $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.



From the diagram,

$$\theta = \theta_1 - \theta_2 \ (\theta_1 > \theta_2) \text{ or } \theta = \theta_2 - \theta_1 \ (\theta_2 > \theta_1)$$
$$\therefore\ \theta = \pm(\theta_1 - \theta_2)$$
$$\therefore\ \tan\theta = \tan\pm(\theta_1 - \theta_2) = \pm\tan(\theta_1 - \theta_2)$$
$$\text{i.e. } \tan\theta = \pm\frac{(\tan\theta_1 - \tan\theta_2)}{1 + \tan\theta_1\tan\theta_2} = \pm\frac{(m_1 - m_2)}{1 + m_1m_2}.$$

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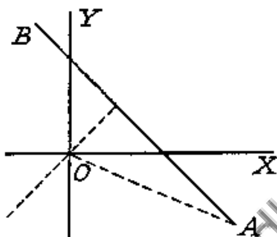
When the lines are perpendicular, $\theta = 90°$ and $\tan\theta = \infty$.
In this case

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \infty \quad \therefore\ 1 + m_1 m_2 = 0.$$

$$\text{i.e. } m_1 m_2 = -1.$$

When the lines are parallel $\theta = 0$, $\tan\theta = 0$ $\therefore m_1 = m_2$, which is obvious from geometry.

EXAMPLE. Two vertices of a triangle are at the points $A(3, -1)$, $B(-2, 3)$ and the orthocentre is at the origin. Find the co-ordinates of the remaining vertex.



Let C be the remaining vertex. Then C will lie on a line through O perpendicular to AB, and also on a line through B perpendicular to AO, since the orthocentre is the intersection of the altitudes of a triangle.

The slope of AB is

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$$\frac{3 - (-1)}{-2 - 3} = \frac{-4}{5}. \qquad \left(\frac{y_1 - y_2}{x_1 - x_2}\right)$$

Therefore slope of line perpendicular to AB is $5/4$. $\quad (m_1 m_2 = -1)$
Therefore equation of the line through O perpendicular to AB is

$$y = \frac{5}{4}x \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The slope of AO is $-\frac{1}{3}$, therefore slope of line perpendicular to AO is 3, and the equation of the line through B perpendicular to OA is

$$y - 3 = 3(x + 2), \quad \text{i.e. } y = 3x + 9 \ldots\ldots\ldots\ldots (2)$$

C will be the point of intersection of the lines (1) and (2) (i.e. the lines represented by equations (1) and (2)).

Solving (1) and (2),

$$\frac{5}{4}x = 3x + 9,$$

$$\therefore\ 5x = 12x + 36, \ \therefore\ 7x = -36$$

$$\therefore\ x = -36/7.$$

From (1) $\qquad y = -45/7,$

$$\therefore\ C \equiv \left(\frac{-36}{7}, \frac{-45}{7}\right).$$

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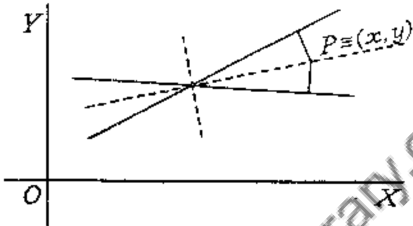
Theorem. *To find the equations of the bisectors of the angles between the lines*

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0.$$

$P \equiv (x, y)$ is any point on one of the bisectors of the angles (shown by dotted lines in the diagram).



By geometry, the perpendiculars from P on the two given lines must be equal in length, and for one bisector they will be of the same sign, and of opposite signs for the other bisector. Therefore equation of the bisectors is given by

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{(a_2^2 + b_2^2)}}.$$

The positive sign gives one bisector and the negative sign gives the other bisector.

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In any particular example in which a specific bisector of an angle is required, it is advisable to draw a fairly accurate diagram showing the bisector required and note the sign of its slope from the diagram, and this will determine the ambiguity in sign.

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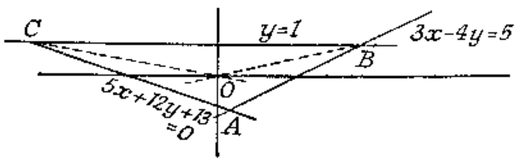
EXAMPLE. Find the incentre of the triangle formed by the straight lines $y = 1$, $3x - 4y = 5$, and $5x + 12y + 13 = 0$.



Let BC be the line $y = 1$ .... (1),

AB be the line $3x - 4y = 5$ .... (2),

AC be the line $5x + 12y + 13 = 0$ .... (3),

as shown in the diagram.

The bisector of $\angle C$ is

$$\frac{y - 1}{1} = \pm\frac{(5x + 12y + 13)}{\sqrt{(5^2 + 12^2)}},$$

i.e. $13(y - 1) = \pm(5x + 12y + 13)$ .... (4)

From the diagram the slope of this line is negative, therefore the negative sign is taken in (4) since it gives a negative slope, i.e. the equation of the internal bisector through C is

$$13y - 13 = -(5x + 12y + 13),$$
$$\text{i.e. } 5x + 25y = 0,$$
$$\text{i.e. } x + 5y = 0 \qquad\qquad (5).$$

The equation of the bisectors of $\angle B$ are given by

$$y - 1 = \pm\frac{(3x - 4y - 5)}{\sqrt{(3^2 + 4^2)}} = \pm\frac{(3x - 4y - 5)}{5},$$
$$\text{i.e. } 5(y - 1) = \pm(3x - 4y - 5) \qquad\qquad (6).$$

From the diagram the slope of the internal bisector of $\angle B$ is positive, and this requires that the positive sign be taken in (6), giving the equation

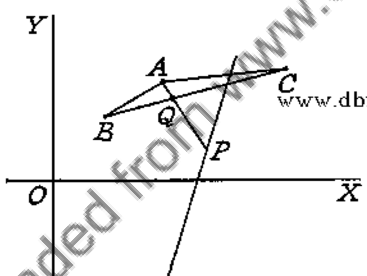
$$5y - 5 = 3x - 4y - 5,$$
$$\text{i.e. } 9y = 3x,$$
$$\text{i.e. } x - 3y = 0 \qquad\qquad (7).$$

The point (0, 0) clearly lies on lines (5) and (7), therefore the origin is the required incentre.

Example (L.U.). Find the co-ordinates of the point that divides the line joining (x_1, y_1) to (x_2, y_2) in the ratio $k : 1$.

The co-ordinates of three points A, B, C are respectively (4, 4), (3, 2), (9, 5). A point P lies on the line $3x - y = 20$ and AP meets BC in Q.

If $AQ = \frac{1}{3}AP$, find the co-ordinates of P.



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The first part of the question has been proved as a theorem using a ratio $\lambda_2 : \lambda_1$, and, if $(\bar{x}, \bar{y})$ be the required point,

$$\bar{x} = \frac{x_1 + kx_2}{1 + k}, \quad \bar{y} = \frac{y_1 + ky_2}{1 + k}.$$

Let $P \equiv (x_1, y_1)$.

Since P lies on the line $3x - y = 20 \qquad\qquad (1)$,

$$3x_1 - y_1 = 20,$$
$$\therefore y_1 = 3x_1 - 20 \qquad\qquad (2).$$

Since BC passes through (3, 2), (9, 5), its equation is

$$\frac{y - 5}{5 - 2} = \frac{x - 9}{9 - 3},$$
$$\text{i.e. } \frac{y - 5}{3} = \frac{x - 9}{6},$$
$$\text{i.e. } 2(y - 5) = x - 9,$$
$$\text{i.e. } 2y = x + 1 \qquad\qquad (3).$$

Let $(\bar{x}, \bar{y})$ be the co-ordinates of Q.

Now $AQ = \frac{1}{3}AP, \quad \therefore \quad QP = \frac{2}{3}AP, \quad \therefore \quad \frac{QP}{AQ} = \frac{2}{1}.$

Using the first part of the question with $k = 2$, etc.,

$$\bar{x} = \frac{x_1 + 2 \times 4}{1 + 2} = \frac{x_1 + 8}{3}, \quad \bar{y} = \frac{y_1 + 2 \times 4}{1 + 2} = \frac{y_1 + 8}{3}.$$

But Q lies on the line (3).

$$\therefore \frac{2y_1 + 16}{3} = \frac{x_1 + 8}{3} + 1,$$

$$\text{i.e. } 2y_1 + 16 = x_1 + 8 + 3,$$

$$\therefore 2y_1 = x_1 - 5 \ldots\ldots\ldots\ldots (4).$$

Using (2) in (4), $6x_1 - 40 = x_1 - 5$,

$$\therefore 5x_1 = 35, \quad \therefore \quad x_1 = 7,$$

and from (4), $2y_1 = 2 \quad \therefore \quad y_1 = 1.$

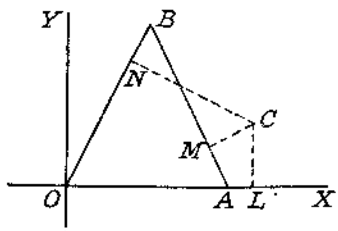
Hence, $P \equiv (7, 1)$.

EXAMPLE (L.U.). Find the co-ordinates of the feet of the perpendiculars from the point (9, 3) to the sides of the triangle whose vertices are at the points (0, 0), (8, 0), (4, 8).

Prove that the points so determined lie on a straight line and find its equation.

Let $A \equiv (8, 0)$, $B \equiv (4, 8)$, $C \equiv (9, 3)$, and L, M, N be the feet of the perpendiculars from C on OA, AB, OB respectively.

Since OA is the x-axis, $L \equiv (9, 0)$.



The equation of AB is

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$$\frac{y - 8}{8 - 0} = \frac{x - 4}{4 - 8},$$

$$\text{i.e. } \frac{y - 8}{8} = \frac{x - 4}{-4},$$

$$\text{i.e. } y - 8 = -2(x - 4),$$

$$\text{i.e. } 2x + y = 16 \ldots\ldots\ldots (1).$$

The slope of this line is -2, and therefore the slope of CM (perpendicular to AB) is $\frac{1}{2}$. Therefore equation of CM is

$$y - 3 = \tfrac{1}{2}(x - 9),$$

$$\text{i.e. } 2y - 6 = x - 9,$$

$$\text{i.e. } 2y = x - 3 \ldots\ldots\ldots (2).$$

Solving (1) and (2) for the point M,

$$2(16 - 2x) = x - 3, \quad \text{i.e. } 32 - 4x = x - 3,$$

$$\text{i.e. } 35 = 5x, \quad \therefore x = 7.$$

From (1) $14 + y = 16, \therefore y = 2$, and $M \equiv (7, 2)$.

The equation of OB is $y = \frac{8}{4}x = 2x \ldots\ldots\ldots (3),$

(slope of $OB = \frac{8}{4} = 2$)

therefore slope of the perpendicular CN is $-\frac{1}{2}$, therefore equation of CN is

$$(y - 3) = -\tfrac{1}{2}(x - 9),$$

$$\text{i.e. } 2y - 6 = -x + 9,$$

$$\text{i.e. } 2y + x = 15 \ldots\ldots\ldots (4).$$

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Solving (3) and (4) for N,

$$4x + x = 15, \quad \text{i.e. } 5x = 15 \quad \therefore x = 3,$$

and from (3),

$$y = 6.$$

$$\therefore N \equiv (3, 6).$$

The equation of LM is

$$\frac{y - 0}{0 - 2} = \frac{x - 9}{9 - 7},$$

$$\text{i.e. } \frac{y}{-2} = \frac{x - 9}{2}, \quad \text{i.e. } y = -(x - 9),$$

$$\text{i.e. } y + x = 9.$$

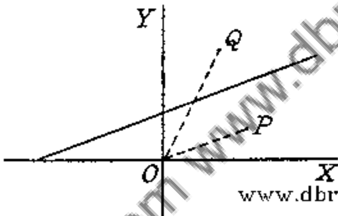
Since $6 + 3 = 9$, the point (3, 6) lies on this line.

Hence, L, M, N are collinear and lie on the line $y + x = 9$.

EXAMPLE (L.U.). Prove that the point Q (4, 8) is the image of the point P (6, 2) with respect to the line $x - 3y + 10 = 0$.

Show also that PQ subtends an angle of 45° at the origin.

(NOTE. The *image* Q of a point P in a given line is a point such that PQ is bisected at right angles by the given line.)



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The mid-point of PQ is

$$\left(\frac{4 + 6}{2}, \quad \frac{8 + 2}{2}\right), \quad \text{i.e. } (5, 5).$$

When $x = 5$ and $y = 5$, $x - 3y + 10 = 5 - 15 + 10 = 0$. Therefore (5, 5) lies on the line

$$x - 3y + 10 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The slope of $PQ = (8 - 2)/(4 - 6) = 6/(-2) = -3$.

The slope of line (1) is $\frac{1}{3}$.

The product of these slopes is $\frac{1}{3}(-3) = -1$, therefore the lines PQ and (1) are perpendicular, therefore PQ is bisected at right angles by line (1), i.e. Q is the image of P in the line (1).

The slope of $OP = \frac{2}{6} = \frac{1}{3}$, and the slope of $OQ = \frac{8}{4} = 2$.

If $\theta = \angle POQ$,

$$\tan\theta = \pm\frac{(2 - \frac{1}{3})}{1 + 2 \times \frac{1}{3}} \qquad \left(\pm\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

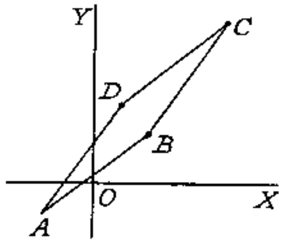
$$= \pm\frac{\frac{5}{3}}{\frac{5}{3}} = \pm 1,$$

therefore acute angle θ is 45°.

EXAMPLE (L.U.). If the points $(-2, -1)$, $(2, 2)$ and $(5, 6)$ are three vertices of a rhombus, find the co-ordinates of the fourth vertex, and the area of the rhombus.

Let $A \equiv (-2, -1)$, $B \equiv (x_1, y_1)$ be the fourth vertex, $C \equiv (5, 6)$, $D \equiv (2, 2)$.

Since $ABCD$ is a rhombus and therefore a parallelogram, the mid-point of AC is the mid-point of BD.



Now the mid-point of AC is $(\frac{3}{2}, \frac{5}{2})$ and the mid-point of BD is

$$\left(\frac{2 + x_1}{2}, \frac{y_1 + 2}{2}\right)$$

$$\therefore \frac{2 + x_1}{2} = \frac{3}{2}, \text{ i.e. } 2 + x_1 = 3 \quad \therefore x_1 = 1$$

$$\frac{y_1 + 2}{2} = \frac{5}{2}, \text{ i.e. } y_1 + 2 = 5 \quad \therefore y_1 = 3$$

therefore fourth vertex B is $(1, 3)$.

$$AD = \sqrt{[(2 + 2)^2 + (2 + 1)^2]} = \sqrt{(16 + 9)} = \sqrt{25} = 5.$$
$$CD = \sqrt{[(5 - 2)^2 + (6 - 2)^2]} = \sqrt{(9 + 16)} = \sqrt{25} = 5.$$

Therefore figure $ABCD$ is a parallelogram with two adjacent sides equal, i.e. $ABCD$ is a rhombus.

$$\text{Area } \triangle ABD = \text{area } \triangle BCD$$
$$= \tfrac{1}{2}BD \times (\tfrac{1}{2}AC)$$

(diagonals of rhombus bisect at right angles)

therefore area of rhombus

$$= 2 \times \text{area } \triangle ABD = 2 \times \tfrac{1}{2}BD \times \tfrac{1}{2}AC = \tfrac{1}{2}BD \times AC.$$

Now $AC = \sqrt{[(5 + 2)^2 + (6 + 1)^2]} = \sqrt{98} = 7\sqrt{2}.$

$BD = \sqrt{[(2 - 1)^2 + (2 - 3)^2]} = \sqrt{(1 + 1)} = \sqrt{2}.$

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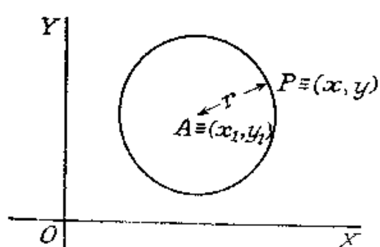
Therefore area of rhombus $= \frac{1}{2} \times \sqrt{2} \times 7\sqrt{2} = 7$ square units.

Theorem. *To find the equation of a circle whose centre is the point* (x_1, y_1), *and radius* r.



Let $A \equiv (x_1, y_1)$ be the centre of the circle, and $P \equiv (x, y)$ any point on the circle.

Then $AP = r$ and $AP^2 = r^2$.

Using the formula for the distance between two points,

$$AP^2 = (x - x_1)^2 + (y - y_1)^2.$$
$$\therefore (x - x_1)^2 + (y - y_1)^2 = r^2,$$

which is the required equation since P is any point on the circle.

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If $x_1 = y_1 = 0$, the equation of a circle centre O, radius r, is
$$x^2 + y^2 = r^2.$$

EXAMPLE. Find the equation of the circle, centre (3, 1), radius 2 units.

Using the formula of the previous theorem, the required equation is
$$(x - 3)^2 + (y - 1)^2 = 2^2,$$
$$\text{i.e. } x^2 - 6x + 9 + y^2 - 2y + 1 = 4,$$
$$\text{i.e. } x^2 + y^2 - 6x - 2y + 6 = 0.$$

Theorem. *To find the most general form of the equation of a circle.*

Consider a circle, centre (x_1, y_1), radius r. Its equation will be
$$(x - x_1)^2 + (y - y_1)^2 = r^2,$$
$$\text{i.e. } x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = r^2,$$
$$\text{i.e. } x^2 + y^2 - 2xx_1 - 2yy_1 + (x_1^2 + y_1^2 - r^2) = 0.$$

Let $x_1 = -g$, $y_1 = -f$ and $x_1^2 + y_1^2 - r^2 = c$, then the equation becomes the general form of the equation to a circle, viz.
$$x^2 + y^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots\ldots (1)$$

NOTE. From the substitutions used it can be seen that the circle whose equation is (1) has its centre at the point $(-g, -f)$, and its radius r is given by $x_1^2 + y_1^2 - r^2 = c$, where $x_1 = -g, y_1 = -f$, i.e. $g^2 + f^2 - r^2 = c$, i.e. $r^2 = g^2 + f^2 - c$, i.e. $r = \sqrt{(g^2 + f^2 - c)}$.

The general equation (1) of a circle can be seen to be an equation of the second degree in x and y involving the following details:

(i) the coefficients of x^2 and y^2 are equal,

(ii) there is no term in xy present.

Hence, in order that an equation of the second degree in x and y shall represent a circle, it is necessary that *the coefficients of x^2 and y^2 shall be equal, and also that there shall be no term in xy present.*

Theorem. *To find the equation of a circle passing through the points (x_1, y_1), (x_2, y_2), and (x_3, y_3).*

There are two methods that can be used.

Method (i). Let the equation of the circle be
$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since the three points lie on the circle,
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \ldots\ldots\ldots\ldots (1)$$
$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \ldots\ldots\ldots\ldots (2)$$
$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \ldots\ldots\ldots\ldots (3)$$

By solving these as simultaneous equations in f, g, c, the values of these quantities will be found, and hence the required equation.

Method (ii). Find the equations of the perpendicular bisectors of two of the sides of the triangle formed by the three given points, and their point of intersection $(\bar{x}, \bar{y})$ will be the centre of the required circle, whilst its radius r is given by $r^2 = (\bar{x} - x_1)^2 + (\bar{y} - y_1)^2$. The required equation with the determined values will then be
$$(x - \bar{x})^2 + (y - \bar{y})^2 = r^2.$$

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Example. Find the equation of the circle circumscribing the triangle whose vertices are (2, 1), (0, 2), (1, 0), and also find its centre and radius.

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since the given points lie on this circle,

$$4 + 1 + 4g + 2f + c = 0,$$
$$\text{i.e. } 5 + 4g + 2f + c = 0 \quad \ldots\ldots(1),$$
$$4 + 4f + c = 0 \quad \ldots\ldots(2),$$
$$1 + 2g + c = 0 \quad \ldots\ldots(3).$$

(2) − (1) gives, $-1 - 4g + 2f = 0,$

i.e. $4g - 2f = -1 \quad \ldots\ldots(4).$

(1) − (3) gives, $4 + 2g + 2f = 0,$

i.e. $2g + 2f = -4 \quad \ldots\ldots(5).$

(4) + (5) gives, $6g = -5 \therefore g = -\frac{5}{6}.$

Using this in (4),

$$2f = -\frac{10}{3} + 1 = -\frac{7}{3} \quad \therefore f = -\frac{7}{6}.$$

From (3), $1 - \frac{5}{3} + c = 0 \quad \therefore \quad c = \frac{2}{3}.$

Hence, the required equation is

$$x^2 + y^2 - \tfrac{5}{3}x - \tfrac{7}{3}y + \tfrac{2}{3} = 0.$$
$$\text{i.e. } 3x^2 + 3y^2 - 5x - 7y + 2 = 0$$

The centre of the circle is $(-g, -f)$, i.e. $(\frac{5}{6}, \frac{7}{6})$, and its radius

$$= \sqrt{(g^2 + f^2 - c)} = \sqrt{[(\tfrac{5}{6})^2 + (\tfrac{7}{6})^2 - \tfrac{2}{3}]}$$

$$= \sqrt{\left(\frac{25}{36} + \frac{49}{36} - \frac{2}{3}\right)} = \sqrt{\frac{50}{36}} = \frac{5\sqrt{2}}{6}.$$

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Theorem. *To find the equation of the tangent at the point (x_1, y_1) of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.*

Let $P \equiv (x_1, y_1)$, and $A \equiv (-g, -f)$ be the centre of the given circle. The tangent at P will be perpendicular to AP. The slope of AP is $(y_1 + f)/(x_1 + g)$, and, hence, the slope of the tangent at P is $-(x_1 + g)/(y_1 + f)$.

The equation of the tangent at P will be

$$y - y_1 = -\frac{(x_1 + g)}{y_1 + f}(x - x_1).$$

$$\text{i.e. } (y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1).$$
$$\text{i.e. } yy_1 + fy - y_1^2 - fy_1 = -xx_1 + x_1^2 - gx + gx_1.$$
$$\text{i.e. } xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1.$$

Adding $gx_1 + fy_1 + c$ to each side, the equation becomes,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$
$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

Since the point (x_1, y_1) lies on the given circle,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0.$$

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Using this in the equation of the tangent, it becomes,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

The *general rule* for writing down the equation of the tangent at the point (x_1, y_1) of the curve represented by the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

is:

Replace x^2 by xx_1, y^2 by yy_1, $2xy$ by $(xy_1 + yx_1)$, $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$, and leave the constant term unchanged.

EXAMPLE. Find the equation of the tangent at the point $(-2, 3)$ to the circle $x^2 + y^2 - 4x + 2y - 27 = 0$.

The equation at (x_1, y_1) of the tangent to this circle is

$$xx_1 + yy_1 - 2(x + x_1) + (y + y_1) - 27 = 0.$$

Therefore at the point $(-2, 3)$ the equation of the tangent is

$$-2x + 3y - 2(x - 2) + (y + 3) - 27 = 0,$$
$$\text{i.e. } -2x + 3y - 2x + 4 + y + 3 - 27 = 0,$$
$$\text{i.e. } -4x + 4y - 20 = 0,$$
$$\text{i.e. } x - y + 5 = 0 \ldots\ldots\ldots (1).$$

Check. By inspection, the centre of the given circle is $(2, -1)$ and its radius $\sqrt{2^2 + 1^2 + 27} = \sqrt{32} = 4\sqrt{2}$.

The perpendicular from the centre on the line (1) is

$$\pm\frac{(2 + 1 + 5)}{\sqrt{(1^2 + 1^2)}} = \pm\frac{8}{\sqrt{2}} = \pm 4\sqrt{2}.$$

Theorem. *To find the points of intersection of a straight line and a circle.*

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Let the equation of the line be

$$y = mx + c \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c_1 = 0 \ldots\ldots\ldots\ldots (2).$$

Since the points of intersection lie on both the line and the circle, they will be given by the solutions of equations (1) and (2) considered as simultaneous equations in x and y.

Substituting from (1) in (2) for y,

$$x^2 + (mx + c)^2 + 2gx + 2f(mx + c) + c_1 = 0$$
$$\text{i.e. } x^2(1 + m^2) + 2x(g + mf + mc) + c^2 + 2fc + c_1 = 0 \ldots. (3)$$

The equation (3) is a quadratic in x and will give two values for x, and from equation (1) the two corresponding values for y are found.

The points of intersection will be coincident (i.e. the line (1) will be a tangent) if the roots of equation (3) are coincident.

The condition for this is

$$[2(g + mc + fm)]^2 = 4(1 + m^2)(c^2 + 2fc + c_1)$$
$$\text{i.e. } (g + mc + fm)^2 = (1 + m^2)(c^2 + 2fc + c_1).$$

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When $g = f = 0$ and $c_1 = -r^2$, the equation of the circle becomes $x^2 + y^2 = r^2$, and the condition that the line $y = mx + c$ be a tangent to this circle will be

$$\begin{aligned} m^2c^2 &= (1 + m^2)(c^2 - r^2) \\ &= c^2 + c^2m^2 - r^2 - r^2m^2 \\ \text{i.e. } c^2 &= r^2(1 + m^2) \\ \therefore\ c &= \pm r\sqrt{(1 + m^2)}. \end{aligned}$$

Hence, the lines $y = mx \pm r\sqrt{(1 + m^2)}$ are tangents to the circle $x^2 + y^2 = r^2$ for all values of m, and this is known as the *slope equation* of the tangent.

This result could also be obtained as follows: The perpendicular from the centre O of the circle $x^2 + y^2 = r^2$ on the straight line $y = mx + c$ is $\pm c/\sqrt{(1 + m^2)}$.

If the line is to be a tangent to the circle this perpendicular is equal to the radius r of the circle.

$$\therefore\ r = \pm c/\sqrt{(1 + m^2)} \quad \therefore\ c = \pm r\sqrt{(1 + m^2)}.$$

NOTE. This method of using the perpendicular from the centre on the tangent as being equal to its radius frequently simplifies the working in questions dealing with tangents to a circle.

Theorem. *To find the length of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.*

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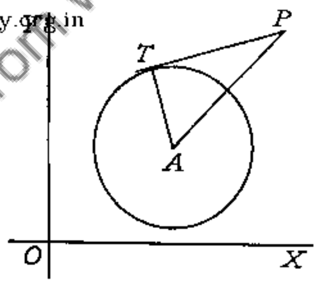

$A \equiv (-g, -f)$ is the centre of the circle, and $P \equiv (x_1, y_1)$; PT is the tangent from P to the circle,

$$\begin{aligned} \therefore\ AT^2 &= (\text{radius})^2 \\ &= g^2 + f^2 - c. \end{aligned}$$

Taking the distance between the two points A and P,

$$AP^2 = (x_1 + g)^2 + (g_1 + f)^2.$$

Using Pythagoras' theorem,

$$\begin{aligned} PT^2 &= AP^2 - AT^2 \\ &= [(x_1 + g)^2 + (y_1 + f)^2] - (g^2 + f^2 - c) \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c. \end{aligned}$$

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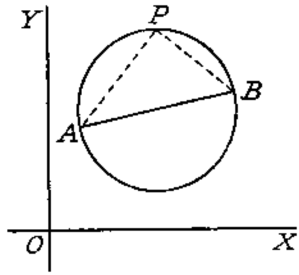
Thus, if the equation of the circle be $f(x, y) = 0$, with the *coefficients of x^2 and y^2 each unity* the square of the tangent from (x_1, y_1) to the circle will be $f(x_1, y_1)$.

Theorem. *To find the equation of the circle on the line joining (x_1, y_1) and (x_2, y_2) as diameter.*



Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, and $P \equiv (x, y)$ be any point on the required circle.

The slope of $AP = (y - y_1)/(x - x_1)$, and the slope of $BP = (y - y_2)/(x - x_2)$.

Since $\angle APB$ is the angle of a semicircle, AP and PB are perpendicular,

$$\therefore \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1,$$

$$\text{i.e. } (y - y_1)(y - y_2) = -(x - x_1)(x - x_2),$$

$$\text{i.e. } (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0,$$

which is the required equation since P is any point on the circle.

EXAMPLE (L.U.). Find the value of m such that the line $y = mx$ is a tangent to the circle $x^2 + y^2 + 2fy + c = 0$.

Find the equations of the tangents from the origin to the circle $x^2 + y^2 - 10y + 20 = 0$, and determine the points of contact.

Substituting $\qquad y = mx \ldots\ldots\ldots\ldots\ldots\ldots(1),$

in the equation $\qquad x^2 + y^2 + 2fy + c = 0 \ldots\ldots\ldots\ldots(2),$

the abscissæ of the points of intersection of the line (1) and the circle (2) are given by

$$x^2 + m^2x^2 + 2fmx + c = 0$$

$$\text{i.e. } x^2(1 + m^2) + 2fmx + c = 0 \ldots\ldots\ldots\ldots (3)$$

The line (1) is a tangent to the circle (2) if the roots of equation (3) in x are coincident;

$$\text{i.e. if } 4f^2m^2 = 4c(1 + m^2),$$

$$\text{i.e. if } m^2(f^2 - c) = c,$$

$$\text{i.e. if } m^2 = c/(f^2 - c)$$

$$\text{i.e. if } m = \pm\sqrt{\frac{c}{f^2 - c}}.$$

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Let the equation of the tangent from O to the circle

$$x^2 + y^2 - 10y + 20 = 0 \quad \ldots\ldots (4)$$

be the line (1).

From the first part of the question, using $c = 20,\ f = -5$,

$$m = \pm\sqrt{\frac{20}{25 - 20}} = \pm\sqrt{4} = \pm 2.$$

Therefore the tangents from O to the circle (4) are $y = \pm 2x$.

With $m = +2,\ f = -5,\ c = 20$, the equation (3) becomes,

$$5x^2 - 20x + 20 = 0,$$
$$\text{i.e. } 5(x - 2)^2 = 0 \quad \therefore \quad x = 2 \text{ (twice)}.$$

Using $y = 2x,\ y = 4$, therefore point of contact of line $y = 2x$ is (2, 4).

With $m = -2,\ f = -5,\ c = 20$, the equation (3) becomes,

$$5x^2 + 20x + 20 = 0,$$
$$\text{i.e. } 5(x + 2)^2 = 0, \quad \therefore \quad x = -2 \text{ (twice)},$$

and since $y = -2x,\ y = +4$. Therefore point of contact of line $y = -2x$ is (−2, 4).

EXAMPLE (L.U.). Find the equations of the two circles of radius 5 units which pass through the origin and whose centres lie on the line

$$x + y - 1 = 0.$$

Show that the point (2, 0) is inside one of these circles, and find the length of the tangent from this point to the other circle.

Let the equation of a circle be

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$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots (1)$$

Since the circle passes through the origin, $c = 0$.

With $c = 0$, the radius of the circle $= \sqrt{(g^2 + f^2)}$

$$\therefore \sqrt{(g^2 + f^2)} = \pm 5$$
$$\therefore g^2 + f^2 = 25 \quad \ldots\ldots (2)$$

The centre of the circle (1) is $(-g, -f)$, and since this lies on the line $x + y - 1 = 0$,

$$-g - f - 1 = 0, \text{ i.e. } f = -(g + 1) \quad \ldots\ldots (3)$$

Using (3) in (2) $\quad g^2 + g^2 + 2g + 1 = 25,$

$$\therefore 2g^2 + 2g - 24 = 0,$$
$$\therefore (g + 4)(g - 3) = 0,$$
$$\therefore g = 3 \text{ or } -4.$$

From (3) $\quad f = -4 \text{ or } +3.$

Hence, the equations of the circles are

$$\left.\begin{array}{l} x^2 + y^2 + 6x - 8y = 0 \quad \ldots\ldots (4). \\ x^2 + y^2 - 8x + 6y = 0 \quad \ldots\ldots (5). \end{array}\right\}$$

The centre of the circle (4) is (−3, 4) and its radius = 5.

The distance of (2, 0) from the centre (−3, 4) is

$$\sqrt{(5^2 + 4^2)} > 5,$$

therefore (2, 0) lies outside the circle (4).

The centre of circle (5) is (4, −3) and its radius is 5.

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The distance of (2, 0) from the centre (4, −3) is

$$\sqrt{(2^2 + 3^2)} = \sqrt{13} < 5$$

therefore (2, 0) lies inside the circle (5).

If t be the length of tangent from (x_1, y_1) to the circle (4), then

$$t = \sqrt{(x_1^2 + y_1^2 + 6x_1 - 8y_1)}.$$

Hence, when $x_1 = 2$ and $y_1 = 0$, $t = \sqrt{(4 + 12)} = 4$.

EXAMPLE (L.U.). Find the equation of the circle which has the line joining (−4, 3) and (8, −2) for a diameter. Write down the co-ordinates of the centre and the radius. Obtain the equation of the chord of which the origin is the mid-point.

The equation of the circle on the line joining (x_1, y_1), (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, therefore equation of the circle on the line joining (−4, 3), and (8, −2) as diameter is

$$(x + 4)(x - 8) + (y - 3)(y + 2) = 0,$$

$$\text{i.e. } x^2 + y^2 - 4x - y - 38 = 0 \quad \ldots\ldots\ldots\ldots (1).$$

The centre of this circle is $(2, \frac{1}{2})$ and its radius

$$= \sqrt{(4 + \tfrac{1}{4} + 38)} = \sqrt{\tfrac{169}{4}} = \tfrac{13}{2}.$$

Any line through the origin O is

$$y = mx \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Where this cuts the circle (1)

$$x^2 + m^2x^2 - 4x - mx - 38 = 0,$$

$$\text{i.e. } x^2(1 + m^2) - x(4 + m) - 38 = 0 \quad \ldots\ldots\ldots\ldots (3).$$

If x_1 and x_2 be the roots of equation (3),

$$x_1 + x_2 = \frac{4 + m}{1 + m^2}.$$

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If O be the mid-point of the chord (2), then

$$\frac{x_1 + x_2}{2} = 0, \quad \therefore \quad \frac{4 + m}{1 + m^2} = 0,$$

$$\therefore m = -4.$$

Hence, the chord of circle (1) bisected at O has as its equation

$$y = -4x,$$

$$\text{i.e. } y + 4x = 0.$$

EXAMPLE (L.U.). Find the points of intersection of the circle

$$x^2 + y^2 - 4x + 8y + 10 = 0$$

with the straight line $x - 2y - 5 = 0$.

Denoting these points by A, B, show that OA, OB are the tangents from the origin O. Find the equations of these tangents and prove that they are at right angles.

The equation of the line can be written

$$x = 2y + 5 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and where this line intersects the circle

$$x^2 + y^2 - 4x + 8y + 10 = 0 \quad \ldots\ldots\ldots\ldots (2),$$

by substituting from (1) in (2) for x,

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$$(2y + 5)^2 + y^2 - 4(2y + 5) + 8y + 10 = 0,$$
$$\text{i.e. } 4y^2 + 20y + 25 + y^2 - 8y - 20 + 8y + 10 = 0,$$
$$\therefore 5y^2 + 20y + 15 = 0,$$
$$\text{i.e. } y^2 + 4y + 3 = 0,$$
$$\text{i.e. } (y + 3)(y + 1) = 0,$$
$$\therefore y = -3 \text{ or } -1.$$

Using these in (1), $x = -1$ or 3,

$$\therefore A \equiv (-1, -3), \quad B \equiv (+3, -1).$$

The equation of OA is

$$y = \frac{-3}{-1}x,$$
$$\text{i.e. } y = 3x \ldots\ldots\ldots\ldots (3),$$

and the equation of OB is

$$y = \frac{-1}{3}x,$$
$$\text{i.e. } x + 3y = 0 \ldots\ldots\ldots\ldots (4).$$

The centre of circle (2) is $(2, -4)$, and its radius

$$= \sqrt{[2^2 + (-4)^2 - 10]} = \sqrt{(4 + 16 - 10)} = \sqrt{10}.$$

The perpendicular from $(2, -4)$ on the line (3) is of length

$$\pm\frac{(6 + 4)}{\sqrt{(1^2 + 3^2)}} = \pm\frac{10}{\sqrt{10}} = \pm\sqrt{10} = \text{radius of circle (2)},$$

and the perpendicular from $(2, -4)$ on the line (4) is of length

$$\pm\frac{(2 - 12)}{\sqrt{(1^2 + 3^2)}} = \pm\frac{(-10)}{\sqrt{10}} = \pm\sqrt{10} = \text{radius of circle (2)},$$

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therefore OA and OB whose equations are $y = 3x$ and $x + 3y = 0$ are tangents to the circle (2).

The slope of OA is 3, and the slope OB is $-\frac{1}{3}$, therefore product of slopes of OA and OB is $3 \times (-\frac{1}{3}) = -1$, therefore OA and OB are at right angles.

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EXAMPLE (L.U.). Prove that, for all numerical values of λ, the equation

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 + \lambda(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$$

represents a circle passing through the points of intersection of the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Find the equation of a circle which passes through the origin and the points of intersection of the circles

$$x^2 + y^2 - 2x - 6y + 2 = 0, \quad x^2 + y^2 - 5x - 8y + 3 = 0.$$

Prove that the circle so determined intersects the first of the two given circles at right angles.

Let (x_1, y_1) be a point of intersection of the circles

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \ldots\ldots\ldots (1),$$
$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \ldots\ldots\ldots (2).$$

Then,
$$x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 = 0 \ldots\ldots\ldots (3)$$
$$x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2 = 0 \ldots\ldots\ldots (4)$$

Consider the equation (λ any constant)

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 + \lambda(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0 \ldots (5)$$

If $\lambda \neq -1$, the coefficients of x^2 and y^2 in (5) will be equal (each equal to $1 + \lambda$), and there is no term in xy in the equation. Hence, equation (5) represents a circle.

Replacing x by x_1, y by y_1, the L.H.S. of (5)

$= x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1 + \lambda(x_1^2 + y_1^2 + 2g_2x_1 + 2f_2y_1 + c_2)$
$= 0$, using (3) and (4).

Hence, the circle represented by (5) passes through the points of intersection of the circles (1) and (2). Thus, the equation (5) *represents a circle passing through the points of intersection of circles* (1) *and* (2), except if $\lambda = -1$, when equation (5) is linear in x and y and represents the common chord of the two circles when they intersect, and *the radical axis* when they do not intersect.

Any circle passing through the intersections of the circles

$$x^2 + y^2 - 2x - 6y + 2 = 0 \ldots\ldots\ldots\ldots\ldots (6),$$
$$x^2 + y^2 - 5x - 8y + 3 = 0$$

is given by $x^2 + y^2 - 2x - 6y + 2 + \lambda(x^2 + y^2 - 5x - 8y + 3) = 0$, where λ is any constant not equal to -1.

If this circle pass through O, the constant term must be zero, i.e. $2 + 3\lambda = 0$, therefore $\lambda = -\frac{2}{3}$. Therefore required circle has as its equation

$$x^2 + y^2 - 2x - 6y + 2 - \tfrac{2}{3}(x^2 + y^2 - 5x - 8y + 3) = 0$$
i.e. $3x^2 + 3y^2 - 6x - 18y + 6 - 2x^2 - 2y^2 + 10x + 16y - 6 = 0$
i.e. $x^2 + y^2 + 4x - 2y = 0$. (7)

When two circles intersect at right angles the tangents (and therefore the radii) at a point of intersection are at right angles.

Thus, by Pythagoras' theorem, *the sum of the squares of their radii must equal the square on the distance between their centres.*

The circle (6) has centre (1, 3), radius $\sqrt{(1^2 + 3^2 - 2)} = \sqrt{8}$.

The circle (7) has centre $(-2, 1)$, radius $\sqrt{(2^2 + 1^2)} = \sqrt{5}$. The distance d between their centres is given by $d^2 = 3^2 + 2^2 = 13$.

The sum of the squares of their radii $= 8 + 5 = 13$, therefore the sum of the squares of their radii = square on their line of centres, therefore the circles (6) and (7) cut at right angles.

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EXAMPLES VIII

1. Prove that the lines $2x + y + 3 = 0$, $x - 2y - 1 = 0$ are perpendicular. If these two lines be taken as the sides of a rectangle whose other sides intersect at (3, 4), find the equations of these other sides and the area of the rectangle.

2. Prove that the perpendicular distance of the point (h, k) from the line $ax + by + c = 0$ is $(ah + bk + c)/\sqrt{(a^2 + b^2)}$.

Show that the lines $x = 5$, $y = 5$, $5x + 12y = 65$ all touch a circle whose centre is the origin.

3. A is the point (2, 3) and B the point $(0, -1)$. The angle BAC is a right angle and $BC = 5$ units.

Find the co-ordinates of the two possible positions of C.

4. Find the equations of the lines through the point (2, 3) which makes angles of 45° with the line $(x - 2y - 1) = 0$, and find also the area of the triangle formed by the three lines.

5. The line $ax + by + c = 0$ meets the x-axis at A and the y-axis at B, the triangle OAB, where O is the origin, being in the first quadrant. A point (h, k) is taken inside the triangle OAB.

Prove that $k \cdot OA + h \cdot OB + p \cdot AB = OA \cdot OB$, where p is the perpendicular from P to AB. Use the above result to prove that

$$p = (ah + bk + c)/\sqrt{(a^2 + b^2)}.$$

6. Find the co-ordinates of the point that divides the line joining (x_1, y_1) and (x_2, y_2) in the ratio of $l : m$.

The co-ordinates of three points A, B, C are respectively (2, 1), (1, − 1), (7, 2). A point P lies on the line $3x - y = 17$, and AP meets BC at Q. If $AP = 2AQ$, find the co-ordinates of P.

7. The origin O and the points $A(1, 0)$, $C(0, 7)$ are three vertices of the convex quadrilateral $OABC$; the angle BAC is 45° and AB is 5 units in length. Find the co-ordinates of B.

If G be the intersection of the diagonals, show that G divides AC in the ratio 1 : 7.

8. Find the equation of the straight line through the point (h, k) perpendicular to $ax + by + c = 0$.

The points A(−7, −8), $B(18, -8)$ are two vertices of an isosceles triangle ABC, in which $AB = BC$.

The altitude BM from B to the base AC has the equation $4x + 3y = 48$ Find the co-ordinates of M and C.

9. If the co-ordinates of a point P are (x_1, y_1) and a point Q (x_2, y_2), find x_1 and y_1 in terms of x_2, y_2 if PQ is bisected at right-angles by the line $x + 2y - 1 = 0$.

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Show that, if P describes the line $x - y = 0$, Q will describe the line $7x - y - 2 = 0$, and find the co-ordinates of the point common to the three lines.

10. Explain the geometrical meaning of the constants m and c in the equation of a straight line expressed in the form $y = mx + c$.

Find the equation of the line joining the point (3, 2) to the point of intersection of the lines $x - y + 4 = 0$, $y - 2x - 5 = 0$, and determine the inclination of each of these lines to the line so found.

11. The equations of the straight lines OA, OB are $y + \frac{1}{2}x = 0$, and $y - \frac{1}{2}x = 0$ respectively. From the point $P \equiv (x_1, y_1)$ a line is drawn parallel to OA meeting OB at M, and another line is drawn from P parallel to OB meeting OA at L. Through L and M lines are drawn perpendicular to OA and OB respectively meeting at Q.

Prove that the co-ordinates of Q are $(\frac{5}{8}x_1, \frac{5}{2}y_1)$. Hence, show that, if P describes a straight line inclined at 45° to the x-axis, Q also describes a straight line and this line is inclined at $\tan^{-1} 4$ to the x-axis.

12. The equation of a straight line is $x - 2y + 3 = 0$. Given a point P whose co-ordinates are (x_1, y_1), find in terms of x_1, y_1 the co-ordinates of the point Q such that PQ is bisected at right angles by the given line.

If P describes the circle $x^2 + y^2 = 1$, find the equation of the curve described by Q.

13. O is the origin and A the point (4, 3). $OABC$ is a parallelogram in the positive quadrant and the equation of OC is $y = 7x$. Find the co-ordinates of B and C if the area of the parallelogram is 50 square units; find also the co-

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ordinates of P on BC (produced if necessary) such that POA is an isosceles triangle.

14. A parallelogram whose sides are equally inclined to the co-ordinate axes has a vertex at $(2, -1)$, a diagonal along the line $x - 4y + 10 = 0$ and a pair of sides parallel to $3x + 4y = 0$. Find the equations of the sides and the co-ordinates of the remaining vertices.

Show that the lengths of the sides of the parallelogram are in the ratio $1 : 2$.

15. If A be the point $(2, 1)$ and C the point $(5, 2)$, find the equation of the line bisecting AC at right angles and the co-ordinates of the point B where this line meets the y-axis.

Find the co-ordinates of the point of intersection of the altitudes of the triangle ABC.

16. The sides BC, CA, AB of a triangle ABC lie along the lines $3x + 4y = 1$, $5x + y = 13$, $2x - 3y + 5 = 0$ respectively.

Find the co-ordinates of the point of intersection of the perpendiculars from the angular points to the opposite sides.

Show that the locus of a point P such that $CP^2 - BP^2 = 13$ is the line through A perpendicular to BC.

17. Explain how you can see, by inspection, that the four straight lines represented by the equations $2x - 3y - 1 = 0$, $2x - 3y + 5 = 0$, $5x + 6y = 0$, $5x + 6y + 2 = 0$, form a parallelogram.

Find the equations of its diagonals and its area.

18. Find the co-ordinates of (*a*) the foot of the perpendicular from the point $(2, -1)$ to the line $3x - 4y + 5 = 0$, and (*b*) the points on the line $3x - 4y + 5 = 0$ distant 5 units from the point $(2, -1)$.

19. Show that, if the perpendicular from the origin to a straight line is of length p and makes an angle α with the x-axis, the equation of the line is $x \cos \alpha + y \sin \alpha = p$.

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The perpendiculars from the origin to two straight lines are equal. Show that, if the sum of the angles the perpendiculars make with the positive direction of the x-axis is constant, the locus of their points of intersection is a straight line.

20. Write down the equations of the lines AP, BQ of slope m through the points A $(5, 0)$ and B $(-5, 0)$.

Find the value of m if these lines meet the line $4x + 3y = 25$ in points P and Q such that the distance PQ is 5 units.

21. If two points A, B have positive co-ordinates (x_1, y_1), (x_2, y_2) with respect to two perpendicular axes through a point O, prove that the area of the triangle OAB is equal to the numerical value of the expression $\frac{1}{2}(x_1y_2 - x_2y_1)$.

Find the area of the triangle whose vertices are $(1, 0)$, $(2, 1)$, $(4, 5)$.

22. Prove that the distance of the point (h, k) from the straight line $ax + by + c = 0$ is $(ah + bk + c)/(a^2 + b^2)^{\frac{1}{2}}$.

The co-ordinates of the vertices of the triangle ABC are $(2, 4)$, $(1, 1)$, and $(3, 2)$. The triangle DEF is drawn entirely outside the triangle ABC and with its sides parallel to and at unit distance from the sides of ABC. Find the equations of the sides of the triangle DEF.

23. The co-ordinates of the angular points A, B, C of a triangle are $(4, -3)$, $(13, 0)$, $(-2, 9)$ and points D, E, F are taken upon the sides so that

$$\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 2.$$

Find the areas of the triangles ABC, DEF and prove that their ratio is $3 : 1$.

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24. The point (p_2, q_2) is the image of the point (p_1, q_1) in the line

$$ax + by + c = 0.$$

Show that
$$a(q_1 - q_2) - b(p_1 - p_2) = 0,$$
$$a(p_1 + p_2) + b(q_1 + q_2) + 2c = 0.$$

Hence, express p_2, q_2 in terms of p_1, q_1, a, b, c, and find the co-ordinates of the image of the point (3, 1) in the line $x + 2y + 1 = 0$.

25. If the rectangular co-ordinates of the points A, B are (x_1, y_1), (x_2, y_2) respectively, and a point P be taken on AB such that $AP = \lambda \cdot PB$, show that the co-ordinates of P are

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}\right).$$

Find the ratio in which the line joining the points A (1, 5), B (7, −4) is divided by the line $2x = 3y$.

Find also the equation of the line through the origin which divides AB externally in the same ratio.

26. Prove that the lines $ax + by + c = 0$, $a'x + b'y + c' = 0$ are at right angles if $aa' + bb' = 0$.

The angular points of a triangle are A (3, 0), B (4, 6), C (−2, 3). Find the equations of the lines drawn through the angular points at right angles to the opposite sides, and show that these lines are concurrent.

27. Obtain a formula giving the inclination of two straight lines whose equations are given.

Find the equations of two straight lines through the point (−1, 1) which are inclined at an angle of 45° to the line $2x + y - 2 = 0$, and verify from their equations that they are at right angles.

Find, also, the length of the segment they intersect on the given line.

28. Find an expression for the area of a triangle in terms of the co-ordinates of its vertices, explaining the convention of signs adopted in connection with areas.

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The points A, B, C have co-ordinates (−1, 2), (3, 1), (0, 5) respectively. Find the co-ordinates of the point P on the line $y = 2x$, such that the triangles ABC, PAB are equal in area, taking into account the signs of the areas.

29. Explain carefully the meaning of the constants p, α in the equation $x \cos \alpha + y \sin \alpha = p$, and write down the co-ordinates of the foot of the perpendicular from the origin to the line represented by the equation.

L, M are the feet of the perpendiculars from the origin to the lines $3x + 4y = 10$, $12x - 5y = 13$. Find the area of the triangle OLM, the length LM, and the equation of the line LM.

30. Show that the area of a triangle whose angular points have co-ordinates (x_1, y_1), (x_2, y_2), (x_3, y_3) is $\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$.

The angular points A, B, C of a triangle are (−2, 1), (1, 2), (0, 4). Find the co-ordinates of the angular points of the triangle DEF having A, B, C as the mid-points of its sides, and thence show that the area of DEF is four times that of ABC.

31. $ABCD$ is a parallelogram and the equations of the lines AB, BC, CA are $2x + y - 8 = 0$, $x - y + 2 = 0$ and $x + 2y - 2 = 0$ respectively.

Find the equations of AD, DC and the co-ordinates of D, and verify that AC, BD have the same mid-point.

32. Prove that the co-ordinates of the point R, which divides the line PQ so that $p \cdot PR = q \cdot RQ$, are $\left(\frac{px_1 + qx_2}{p + q}, \frac{py_1 + qy_2}{p + q}\right)$, where (x_1, y_1), (x_2, y_2) are the co-ordinates of P and Q respectively.

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What is the position of R relative to P and Q when p and q have opposite signs?

O is the origin and the co-ordinates of the angular points B, C of a triangle ABC are $(-4, 1)$, $(5, -5)$ respectively. AO meets BC at D. If $2BD = DC$ and $3DO = OA$, find the co-ordinates of A.

Find also the co-ordinates of E and F, the points where BO and CO meet CA and AB respectively.

33. Find an expression for the angle between the two lines $y = m_1 x + c_1$, $y = m_2 x + c_2$, and obtain the condition that the two lines $Ax + By + C = 0$, $A'x + B'y + C' = 0$ should be perpendicular to each other.

Through a fixed point P of co-ordinates (x_0, y_0), two perpendicular lines are drawn. One of these lines meets the x-axis in A, and the other meets the y-axis in B. Show that, if O be the origin, the locus of the centroid (mean centre) of the four points O, A, P, B is a straight line and find its equation.

34. Obtain the condition that the lines

$$a_1x + b_1y + c_1 = 0, \text{ and } a_2x + b_2y + c_2 = 0$$

are perpendicular.

AB is a straight line whose equation is $5x - 12y + 7 = 0$. Find the equation of the perpendicular to AB from the point C $(-4, 13)$. If the foot of this perpendicular is D, find the point P on CD such that $CP = 4CD$, C and P being on opposite sides of AB.

35. Three points A $(2, 1)$, B $(6, 4)$, C $(-3, -8)$ are given. Find the equations of AB and BC.

A point P is such that its perpendicular distance from BC is twice its perpendicular distance from AB. Show that P lies on one or other of two straight lines and find their equations.

36. $OABC$ is a parallelogram, OA lying along the x-axis, OC along the line $y = 2x$, and B being the point $(4, 2)$.

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Find (i) the co-ordinates of A and C;

(ii) the equation of the diagonal AC;

(iii) the co-ordinates of a point P on AB produced such that the area of the triangle OPA is twice the area of the parallelogram $OABC$.

37. Show that the points whose rectangular Cartesian co-ordinates are $(3, 2)$, $(2, -1)$, $(8, -3)$ are three vertices of a rectangle. Find the co-ordinates of the fourth vertex, and the area of the rectangle.

38. Explain what is meant by the projection of a segment of a line on another line OX, and obtain an expression for it in terms of the length of the segment AB and the angle between the lines.

The vertices of a quadrilateral are the points A $(0, 0)$, B $(3, 1)$, C $(4, 2)$, D $(1, 6)$. Find the projections of the sides AB, BC, CD, DA on the positive directions of the axes of co-ordinates, taking the senses of the segments in the order of the letters.

39. A circle radius three units is to be drawn with its centre on the line $y = x - 1$ and passing through the point $P \equiv (7, 3)$. Show that two such circles are possible and find their equations. Show that one of these circles touches the axis of x, and, if Q is the point of contact, find the length of the minor arc PQ of this circle.

40. Show that $y = mx$ is a tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

if $(g + mf)^2 = c(1 + m^2)$. Find the equations of the tangents from the origin to the circle $x^2 + y^2 - 6x - 3y + 9 = 0$, and the co-ordinates of their point of contact.

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41. Show that the circles, whose equations are $x^2 + y^2 + 4x - 2y - 11 = 0$, and $x^2 + y^2 - 4x - 8y + 11 = 0$, intersect at right angles and find the length of the common chord.

42. Show that the equation $x^2 + y^2 + 2gx + 2fy + g^2 = 0$ represents a circle touching the x-axis.

Find the equation of a circle touching the x-axis at the point (5, 0) and passing through the point (7, 4). What are the co-ordinates of the point on the circle other than (5, 0), the tangent at which passes through the origin?

43. If O be the origin and P, Q are the intersections of the circle

$$x^2 + y^2 + 4x + 2y - 20 = 0,$$

and the straight line $x - 7y + 20 = 0$, show that OP and OQ are perpendicular. Find the equation of the circle through O, P, Q.

44. Find the equation of the straight line through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 + 8x - 4y + 6 = 0$.

Find also the equation of the circle through the origin and through the points of intersection of the given circles, and state its radius and the co-ordinates of its centre.

45. Find the equation of the circle having the two points $(-3, \quad 1)$, $(5, 5)$ as extremities of a diameter.

Find the equations of the lines, parallel to $3x - 4y = 0$, on which the circle intercepts a chord of length 8 units.

46. If A, B are the points (3, 0), (0, 4) respectively, show that the locus of a point P such that $4PA = 3PB$ is the circle $7(x^2 + y^2) - 96x + 72y = 0$.

Find the equation of the tangent at the origin to this circle, and show that the above circle cuts at right angles the circle $(2x - 3)^2 + (2y - 4)^2 = 25$.

47. A circle with centre (6, 0) passes through the intersections of the circle $x^2 + y^2 - 4x = 0$ and the line $x = 3$. Find its equation, and show that the two circles intersect at right angles.

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48. Find the equations of the two circles which touch the line $3x + 4y = 15$, and the axis of y, and pass through the point (1, 2). Determine the distance between their centres.

49. Find the centres and radii of the circles which touch the line $4x - 3y = 0$, and the axis of x, and which pass through the point (2, 2). Determine the equation of their common chord.

50. Find the equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

at the point (x_1, y_1).

Find the equations of the tangents from the origin to the circle

$$x^2 + y^2 - 40x - 40y + 80 = 0$$

51. Find a formula for the perpendicular distance at the point (x_1, y_1) from the line $ax + by + c = 0$.

Circles are drawn with centre (5, 1) to cut the two lines $3x - 4y - 1 = 0$ and $12x + 5y + k = 0$.

Show that, if k has one of two values, which are to be found, the intercepts made by any one of the circles on the two lines will be of equal length.

Find also the equation of the circle which makes an intercept of length 4 units on each line.

52. Show that the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle, and find its centre and radius.

Prove, geometrically or otherwise, that the circles $x^2 + y^2 = 4$, $x^2 + y^2 + \frac{10}{3}x = \frac{56}{9}$, have only two common tangents and that their equations are $3x \pm 4y = 10$.

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53. Find the equation of the circle which has its centre at the point (1, 2) and touches the line $5x + 12y = 42$.

Show that it touches the axis of y, and find the equation of the other tangent from the origin.

54. Find the equation of the chord AB of the circle

$$x^2 + y^2 + 2x - 4y - 4 = 0$$

which is bisected at the point (1, 3).

Calculate the length of the tangents to the circle from the point (3, 5) and the angle between those tangents.

55. Prove that the general equation to all circles which make an intercept of 3 units on the x-axis and touch the y-axis is

$$x^2 + y^2 - (2a + 3)x - 2by + b^2 = 0,$$

where $b^2 = a(a + 3)$.

Show that their centres lie on the curve whose equation is $4x^2 - 4y^2 = 9$.

56. (i) Find the centre and radius of the circle which passes through the points (7, 5), (6, −2), (−1, −1).

(ii) The line joining (5, 0) to $(10 \cos\theta, 10 \sin\theta)$ is divided internally in the ratio 2 : 3 at P.

If θ varies, prove that the locus of P is a circle, and find its centre and radius.

57. Find the equation of the circle whose centre is at the point (5, −7) and which passes through the origin.

If this circle intercepts on the straight line $3x + 4y = c$, a segment of length 14 units, find the value or values of c.

58. Given the three circles $x^2 + y^2 - 16x + 60 = 0$, $x^2 + y^2 - 12x + 20 = 0$ $x^2 + y^2 - 16x - 12y + 84 = 0$, find (i) the co-ordinates of the point such that the lengths of the tangents from it to each of the three circles are equal; (ii) the length of each tangent.

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59. Find the length of the chord $x + 2y = 5$ of the circle whose equation is $x^2 + y^2 = 9$.

Determine also the equation of the circle described on this chord as diameter.

60. If O be the origin and P and Q the intersections of the circle

$$x^2 + y^2 + 4x + 2y - 20 = 0$$

and the straight line $x - 7y + 20 = 0$, show that OP and OQ are perpendicular.

Find the equation of the circle through O, P, Q.

61. Find the co-ordinates of the points of contact of the tangents from (−16, 0) to the circle $x^2 + y^2 = 16$, and prove that each of these tangents also touches the circle $x^2 + y^2 - 24x + 95 = 0$.

62. P is a point on the straight line $3x + 4y = 11$, whose x-co-ordinate is unity. Find the equation to the circle radius 3 units, which touches the given line at P and which lies on that side of the line which is furthest from the origin. Find also the length cut off from the axis OY by the circle.

63. Prove that the four points (0, 2), (0, 9), (3, 0), (6, 0) lie on a circle. Find its equation and the length of its radius.

64. A semicircle is drawn with the line joining the two points (4, 2) and (2, 5) as its diameter, on the side of the diameter away from the origin.

Find the co-ordinates of the point on this semicircle whose distance from the point (4, 2) is 2 units.

65. The co-ordinates of the extremities A, B of a diameter of a circle are (2, 5), (6, 2) respectively.

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Find the equation of the circle, and the angle between the tangents which pass through the point $(-2, 1)$.

66. Show that the equation of any circle may be put in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Obtain the equations of the two circles passing through the two points $(3, 0)$, $(7, 0)$ and touching the y-axis. Determine their centres and radii.

67. Find the condition that the line $y = mx + c$ may touch the circle $x^2 + y^2 = a^2$.

Find the equations of the two tangents from the point $(3, -2)$ to the circle $x^2 + y^2 = 4$, and deduce the value of the angle between them.

Verify your result by calculating this angle by some other method.

68. $OABC$ is a square of side $2a$. Taking OA, OC as co-ordinate axes, find the equation of the circle inscribed in the square.

Any tangent to this circle meets OA in P, and OC in Q. Prove that

$$PA \cdot QC = 2a^2.$$

69. Obtain the equation of the tangent at the point (x_1, y_1) of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, and show that it passes through the origin if $gx_1 + fy_1 + c = 0$.

Hence, or otherwise, find the points of contact of the tangents from the origin to the circle $x^2 + y^2 - 6x - 2y + 9 = 0$, and find the equations of these tangents.

70. Find the equation of the circle described on the line joining $(15, 5)$ to $(7, -1)$ as diameter.

Show that the abscissae of the points in which the line $y = mx$ cuts the above circle are given by the equation $x^2(1 + m^2) - 2x(2m + 11) + 100 = 0$.

Hence, or otherwise, obtain the equations of the tangents from the origin to the circle.

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71. Find the equation of the circle through the points $(2, 2)$, $(-3, 1)$, $(5, 2)$ and obtain the co-ordinates of its centre and the length of its radius.

Find also the co-ordinates of the point of intersection of the tangents to the circle at the points $(2, 2)$ and $(5, 2)$.

72. Show that the equation $x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle, and find its radius and the co-ordinates of its centre.

Any straight line is drawn through the point $(3, -2)$ and K is the foot of the perpendicular to it from the point $(-2, 1)$. Find the equation of the locus of K and state what curve it represents.

73. Find the equations of the two tangents to the circle $x^2 + y^2 = a^2$ that have gradient m.

One diagonal of a square that circumscribes the circle $x^2 + y^2 = 5$ has gradient 3. Find the equations of the four sides of the square.

74. If the circles

$$x^2 + y^2 + ax + by + c = 0$$
$$x^2 + y^2 + px + qy + r = 0$$

intersect, find the equation of the common chord.

The circle $x^2 + y^2 - 20y + 15 = 0$ meets the line $x - y = 1$ at the points A, B. Find the equation of the circle that passes through A, B and the point $(3, 1)$.

75. Find the equation of the circle described on the line joining the two points $(9, 2)$ and $(21, 18)$ as diameter and show that it touches the x-axis.

Determine the equation of another circle that passes through the same two points and touches the x-axis.

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76. Show that the line $lx + my + n = 0$ touches the circle

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

if $(lx_0 + my_0 + n)^2 = R^2(l^2 + m^2)$.

A circle has unit radius. Its centre lies in the first quadrant and it touches the x-axis and the line $3y = 4x$. Find its equation and show that the line $3x + 4y = 15$ touches it.

Determine the radius of a second circle whose centre lies in the first quadrant and which also touches the x-axis, the line $3y = 4x$, and the line $3x + 4y = 15$.

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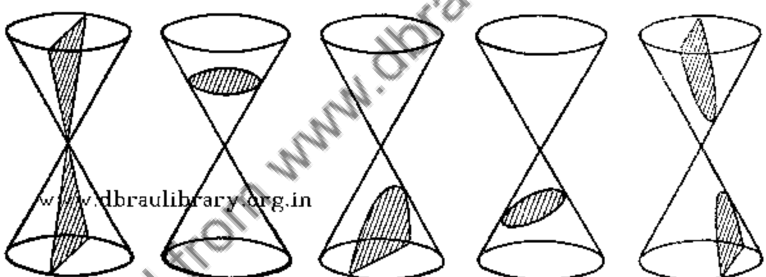
CHAPTER IX

Conic Sections—The Parabola, Ellipse, and Hyperbola

Conic sections are the sections of a double cone (i.e. two equal circular cones placed with their vertices in contact and having the same axis) made by a plane.

These conic sections or conics consist of the following curves: (i) a pair of intersecting straight lines; (ii) a circle; (iii) a parabola; (iv) an ellipse; (v) a hyperbola.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(i) (ii) (iii) (iv) (v)

(i) *A pair of intersecting straight lines* is formed by a plane section of the double cone through the common vertex, and these straight lines will be generators of the double cone.

(ii) *A circle* is formed by a plane section of the double cone perpendicular to the common axis.

(iii) A *parabola* is formed by a plane section of the double cone parallel to a generator.

(iv) An *ellipse* is formed by a plane section of the double cone, cutting only one half of the double cone, but neither perpendicular to the common axis nor parallel to a generator.

(v) A *hyperbola* is formed by a plane section cutting both halves of the double cone, but not passing through the common vertex.

The mathematical definition of a conic section is given in the following:

A *conic section* is the locus of a point that moves in a plane so that its distance from a fixed point (*focus*) in the plane bears a constant ratio (*eccentricity*) to its distance from a fixed straight line (*directrix*) in the plane.

The magnitude of the eccentricity (usually denoted by e) determines the type of curve, thus:

(i) for a pair of intersecting straight lines, eccentricity $e = \infty$;
(ii) for a circle, $e = 0$;
(iii) for a parabola, $e = 1$;
(iv) for an ellipse, $e < 1$;
(v) for a hyperbola, $e > 1$.

The pair of intersecting straight lines can be seen to be a special (degenerate) form of the hyperbola, and the circle is a degenerate form of the ellipse.

Definitions. The *latus rectum* of any conic is a chord through its focus parallel to the directrix.

The *centre* of a conic is the point that bisects all chords of the conic passing through it.

THE PARABOLA

Definition. A *parabola* is a conic section whose eccentricity is unity.

This would be stated more fully as: A *parabola* is the locus of a point that moves in a plane so that its distance from a fixed point (focus) in the plane is equal to its distance from a fixed straight line (directrix) in the plane.

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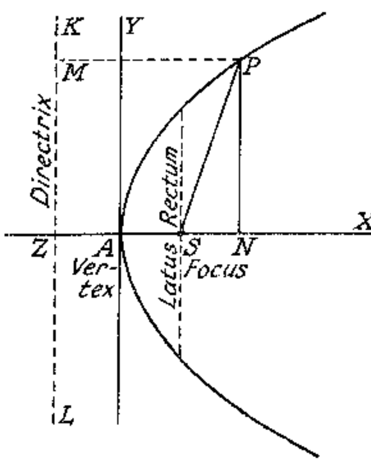
Theorem. *To find the standard (canonical) form of the equation of a parabola, i.e. the simplest form with the most convenient choice of origin and axes.*

NOTE. Standard notation is used throughout.

Let S (the fixed point) be the focus, and KL (fixed straight line) be the directrix. Draw SZ perpendicular to KL and choose ZS produced as the axis of x. Let A, the mid-point of SZ, be the origin of co-ordinates, and SZ be of length $2a$. Hence A lies on the parabola. AY perpendicular to SZ is the y-axis, and $P = (x, y)$ is any point on the parabola with this choice of origin and axes. PM is the perpendicular from P on KL, and PN is the ordinate of P.



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Now the focus $S \equiv (a, 0)$.

From the diagram

$$PM = NZ \text{ (opposite sides of rectangle)}$$
$$= AZ + AN = a + x.$$

Using the formula for the distance between two points,

$$PS^2 = (x - a)^2 + y^2.$$

From the definition of the parabola,

$$PS = PM \quad \therefore PS^2 = PM^2.$$
$$\text{i.e. } (x - a)^2 + y^2 = (a + x)^2$$
$$\text{i.e. } x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2$$
$$\text{i.e. } y^2 = 4ax,$$

which is the required canonical (simplest) form of the equation of the parabola, since P is any point on the locus.

NOTE. This equation should always be used as the equation of the parabola if choice be permitted as it is the simplest form possible.

Simple Properties of the Parabola $y^2 =$. Since y^2 is always positive, it follows that x is always positive if a is positive and negative if a is negative. Therefore, when a is positive, the curve lies entirely to the right of AY, and, when a is negative, it lies entirely to the left of AY.

As $x \to \infty$, so $y \to \pm\infty$ ($\to$ means 'approaches') and therefore both ends of the curve extend to infinity.

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For any value of x there are two equal and opposite values of y, therefore the curve is symmetrical about the x-axis, which is known as the *axis* of the curve.

The origin A is known as the *vertex* of the parabola.

Theorem. *To prove that the latus rectum of the parabola $y^2 = 4ax$ is of length $4a$.*

For the parabola $y^2 = 4ax$, when $x = a$, i.e. at the focus S, $y^2 = 4a^2 \quad \therefore y = \pm 2a$. Hence the length of the latus rectum is $2a + 2a = 4a$, and the length of the semi-latus rectum is $2a$.

Theorem. *To find the equation of the chord of the parabola $y^2 = 4ax$, joining the points $P \equiv (x_1, y_1)$, $Q \equiv (x_2, y_2)$, and deduce the equation of the tangent at the point (x_1, y_1) of the parabola.*

Since the points P and Q lie on the parabola $y^2 = 4ax$,

$$y_1^2 = 4ax_1 \quad \ldots\ldots (1)$$
$$y_2^2 = 4ax_2 \quad \ldots\ldots (2)$$

(1) − (2) gives $\quad y_1^2 - y_2^2 = 4a(x_1 - x_2) \quad \ldots\ldots (3)$

The equation of PQ is $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2} \quad \ldots\ldots (4)$

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From (3) × (4) $(y - y_1)(y_1 + y_2) = 4a(x - x_1)$
which is the required equation of PQ.

Replacing y_2 by y_1 the equation of the tangent at (x_1, y_1) is

$$\begin{aligned} 2y_1(y - y_1) &= 4a(x - x_1) \\ \text{i.e. } yy_1 - y_1^2 &= 2ax - 2ax_1 \\ \text{i.e. } yy_1 &= 2ax - 2ax_1 + 4ax_1 \text{ (using (1))} \\ \text{i.e. } yy_1 &= 2a(x + x_1) \end{aligned}$$

NOTE. This result could be obtained more simply by using calculus.

When $x_1 = y_1 = 0$, the equation is that of the tangent at the vertex A, and is $x = 0$, i.e. the y-axis.

Theorem. *To find the points in which the line $y = mx + c$ cuts the parabola $y^2 = 4ax$, and hence to deduce the equation of the tangent of slope m to the parabola.*

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$y^2 = 4ax \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

The abscissae of the points of intersection of the line (1) and the parabola (2) are obtained by substituting from (1) for y in (2), giving

$$(mx + c)^2 = 4ax$$

$$\text{i.e. } m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad \ldots\ldots\ldots \quad (3)$$

The equation (3) is a quadratic in x and will give two values of x (real, coincident, or complex), and the two corresponding values of y will be obtained by using these in equation (1).

If the line (1) is to touch the parabola (2) it is necessary that the roots of equation (3) in x are equal, and the condition for this is

$$\begin{aligned} 4(mc - 2a)^2 &= 4m^2c^2 \\ \text{i.e. } m^2c^2 - 4amc + 4a^2 &= m^2c^2 \\ \text{i.e. } 4amc &= 4a^2 \\ \text{i.e. } c &= a/m. \qquad \text{(since } a \neq 0) \end{aligned}$$

Thus, the line $y = mx + a/m$ is a tangent to the parabola $y^2 = 4ax$ for all values of m and this is known as the *slope equation* of the parabola.

If this line pass through the point (h, k), then

$$k = mh + a/m$$

$$\text{i.e. } m^2h - mk + a = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

This shows, being a quadratic in m, that two tangents (real, coincident, or imaginary) can be drawn from a point to a parabola.

The product of the slopes of these tangents, using the theory of quadratics, is $+a/h$, and the tangents will be perpendicular if this is equal to -1,

$$\begin{aligned} \text{i.e. if } a/h &= -1 \\ \text{i.e. if } h &= -a, \end{aligned}$$

which means that the point (h, k) must lie on the directrix.

Thus perpendicular tangents to a parabola must meet on the directrix.

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Theorem. *To find the equation of the normal at the point (x_1, y_1) to the parabola $y^2 = 4ax$.*

The equation of the tangent to the parabola at (x_1, y_1) is $yy_1 = 2a(x + x_1)$, and its slope is $2a/y_1$. Hence the slope of the normal is $-y_1/2a$, and the equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1) \quad \ldots\ldots\ldots\ldots \text{(1)}$$

The slope equation of the normal can be deduced from this in the following manner.

Let the slope of the normal be m, then

$$m = -y_1/2a, \text{ i.e. } y_1 = -2am.$$

Also, since (x_1, y_1) lies on the parabola $y^2 = 4ax$

$$x_1 = y_1^2/4a = 4a^2m^2/4a = am^2.$$

Using these results for y_1 and x_1 in (1), the slope equation of the normal is

$$y + 2am = m(x - am^2)$$
$$\text{i.e. } y = mx - 2am - am^3 \quad \ldots\ldots\ldots \text{(2)}$$

If the normal (2) pass through the point (h, k), then

$$k = mh - 2am - am^3,$$
$$\text{i.e. } am^3 + m(2a - h) + k = 0.$$

Since this is a cubic equation in m there will be three values of m satisfying it, and this shows that three normals can be drawn from any point to a parabola, one of which must be real (the other two may be real, coincident, or imaginary).

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Example. Find the points of intersection of the line $2y = x + 6$ and the parabola $y^2 = 8x$, and the equations of the tangents and normals to the parabola at these points of intersection.

The line is given by

$$2y = x + 6,$$
$$\text{i.e. } x = 2y - 6 \quad \ldots\ldots\ldots\ldots \text{(1)},$$

and the parabola is given by

$$y^2 = 8x \quad \ldots\ldots\ldots\ldots \text{(2)}.$$

Substituting from (1) in (2) for x, the ordinates of the points of intersection are given by

$$y^2 = 16y - 48,$$
$$\text{i.e. } y^2 - 16y + 48 = 0,$$
$$\text{i.e. } (y - 4)(y - 12) = 0,$$
$$\therefore y = 4 \text{ or } 12.$$

Using these in (1) $\quad x = 2 \text{ or } 18.$

Thus the required points of intersection are (2, 4), (18, 12).

The equation of the tangent to the parabola (2) at the point (x_1, y_1) is

$$yy_1 = 4(x + x_1),$$

therefore the tangent at (2, 4) is

$$4y = 4(x + 2),$$
$$\text{i.e. } y = x + 2 \quad \ldots\ldots\ldots\ldots \text{(3)}$$

and the tangent at (18, 12) is

$$12y = 4(x + 18),$$
$$\text{i.e. } 3y = x + 18 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The line (3) has slope = 1, therefore slope of normal at the point (2, 4) is -1, and the equation of the normal will be

$$y - 4 = -(x - 2),$$
$$\text{i.e. } x + y = 6.$$

The line (4) has slope $\frac{1}{3}$, therefore slope of normal at the point (18, 12) is -3, and the equation of the normal will be

$$y - 12 = -3(x - 18),$$
$$\text{i.e. } 3x + y = 66.$$

EXAMPLE. Find the equations of the two tangents that can be drawn from the point (2, 3) to the parabola $y^2 = 4x$.

The equation of a tangent of slope m to the parabola $y^2 = 4ax$ is

$$y = mx + a/m.$$

Hence, for the parabola $y^2 = 4x$, the equation of the tangent of slope m is

$$y = mx + 1/m \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If this line pass through the point (2, 3), then

$$3 = 2m + 1/m,$$
$$\text{i.e. } 3m = 2m^2 + 1,$$
$$\text{i.e. } 2m^2 - 3m + 1 = 0,$$
$$\text{i.e. } (m - 1)(2m - 1) = 0,$$
$$\therefore m = 1 \text{ or } \tfrac{1}{2}.$$

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When $m = 1$ the tangent is $y = x + 1$, and when $m = \frac{1}{2}$ the tangent is

$$y = \tfrac{1}{2}x + 1/\tfrac{1}{2}, \quad \text{i.e. } y = \tfrac{1}{2}x + 2, \quad \text{i.e. } 2y = x + 4.$$

EXAMPLE. Find the equations of the normals from the point (5, 2) to the parabola $y^2 = 4x$.

The equation of a normal of slope m to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3.$$

If this normal pass through the point (5, 2)

$$2 = 5m - 2am - am^3.$$

In the given parabola $a = 1$ and therefore

$$2 = 5m - 2m - m^3,$$
$$\text{i.e. } m^3 - 3m + 2 = 0.$$

Using the factor theorem this becomes

$$(m - 1)^2(m + 2) = 0$$
$$\therefore m = 1 \text{ (twice), or } -2,$$

and the required equations of the normals are

$$y = x - 2 - 1 \qquad (m = 1 \text{ twice})$$
$$\text{i.e. } y = x - 3 \text{ (twice)}$$

and

$$y = -2x + 4 + 8, \qquad (m = -2)$$
$$\text{i.e. } 2x + y = 12.$$

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EXAMPLE (H.S.C.). A variable tangent to the parabola $y^2 = 4ax$ meets the circle $x^2 + y^2 = r^2$ at P and Q. Prove that the locus of the mid-point of PQ is $x(x^2 + y^2) + ay^2 = 0$.

Let the variable tangent have slope m and its equation will be

$$y = mx + a/m \quad \text{.....(1).}$$

By geometry, the mid-point of PQ is the foot of the perpendicular from O to the variable tangent to the parabola.

The equation of this perpendicular through O is

$$x + my = 0 \quad \text{.....(2),}$$

since its slope will be $-1/m$.

Now the mid-point of PQ lies on both the lines represented by equations (1) and (2) and its locus will be obtained by eliminating m between these two equations.

From (2) $m = -x/y$.

Using this in (1) the required equation is

$$y = -\frac{x^2}{y} + \frac{a}{(-x/y)},$$

$$\text{i.e. } y = \frac{-x^2}{y} - \frac{ay}{x},$$

$$\text{i.e. } xy^2 = -x^3 - ay^2,$$

$$\text{i.e. } xy^2 + x^3 + ay^2 = 0,$$

$$\text{i.e. } x(x^2 + y^2) + ay^2 = 0.$$

EXAMPLE (H.S.C.). The mid-point of a variable chord P_1P_2 of the parabola $y^2 = 4ax$ lies on a fixed line $y = ka$, where k is a constant.

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Show that the locus of the points of intersection of the tangents at P_1 and P_2 is a straight line.

Let $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$.

The mid-point of P_1P_2 has an ordinate $\frac{1}{2}(y_1 + y_2)$ and, since it lies on the line $y = ka$,

$$\tfrac{1}{2}(y_1 + y_2) = ka,$$

$$\text{i.e. } y_1 + y_2 = 2ka \quad \text{.....(1)}$$

The tangents at P_1 and P_2 to the parabola are respectively

$$yy_1 = 2a(x + x_1) \quad \text{.....(2)}$$

$$yy_2 = 2a(x + x_2) \quad \text{.....(3)}$$

For the points of intersection of the tangents (2) − (3) gives

$$y(y_1 - y_2) = 2a(x_1 - x_2).$$

$$\therefore\ y = \frac{2a(x_1 - x_2)}{(y_1 - y_2)} \quad \text{.....(4).}$$

Since P_1 and P_2 lie on the parabola $y^2 = 4ax$,

$$y_1^2 = 4ax_1 \quad \text{.....(5)}$$

$$y_2^2 = 4ax_2 \quad \text{.....(6)}$$

(5) − (6) gives

$$y_1^2 - y_2^2 = 4a(x_1 - x_2),$$

$$\text{i.e. } (y_1 - y_2)(y_1 + y_2) = 4a(x_1 - x_2)$$

$$\text{i.e. } \frac{y_1 + y_2}{4a} = \frac{x_1 - x_2}{y_1 - y_2}.$$

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Using the result (1) this becomes

$$\frac{2ka}{4a} = \frac{x_1 - x_2}{y_1 - y_2}, \quad \text{i.e.} \quad \frac{x_1 - x_2}{y_1 - y_2} = \frac{k}{2}.$$

Using this in (4) for the points of intersection of the tangents

$$y = 2a \times \tfrac{1}{2}k = ak$$

Hence the required locus is the line $y = ak$ which is parallel to the axis of the parabola.

EXAMPLE (H.S.C.). (*a*) Show that the foot of the perpendicular from the focus of a parabola to any tangent lies on the tangent at the vertex.

(*b*) A variable line is drawn through the point $(0, 2a)$ meeting the parabola $y^2 = 4ax$ in the points P, Q. Prove that the equation of the locus of the mid-point of PQ is $y^2 = 2a(x + y)$.

(*a*) Let the parabola have as its equation $y^2 = 4ax$. Then the focus S is $(a, 0)$ and the tangent at the vertex is the y-axis, i.e. $x = 0$.

Let P be the point (x_1, y_1). Then the equation of the tangent at P is

$$yy_1 = 2a(x + x_1) \quad \ldots\ldots\ldots (1)$$

The slope of this tangent is $2a/y_1$, and therefore the slope of the perpendicular to it from S is $-y_1/2a$, and the equation of this perpendicular is

$$y = \frac{-y_1}{2a}(x - a)$$

$$\text{i.e. } yy_1 = \frac{-y_1^2}{2a}(x - a) \quad \ldots\ldots\ldots (2)$$

But, since P lies on the parabola,

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$$y_1^2 = 4ax_1 \quad \therefore \quad y_1^2/2a = 2x_1.$$

Using this in (2) the equation becomes

$$yy_1 = -2x_1(x - a) \quad \ldots\ldots\ldots (3)$$

Taking (1) − (2) for the point of intersection of the tangent and the perpendicular to it through S,

$$\begin{aligned} 0 &= (2ax + 2ax_1) + (2xx_1 - 2ax_1) \\ &= 2x(a + x_1) \\ \therefore\ x &= 0 \qquad\qquad (x_1 \neq -a) \end{aligned}$$

i.e. the foot of the perpendicular from the focus on any tangent lies on the tangent at the vertex.

NOTE. This result is much more easily obtained as follows:

The equation of the variable tangent is $y = mx + a/m$.

The perpendicular from the focus $(a, 0)$ on this tangent will have slope $(-1/m)$ and its equation will therefore be

$$y = -\frac{1}{m}(x - a).$$

Now both these lines can be seen to make the same intercept a/m on the y-axis.

Hence, since the y-axis is the tangent at the vertex, the foot of the perpendicular from the focus to any tangent lies on the tangent at the vertex.

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(*b*) Let m be the slope of the variable line, and its equation will be

$$y - 2a = mx \quad \text{.......... (1).}$$

Where this line meets the parabola

$$y^2 = 4ax \quad \text{.......... (2),}$$

by substituting for x from (1) in (2)

$$y^2 = \frac{4a}{m}(y - 2a)$$

$$\text{i.e. } y^2 - \frac{4a}{m}y + \frac{8a^2}{m} = 0 \quad \text{.......... (3).}$$

If y_1 and y_2 be the roots of (3) and $(\bar{x}, \bar{y})$ be the mid-point of PQ, then

$$\bar{y} = \frac{y_1 + y_2}{2} = \frac{4a}{2m} = \frac{2a}{m} \quad \text{.......... (4)}$$

Using this in (1), since $(\bar{x}, \bar{y})$ lies on line (1)

$$\frac{2a}{m} - 2a = m\bar{x}$$

$$\therefore \bar{x} = 2a\left(\frac{1}{m^2} - \frac{1}{m}\right) \quad \text{.......... (5)}$$

From (4) $1/m = \bar{y}/2a$, which, when used in (5) gives

$$\bar{x} = 2a\left(\frac{\bar{y}^2}{4a^2} - \frac{\bar{y}}{2a}\right)$$

$$= \frac{\bar{y}^2}{2a} - \bar{y},$$

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$$\therefore \bar{y}^2/2a = \bar{x} + \bar{y},$$

$$\text{i.e. } \bar{y}^2 = 2a(\bar{x} + \bar{y}).$$

Changing to running co-ordinates the required locus has as its equation

$$y^2 = 2a(x + y).$$

Parametric Representation of a Parabola. In the case of the parabola $y^2 = 4ax$ it is readily seen that the point $(at^2, 2at)$ lies on the parabola for all values of t.

Thus $x = at^2$, $y = 2at$ are equations which will give a parabola $(y^2 = 4ax)$, and, taken together, are known as the *parametric equation* of the parabola and t is the *parameter*.

Theorem. *To prove that, if t_1 be the parameter of one extremity of a focal chord of the parabola $x = at^2$, $y = 2at$, then $-1/t_1$ will be the parameter of the other end of the focal chord.*

Let P be the point whose parameter is t_1, and Q the other extremity of the focal chord having parameter t_2, i.e. $P \equiv (at_1^2, 2at_1)$, $Q \equiv (at_2^2, 2at_2)$.

The equation of PQ is

$$\frac{y - 2at_1}{2at_1 - 2at_2} = \frac{x - at_1^2}{at_1^2 - at_2^2}$$

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$$\text{i.e. } \frac{y - 2at_1}{2(t_1 - t_2)} = \frac{x - at_1^2}{(t_1 - t_2)(t_1 + t_2)}$$

$$\text{i.e. } \frac{y - 2at_1}{2} = \frac{x - at_1^2}{t_1 + t_2}.$$

Since the focus (a, o) lies on this line

$$\frac{-2at_1}{2} = \frac{a - at_1^2}{t_1 + t_2},$$

$$\therefore\ -t_1 = \frac{1 - t_1^2}{t_1 + t_2}, \qquad (a \neq o)$$

$$\text{i.e. } -t_1^2 - t_1 t_2 = 1 - t_1^2$$

$$\therefore\ -t_1 t_2 = 1, \text{ i.e. } t_2 = -1/t_1.$$

Theorem. *To prove that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.*

Taking the parametric equation of the parabola $x = at^2$, $y = 2at$, with P and Q the extremities of any focal chord, let the parameter of P be t_1. Then, by the previous theorem, the parameter of Q will be $-1/t_1$.

Hence $P \equiv (at_1^2, 2at_1)$, $Q \equiv (a/t_1^2, -2a/t_1)$.

Using the equation $yy_1 = 2a(x + x_1)$ for the equation of the tangent at (x_1, y_1) to the parabola, the equation of the tangent at P will be

$$2at_1 y = 2a(x + at_1^2)$$

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$$\text{i.e. } t_1 y = x + at_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}.$$

Similarly the equation of the tangent at Q to the parabola will be

$$\frac{y}{-t_1} = x + \frac{a}{t_1^2},$$

$$\text{i.e. } -t_1 y = xt_1^2 + a \quad \ldots\ldots\ldots\ldots\ldots \text{(2)}.$$

Adding (1) and (2) for the point of intersection of the tangents at P and Q

$$0 = x(1 + t_1^2) + a(1 + t_1^2)$$

$$\text{i.e. } x + a = 0, \text{ since } 1 + t_1^2 \neq 0.$$

This is the equation of the directrix and shows that the tangents at the ends of a focal chord meet on the directrix.

From (1) and (2) the slopes of the tangents at P and Q are respectively $1/t_1$ and $-t_1$, the product of which is -1, and therefore the tangents at P and Q are at right angles.

Thus, *the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.*

Example (H.S.C.). Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ and show that it meets the parabola again in the point $(at_1^2, 2at_1)$, where $t_1 = -t - 2/t$.

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If a chord of the parabola moves in such a way that the normal at its extremities intersect on the parabola, prove that the mid-point of the chord describes the parabola $y^2 = 2a(x + 2a)$.

The equation of the tangent at the point $(at^2, 2at)$ is found by replacing x_1, y_1 respectively by at^2 and $2at$ in the equation

$$yy_1 = 2a(x + x_1),$$

giving

$$2aty = 2a(x + at^2)$$

$$\text{i.e. } ty = x + at^2.$$

The slope of this tangent is $1/t$, and therefore the slope of the normal is $-t$.

Hence the equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2)$$

$$\text{i.e. } y = -tx + 2at + at^3.$$

If the point $(at_1^2, 2at_1)$ lie on this normal, then

$$2at_1 = -att_1^2 + 2at + at^3,$$

$$\text{i.e. } 2t_1 = -tt_1^2 + 2t + t^3,$$

$$\text{i.e. } tt_1^2 + 2t_1 - t(2 + t^2) = 0,$$

$$\text{i.e. } [tt_1 + (2 + t^2)](t_1 - t) = 0,$$

$$\text{i.e. } tt_1 = -2 - t^2 \text{ or } t_1 = t,$$

$$\therefore t_1 = -t - 2/t, \text{ or } t_1 = t.$$

Hence the normal meets the parabola again in the point $(at_1^2, 2at_1)$, where $t_1 = -t - 2/t$.

Let $P \equiv (at_2^2, 2at_2)$, $Q \equiv (at_3^2, 2at_3)$ be the extremities of the chord.

Then the normals at P and Q meet the parabola again in the point $(at_1^2, 2at_1)$ since they intersect on the parabola.

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By the first part of the question

$$t_1 = -t_2 - \frac{2}{t_2},$$

and

$$t_1 = -t_3 - \frac{2}{t_3},$$

$$\text{i.e. } -t_2 - \frac{2}{t_2} = -t_3 - \frac{2}{t_3},$$

$$\text{i.e. } t_2 - t_3 = \frac{2}{t_3} - \frac{2}{t_2},$$

$$\therefore t_2 - t_3 = \frac{2}{t_2 t_3}(t_2 - t_3),$$

$$\therefore t_2 t_3 = 2 \quad \ldots\ldots(1),$$

since $t_2 \neq t_3$.

The mid-point of PQ is given by (x, y), where

$$x = \tfrac{1}{2}a(t_2^2 + t_3^2) \quad \ldots\ldots(2),$$

$$y = a(t_2 + t_3).$$

$$\therefore y^2 = a^2(t_2^2 + 2t_2t_3 + t_3^2)$$

$$= a^2(t_2^2 + t_3^2) + 2a^2t_2t_3$$

$$= 2ax + 4a^2, \quad \text{(using (1) and (2))}$$

$$\text{i.e. } y^2 = 2a(x + 2a),$$

which is the required equation.

Example (H.S.C.). Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Two perpendicular tangents to the parabola meet at P and intersect the tangent at the vertex in L, M. Show that $LM = PS$, where S is the focus of the parabola.

It has been shown in the previous question that the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$ty = x + at^2.$$

The equations of the tangents at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ will be

$$t_1 y = x + at_1^2 \quad \ldots\ldots (1)$$
$$t_2 y = x + at_2^2 \quad \ldots\ldots (2)$$

The slopes of these tangents are $1/t_1$ and $1/t_2$ respectively, and if they are perpendicular,

$$\frac{1}{t_1} \times \frac{1}{t_2} = -1, \text{ i.e. } t_1 t_2 = -1 \quad \ldots\ldots (3).$$

The equation of the tangent at the vertex of the parabola $y^2 = 4ax$ is $x = 0$, and using this in (1),

$$y = at_1$$
$$\therefore L \equiv (0, at_1),$$

and similarly $\quad M \equiv (0, at_2)$.

$$\therefore LM^2 = (at_1 - at_2)^2 = a^2(t_1 - t_2)^2.$$

At P where the lines (1) and (2) intersect, from (1) − (2)

$$y(t_1 - t_2) = a(t_1^2 - t_2^2),$$
$$\therefore y = a(t_1 + t_2) \quad (t_1 \neq t_2)$$

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From (1) $\quad x = t_1 y - at_1^2 = at_1(t_1 + t_2) - at_1^2$
$$= at_1 t_2$$
$$= -a \quad \text{(by (3))}.$$

Now $\quad S \equiv (a, 0)$

$$\therefore SP^2 = (2a)^2 + a^2(t_1 + t_2)^2$$
$$= 4a^2 + a^2(t_1 + t_2)^2$$
$$= a^2[-4t_1 t_2 + (t_1 + t_2)^2]$$
[using $-1 = t_1 t_2$ from (3)]
$$= a^2[-4t_1 t_2 + t_1^2 + 2t_1 t_2 + t_2^2]$$
$$= a^2[t_1^2 - 2t_1 t_2 + t_2^2]$$
$$= a^2[t_1 - t_2]^2$$
$$\therefore LM^2 = SP^2, \text{ i.e. } LM = SP.$$

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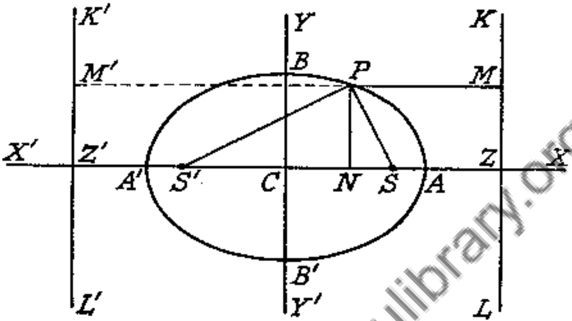
The Ellipse

Definition. The ellipse is a conic section whose eccentricity is less than unity.

The full form of this statement would be: The ellipse is the locus of a point that moves in a plane so that its distance from a fixed point (focus) in that plane bears a constant ratio (eccentricity), less than unity, to its distance from a fixed straight line (directrix) in the plane. The notation used in the following theorem will be used throughout.

Theorem. *To find, with the most convenient choice of origin and axes, the simplest (canonical) form of the equation of an ellipse.*

S is the focus and KL the directrix, with e (<1) the eccentricity. SZ is the perpendicular from S on KL and is produced to form the x axis.



SZ is divided internally at A and externally at A' in the ratio $e : 1$ and the length AA' is taken as $2a$. AA' is bisected at C, which is taken as the origin of co-ordinates, with CY as the y-axis which cuts the ellipse in B and B'.

S' is a point on the x-axis to the left of CY such that $CS' = CS$, and $K'L'$ is a line on the left of CY parallel to CY, and at the same distance from CY as LK, with Z' the point in which it cuts the x-axis.

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With the above choice of origin and axes let $P \equiv (x, y)$ be any point on the ellipse with PN its ordinate and PM, PM' the perpendiculars from P on the lines KL and $K'L'$ respectively.

From the construction,

$$SA = e \,.\, AZ \qquad (1)$$
$$SA' = e \,.\, A'Z \qquad (2)$$

and A and A' are points on the ellipse.

From (1) + (2)

$$\begin{aligned} SA + SA' &= e(AZ + A'Z), \\ \text{i.e. } AA' &= e[(CZ - CA) + (CZ + A'C)], \\ \text{i.e. } 2a &= e[2CZ - a + a] = 2e \,.\, CZ, \\ \therefore\ CZ &= a/e, \text{ and } Z \equiv (a/e, 0). \end{aligned}$$

From (2) − (1),

$$\begin{aligned} SA' - SA &= e(A'Z - AZ), \\ \text{i.e. } (CS + CA') - (CA - CS) &= eAA', \\ \text{i.e. } 2CS + a - a &= 2ae, \\ \therefore\ 2CS &= 2ae, \\ \text{i.e. } CS &= ae, \text{ and } S \equiv (ae, 0). \end{aligned}$$

Now, using the distance between two points,

$$SP^2 = (x - ae)^2 + y^2.$$

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From the diagram,

$$PM = NZ = CZ - CN$$
$$= a/e - x.$$

Using the definition of the ellipse,

$$SP = e\,.\,PM = e(a/e - x) = a - ex,$$
$$\therefore SP^2 = (a - ex)^2.$$

Hence, $$(x - ae)^2 + y^2 = (a - ex)^2,$$

i.e. $$x^2 - 2aex + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2,$$

$$\therefore x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

i.e. $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

where $(1 - e^2)$ is positive and less than unity, since $e < 1$ and positive.

This is the equation to the ellipse since P is any point on the locus, but it is usual in the standard form to replace $a^2(1 - e^2)$ by b^2 giving the result

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $b^2 = a^2(1 - e^2)$.

NOTE. This equation of the ellipse should always be taken if choice be permitted in the particular problem.

Simple Properties of the Ellipse $x^2/a^2 + y^2/b^2 = 1$.

The equation of the ellipse can be written www.dbraulibrary.org.in

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}.$$

If $x^2 > a^2$, it is seen that y is not real. Similarly, if $y^2 > b^2$, it is seen that x is not real. Hence, there is no portion of the curve outside the limits $x = \pm a$, and $y = \pm b$, i.e. the curve is *closed.*

For every value of x such that $x^2 < a^2$, there are two equal and opposite values of y, and for every value of y such that $y^2 < b^2$, there are two equal and opposite values of x.

Thus the curve is symmetrical about the two axes, and from this symmetry it can be seen that S' is a second focus, with $K'L'$ as its corresponding directrix.

The line AA' is known as the *major axis* and the line BB' is the *minor axis* of the ellipse where the ellipse cuts the y- axis in B and B'.

When $x = 0$ in the equation of the ellipse, $y = \pm b$, and therefore $BB' = 2b$.

The point C is known as the *centre* of the ellipse, and can be shown to bisect all chords passing through it.

Theorem. *To find the length of the semi-latus rectum of the ellipse*

$$x^2/a^2 + y^2/b^2 = 1.$$

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The latus rectum is the chord through the focus parallel to the directrix, and the y-values of its extremities will therefore be given when $x = ae$ in the equation

$$x^2/a^2 + y^2/b^2 = 1,$$

since the focus S is the point $(ae, 0)$, i.e. for the latus rectum,

$$e^2 + y^2/b^2 = 1, \;\therefore\; y^2/b^2 = 1 - e^2$$

But $$b^2 = a^2(1 - e^2), \text{ i.e. } (1 - e^2) = b^2/a^2$$
$$\therefore y^2/b^2 = b^2/a^2, \text{ giving } y^2 = b^4/a^2,$$
$$\therefore y = \pm b^2/a,$$

and the length of the semi-latus rectum is b^2/a.

Theorem. *To prove, using standard notation, that $SP + S'P = 2a$, for the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

Using the previous diagram, and geometry,

$$PM' = NZ' = CZ' + CN$$
$$= a/e + x$$
$$\therefore S'P = e \,.\, PM' = e(a/e + x) = a + ex.$$

It was earlier shown that $SP = a - ex$,

$$\therefore SP + S'P = (a - ex) + (a + ex)$$
$$= 2a.$$

This result gives the following mechanical method for drawing an ellipse:

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Take a length of string and fix the two ends to two points (foci) on the paper, and allow a pencil to move along the string, keeping it taut. Then the figure traced out is an ellipse (string must be of greater length than the distance between the two foci).

Theorem. *To find the points in which the line $y = mx + c$ cuts the ellipse $x^2/a^2 + y^2/b^2 = 1$, and to deduce the condition that the line shall be a tangent to the ellipse.*

$$y = mx + c \ldots\ldots\ldots\ldots (1)$$
$$x^2/a^2 + y^2/b^2 = 1 \ldots\ldots\ldots\ldots (2)$$

By substituting for y in terms of x from (1) in (2), the abscissae of the points of intersection are given by

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{i.e. } b^2x^2 + a^2(mx + c)^2 = a^2b^2,$$
$$\text{i.e. } x^2(b^2 + a^2m^2) + 2a^2cmx + (a^2c^2 - a^2b^2) = 0 \ldots\ldots (3)$$

The equation (3) is a quadratic in x, and hence the line (1) cuts the ellipse (2) in two points (real, coincident, or imaginary) whose abscissae are given by the equation (3), and whose corresponding ordinates are then obtained from equation (1).

The line (1) will be a tangent to the ellipse (2) if the roots of equation (3) in x are coincident, and the condition for this is

$$4a^4c^2m^2 = 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2),$$
$$\text{i.e. } a^2c^2m^2 = (b^2 + a^2m^2)(c^2 - b^2), \qquad (a^2 \neq 0)$$
$$\text{i.e. } a^2c^2m^2 = b^2c^2 - b^4 + a^2c^2m^2 - a^2b^2m^2,$$
$$\text{i.e. } b^2c^2 = b^4 + a^2b^2m^2,$$
$$\text{i.e. } c^2 = b^2 + a^2m^2, \qquad (b^2 \neq 0)$$
$$\therefore c = \pm\sqrt{(b^2 + a^2m^2)}.$$

Thus, for all values of m, the straight lines given by

$$y = mx \pm \sqrt{(a^2m^2 + b^2)}$$

are tangents to the given ellipse. This equation is known as the *slope equation* of the tangent.

The perpendiculars from the centre C on these tangents are $\pm\sqrt{(a^2m^2 + b^2)}/\sqrt{(1 + m^2)}$, and, therefore, there are two tangents to an ellipse of given slope m, which are equidistant from the centre C.

Theorem. *To find the equation of the line joining the points* (x_1, y_1), (x_2, y_2) *on the ellipse* $x^2/a^2 + y^2/b^2 = 1$, *and hence obtain the equation of the tangent at the point* (x_1, y_1).

The equation of the line joining the points (x_1, y_1), (x_2, y_2) is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since the points (x_1, y_1), (x_2, y_2) lie on the given ellipse

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$$x_1^2/a^2 + y_1^2/b^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$x_2^2/a^2 + y_2^2/b^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(2) − (3) gives $$\frac{x_1^2 - x_2^2}{a^2} + \frac{y_1^2 - y_2^2}{b^2} = 0$$

$$\text{i.e. } \frac{(x_1 - x_2)(x_1 + x_2)}{a^2} = \frac{-(y_1 - y_2)(y_1 + y_2)}{b^2} \quad \ldots\ldots (4)$$

Combining equations (1) and (4) by multiplication, thus ensuring that the points (x_1, y_1), (x_2, y_2) shall lie on the ellipse, the equation of the line is

$$\frac{(x - x_1)(x_1 + x_2)}{a^2} = -\frac{(y - y_1)(y_1 + y_2)}{b^2},$$

$$\text{i.e. } \frac{x(x_1 + x_2)}{a^2} - \frac{x_1^2}{a^2} - \frac{x_1x_2}{a^2} = -\frac{y(y_1 + y_2)}{b^2} + \frac{y_1^2}{b^2} + \frac{y_1y_2}{b^2},$$

$$\text{i.e. } \frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2}.$$

Using the result (2) in this, the required equation becomes

$$\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = 1 + \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2}.$$

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Replacing x_2 by x_1, y_2 by y_1 in this equation, the equation of the tangent at (x_1, y_1) will be

$$\frac{2xx_1}{a^2} + \frac{2yy_1}{b^2} = 1 + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$= 2 \qquad \text{(using (2)).}$$

Therefore the equation of the tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

NOTE. Taking $x_1 = a$, $y_1 = 0$ in the equation of the tangent, it is seen that the tangent at A is $x = a$, which is a line parallel to the y-axis. Similarly the equations of the tangents at A', B, B' are $x = -a$, $y = b$, $y = -b$ respectively.

Theorem. *To find the condition that the line $lx + my = n$ shall touch the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

$$lx + my = n \ldots\ldots (1)$$
$$x^2/a^2 + y^2/b^2 = 1 \ldots\ldots (2)$$

Let (x_1, y_1) be a point on the ellipse (2),

$$\therefore\ x_1^2/a^2 + y_1^2/b^2 = 1 \ldots\ldots (3).$$

The equation of the tangent at (x_1, y_1) is

$$xx_1/a^2 + yy_1/b^2 = 1 \ldots\ldots (4).$$

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If the line (1) be tangential to the ellipse, then, for some values of x_1 and y_1, the equations (1) and (4) will represent the same straight line, and the equations will be equivalent, giving

$$\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{1}{n}.$$

$$\therefore\ x_1 = \frac{a^2 l}{n} \ldots\ldots (5),$$

$$y_1 = \frac{b^2 m}{n} \ldots\ldots (6).$$

Using (5) and (6) in (3), the required condition is

$$\frac{a^4 l^2}{a^2 n^2} + \frac{b^4 m^2}{b^2 n^2} = 1,$$

i.e. $a^2 l^2 + b^2 m^2 = n^2$.

Theorem. *To find the equation of the normal at the point (x_1, y_1) of the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

The equation of the tangent to the ellipse at (x_1, y_1) is

$$xx_1/a^2 + yy_1/b^2 = 1,$$

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and the slope of this tangent is

$$\frac{-x_1}{a^2}\Big/\frac{y_1}{b^2}.$$

Hence the slope of the normal at (x_1, y_1) will be

$$\frac{y_1}{b^2}\Big/\frac{x_1}{a^2},$$

and the equation of this normal is

$$y - y_1 = \frac{y_1/b^2}{x_1/a^2}(x - x_1),$$

$$\text{i.e.} \frac{y - y_1}{y_1/b^2} = \frac{x - x_1}{x_1/a^2}.$$

EXAMPLE. Find the eccentricity and semi-latus rectum of the ellipse $2x^2 + 3y^2 = 5$.

The equation of the ellipse can be written

$$\frac{x^2}{5/2} + \frac{y^2}{5/3} = 1,$$

and comparing this with the standard form $x^2/a^2 + y^2/b^2 = 1$, $a^2 = 5/2$, $b^2 = 5/3$.

Hence, if e be the eccentricity,

$$b^2 = a^2(1 - e^2),$$

$$\text{i.e. } 5/3 = (5/2)(1 - e^2), \text{ i.e. } 1 - e^2 = 2/3,$$

$$\therefore e^2 = 1/3, \text{ and } e = 1/\sqrt{3} = \sqrt{3}/3.$$

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$$\text{The semi-latus rectum} = \frac{b^2}{a} = \frac{5/3}{\sqrt{(5/2)}} = \frac{5}{3} \times \frac{\sqrt{2}}{\sqrt{5}}$$

$$= \sqrt{10}/3.$$

EXAMPLE. Find where the line $2x + y = 3$ cuts the ellipse $4x^2 + y^2 = 5$, and obtain the equations of the tangents and normals at the points of intersection.

From the equation of the line

$$y = 3 - 2x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Substituting this in the equation of the ellipse

$$4x^2 + y^2 = 5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

for the points of intersection of the line (1) and the ellipse (2)

$$4x^2 + (3 - 2x)^2 = 5,$$

$$\text{i.e. } 4x^2 + 9 - 12x + 4x^2 = 5,$$

$$\text{i.e. } 8x^2 - 12x + 4 = 0,$$

$$\text{i.e. } (2x^2 - 3x + 1) = 0,$$

$$\therefore (x - 1)(2x - 1) = 0,$$

$$\therefore x = 1 \text{ or } 1/2.$$

From (1), $y = 1$ or 2.

Hence the points of intersection are $(1, 1)$, $(\frac{1}{2}, 2)$.

The equation of the tangent to the ellipse (2) at the point (x_1, y_1) is

$$4xx_1 + yy_1 = 5,$$

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Therefore the equations of the tangents at the points (1, 1). ($\frac{1}{2}$, 2) are respectively

$$\left.\begin{aligned} 4x + y &= 5, \\ \text{and} \quad 2x + 2y &= 5. \end{aligned}\right\}$$

The slopes of these tangents are -4, and -1, and the slopes of the corresponding normals will therefore be $\frac{1}{4}$, 1 respectively.

The equation of the normal at (1, 1) is

$$y - 1 = \tfrac{1}{4}(x - 1),$$
$$\text{i.e. } 4y - 4 = x - 1,$$
$$\text{i.e. } 4y = x + 3.$$

The equation of the normal at ($\frac{1}{2}$, 2) is

$$y - 2 = 1(x - \tfrac{1}{2}),$$
$$\text{i.e. } y = x + \tfrac{3}{2},$$
$$\text{i.e. } 2y = 2x + 3.$$

Example. Find the equations of the tangents from the point (1, 2) to the ellipse $x^2/9 + y^2/4 = 1$, and the co-ordinates of the points of contact of these tangents.

The tangents of slope m to the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \quad \ldots\ldots (1)$$

are given by $y = mx \pm \sqrt{(a^2m^2 + b^2)}$.

Thus the equations of the tangents of slope m to the given ellipse are

$$y = mx \pm \sqrt{(9m^2 + 4)}.$$

If these tangents pass through the point (1, 2)

$$2 = m \pm \sqrt{(9m^2 + 4)},$$
$$\text{i.e. } 2 - m = \pm\sqrt{9m^2 + 4)}.$$

Squaring, $4 - 4m + m^2 = 9m^2 + 4,$

$$\text{i.e. } 8m^2 + 4m = 0,$$
$$\text{i.e. } 4m(2m + 1) = 0,$$
$$\therefore m = 0 \text{ or } -\tfrac{1}{2}.$$

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Therefore the equations of the tangents are $y - 2 = m(x - 1)$.

When $m = 0$ $\quad y = 2,$

and when $m = -\frac{1}{2}$ $\quad y - 2 = -\frac{1}{2}(x - 1),$

$$\text{i.e. } 2y - 4 = -x + 1,$$
$$\text{i.e. } x + 2y = 5 \quad \ldots\ldots (2).$$

When $y = 2$ in the equation (1), $x = 0$ (twice). (using $a^2 = 9$, $b^2 = 4$)

Now the tangent at (x_1, y_1) to the ellipse (1) is

$$xx_1/9 + yy_1/4 = 1 \quad \ldots\ldots (3).$$

If the equations (2) and (3) represent the same line, they are equivalent, and, comparing coefficients

$$\frac{x_1/9}{1} = \frac{y_1/4}{2} = \frac{1}{5},$$
$$\text{i.e. } x_1 = 9/5,\ y_1 = 8/5.$$

Hence the points of contact are (0, 2), (9/5, 8/5).

Example (H.S.C.). The equation of a circle is $x^2 + y^2 = a^2$, and P is any point on it. The line joining P to the point $(0, a)$ cuts the axis of x in R, and the line joining R to the point $(0, b)$, where $b < a$, cuts the ordinate through P in Q. Prove that Q always lies on the ellipse

$$x^2/a^2 + y^2/b^2 = 1.$$

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Prove that the normal at P to the circle and the tangent at Q to the ellipse meet on the curve $(x^2/a + y^2/b)^2 = x^2 + y^2$.

Let P be the point (x_1, y_1), $A \equiv (0, a)$, $B \equiv (0, b)$.

The equation of AP is $\dfrac{y - a}{a - y_1} = \dfrac{x}{-x_1}$.

Therefore when $y = 0$, i.e. at R, $\dfrac{-a}{a - y_1} = \dfrac{x}{-x_1}$

$$\therefore x = \frac{ax_1}{a - y_1}, \text{ i.e. } R \equiv \left(\frac{ax_1}{a - y_1}, 0\right).$$

The equation of BR is $\dfrac{y - b}{b} = x\Big/\dfrac{-ax_1}{a - y_1}$.

At Q where $x = x_1$,

$$\frac{y - b}{b} = x_1\Big/\left(-\frac{ax_1}{a - y_1}\right) = \frac{-(a - y_1)}{a}$$

$$\therefore \frac{y}{b} - 1 = -1 + \frac{y_1}{a},$$

$$\therefore \frac{y}{b} = \frac{y_1}{a}, \text{ i.e. } y = by_1/a.$$

Thus, if $(\bar{x}, \bar{y})$ be the point Q

$$\bar{x} = x_1, \quad \bar{y} = by_1/a,$$

$$\text{i.e. } y_1 = a\bar{y}/b \quad \ldots\ldots\ldots\ldots (1).$$

Now (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$,

$$\therefore x_1^2 + y_1^2 = a^2 \quad \ldots\ldots\ldots\ldots (2).$$

Using equations (1) in (2),

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$$\bar{x}^2 + a^2\bar{y}^2/b^2 = a^2,$$

$$\text{i.e. } \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} = 1,$$

i.e. Q lies on the ellipse $x^2/a^2 + y^2/b^2 = 1$.

The equation of the tangent to the ellipse at Q is

$$x\bar{x}/a^2 + y\bar{y}/b^2 = 1,$$

$$\text{i.e. } xx_1/a^2 + yy_1/ab = 1 \quad \ldots\ldots\ldots\ldots (3).$$

The equation of the normal at P to the circle is

$$y = \frac{y_1}{x_1} x \quad \ldots\ldots\ldots\ldots (4).$$

Now if x_1, y_1 are eliminated between equations (3) and (4), using the result (2), the required locus is obtained.

From (3) $$\frac{x}{a} + \frac{y}{b} \times \frac{y_1}{x_1} = \frac{a}{x_1}.$$

Using (4) in this, $$\frac{x}{a} + \frac{y^2}{bx} = \frac{a}{x_1},$$

$$\text{i.e. } x^2/a + y^2/b = ax/x_1,$$

$$\therefore \left(\frac{x^2}{a} + \frac{y^2}{b}\right)^2 = \frac{a^2}{x_1^2} \times x^2 \quad \ldots\ldots\ldots\ldots (5)$$

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Using $y_1 = yx_1/x$ from (4) in (2),

$$x_1^2 + y^2x_1^2/x^2 = a^2,$$

$$\therefore \frac{a^2x^2}{x_1^2} = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Using (6) in (5), the required locus is

$$\left(\frac{x^2}{a} + \frac{y^2}{b}\right)^2 = x^2 + y^2.$$

Theorem. *To find the locus of the points of intersection of perpendicular tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$.*

The equations of tangents of slope m to the given ellipse are given by

$$y = mx \pm \sqrt{(a^2m^2 + b^2)},$$

$$\text{i.e. } y - mx = \pm\sqrt{(a^2m^2 + b^2)}.$$

Squaring this $\qquad (y - mx)^2 = a^2m^2 + b^2,$

$$\text{i.e. } y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2,$$

$$\text{i.e. } m^2(a^2 - x^2) + 2mxy + (b^2 - y^2) = 0 \ldots\ldots\ldots (1).$$

The result (1) is a quadratic equation in m and will give two values for m. Thus two tangents can be drawn from any point (x, y) to the ellipse $x^2/a^2 + y^2/b^2 = 1$, and, from the equation (1), the product of the slopes of these two tangents is $(b^2 - y^2)/(a^2 - x^2)$.

But the two tangents will be perpendicular if the product of the slopes is -1,

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$$\text{i.e. if } \frac{b^2 - y^2}{a^2 - x^2} = -1,$$

$$\text{i.e. if } b^2 - y^2 = -a^2 + x^2,$$

$$\text{i.e. if } x^2 + y^2 = a^2 + b^2.$$

Thus, for two perpendicular tangents, the point of intersection will lie on the circle

$$x^2 + y^2 = a^2 + b^2,$$

which is known as the *director circle* of the ellipse.

Definition. The circle described with the major axis of an ellipse as diameter is known as the *auxiliary circle* of the ellipse.

In the case of the ellipse $x^2/a^2 + y^2/b^2 = 1$, where $a^2 > b^2$, the equation of the auxiliary circle will be $x^2 + y^2 = a^2$.

Theorem. *To prove that corresponding ordinates of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and its auxiliary circle are in the ratio $b : a$.*

If $P \equiv (x, y)$ be any point on the auxiliary circle

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

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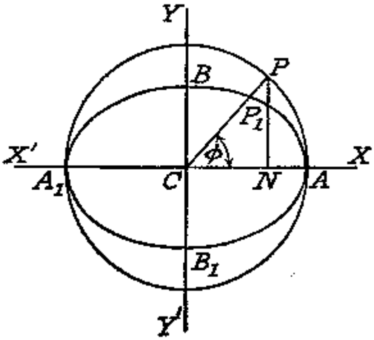
and PN be the ordinate of P cutting the ellipse in the same quadrant in P_1, then P_1 is the *corresponding point* on the ellipse to the point P on the auxiliary circle, and PN, P_1N are corresponding ordinates of the auxiliary circle and the ellipse.



Since P lies on the auxiliary circle (1),

$$\frac{CN^2}{a^2} + \frac{PN^2}{a^2} = 1 \quad\text{.................(2).}$$

Since P_1 lies on the ellipse,

$$\frac{CN^2}{a^2} + \frac{P_1N^2}{b^2} = 1 \quad\text{.................(3).}$$

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(3) − (2) gives

$$\frac{P_1N^2}{b^2} - \frac{PN^2}{a^2} = 0,$$

$$\therefore \frac{P_1N^2}{b^2} = \frac{PN^2}{a^2}, \text{ i.e. } \frac{P_1N^2}{PN^2} = \frac{b^2}{a^2},$$

$$\therefore \frac{P_1N}{PN} = \frac{b}{a} \quad\text{..........................(4).}$$

The angle PCN is usually denoted by φ and is known as the *eccentric angle* of the point P_1 on the ellipse.

From the diagram, $CN = a\cos\varphi$, and $PN = a\sin\varphi$. Thus, from the result (4), $P_1N = b\sin\varphi$.

From this result it can be seen that any point P_1 on an ellipse can be taken as $(a\cos\varphi, b\sin\varphi)$, where φ is the eccentric angle of P_1, and the equations $x = a\cos\varphi$, $y = b\sin\varphi$ are a parametric form of the equation to the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Sometimes the point P_1 is denoted by $[\varphi]$ which means that the eccentric angle of P_1 is φ and $P_1 \equiv (a\cos\varphi, b\sin\varphi)$.

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Theorem. *To find the equations of the tangent and normal at the point P, whose eccentric angle is φ, of the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

Now $P \equiv (a\cos\varphi,\ b\sin\varphi)$, and the equation of the tangent at (x_1, y_1) to the given ellipse is $xx_1/a^2 + yy_1/b^2 = 1$.

Replacing x_1 by $a\cos\varphi$, y_1 by $b\sin\varphi$, the equation of the tangent at P is $x\cos\varphi/a + y\sin\varphi/b = 1$.

The slope of this tangent is

$$-\frac{\cos\varphi}{a}\Big/\frac{\sin\varphi}{b},$$

therefore the slope of the normal at P is

$$\frac{\sin\varphi/b}{\cos\varphi/a},$$

and the equation of this normal is

$$y - b\sin\varphi = \frac{\sin\varphi/b}{\cos\varphi/a}(x - a\cos\varphi),$$

$$\text{i.e. } \frac{b(y - b\sin\varphi)}{\sin\varphi} = \frac{a(x - a\cos\varphi)}{\cos\varphi},$$

$$\text{i.e. } \frac{by}{\sin\varphi} - b^2 = \frac{ax}{\cos\varphi} - a^2,$$

$$\text{i.e. } \frac{ax}{\cos\varphi} - \frac{by}{\sin\varphi} = a^2 - b^2.$$

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EXAMPLE (H.S.C.). Obtain the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a\cos\theta, b\sin\theta)$.

The normal at P to the ellipse meets the x-axis at A and the y-axis at B. Show that the locus of Q, the mid-point of AB is the ellipse whose eccentricity is the same as that of the original ellipse. Also if the eccentric angle of P is $\frac{1}{4}\pi$, show that PQ is a tangent to the second ellipse.

As proved in the text, the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $(a\cos\theta, b\sin\theta)$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2.$$

At A, where $y = 0$,

$$\frac{ax}{\cos\theta} = a^2 - b^2,$$

$$\therefore\ x = \frac{\cos\theta}{a}(a^2 - b^2),$$

$$\text{i.e. } A \equiv \left[\frac{\cos\theta}{a}(a^2 - b^2),\ 0\right].$$

At B, where $x = 0$,

$$-\frac{by}{\sin\theta} = a^2 - b^2,$$

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$$\therefore\ y = -\frac{\sin\theta}{b}(a^2 - b^2),$$

$$\text{i.e. } B \equiv \left[0,\ -\frac{\sin\theta}{b}(a^2 - b^2)\right].$$

Hence, the co-ordinates of $Q \equiv (\bar{x}, \bar{y})$ are given by,

$$\bar{x} = \frac{\cos\theta}{2a}(a^2 - b^2); \qquad \bar{y} = -\frac{\sin\theta}{2b}(a^2 - b^2),$$

$$\text{i.e. } \frac{2a\bar{x}}{a^2 - b^2} = \cos\theta; \qquad \frac{2b\bar{y}}{a^2 - b^2} = -\sin\theta.$$

Squaring and adding these

$$\frac{4a^2\bar{x}^2}{(a^2 - b^2)^2} + \frac{4b^2\bar{y}^2}{(a^2 - b^2)^2} = \cos^2\theta + \sin^2\theta = 1.$$

Thus Q lies on the ellipse

$$\frac{4a^2x^2}{(a^2 - b^2)^2} + \frac{4b^2y^2}{(a^2 - b^2)^2} = 1.$$

If e_1 be the eccentricity of this ellipse

$$\left[\text{taking } b^2 < a^2 \text{ and } \therefore \frac{(a^2 - b^2)^2}{4a^2} < \frac{(a^2 - b^2)^2}{4b^2}\right]$$

$$\frac{(a^2 - b^2)^2}{4a^2} = \frac{(a^2 - b^2)^2}{4b^2}(1 - e_1^2)$$

$$\text{i.e. } b^2 = a^2(1 - e_1^2).$$

Now $b^2 = a^2(1 - e^2)$, where e is the eccentricity of the original ellipse,

$$\therefore\ e_1 = e$$ www.dbraulibrary.org.in

When $\theta = \frac{1}{4}\pi$, Q is the point $\left[\frac{a^2 - b^2}{2a\sqrt{2}},\ \frac{-(a^2 - b^2)}{2b\sqrt{2}},\right]$

and P is the point $(a/\sqrt{2}, b/\sqrt{2})$, therefore the equation of the tangent at Q to the ellipse traced out by Q is

$$\frac{4a^2x}{(a^2 - b^2)^2} \times \frac{(a^2 - b^2)}{2a\sqrt{2}} + \frac{4b^2y}{(a^2 - b^2)^2}\left[-\frac{(a^2 - b^2)}{2b\sqrt{2}}\right] = 1,$$

$$\text{i.e. } \frac{ax\sqrt{2}}{a^2 - b^2} - \frac{by\sqrt{2}}{a^2 - b^2} = 1$$

Using $x = a/\sqrt{2}$, $y = b/\sqrt{2}$, the L.H.S. of this becomes

$$\frac{a^2}{a^2 - b^2} - \frac{b^2}{a^2 - b^2} = 1,$$

therefore the point P lies on the tangent at Q, i.e. PQ is a tangent to the second ellipse.

EXAMPLE (H.S.C.). The tangents at two points P, Q on an ellipse intersect in the point T. Show that the medians of the triangle PTQ meet at a point on the ellipse if the eccentric angles of P and Q differ by 120° or 240°.

Let θ_1, θ_2 be the eccentric angles of P and Q on the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

therefore $P \equiv (a\cos\theta_1, b\sin\theta_1)$ and $Q \equiv (a\cos\theta_2, b\sin\theta_2)$.

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The equations of the tangents at P and Q are respectively

$$x \cos \theta_1/a + y \sin \theta_1/b = 1 \quad \ldots\ldots(1),$$
$$x \cos \theta_2/a + y \sin \theta_2/b = 1 \quad \ldots\ldots(2).$$

Taking (1) $\times \sin \theta_2 -$ (2) $\times \sin \theta_1$, we have for T

$$\frac{x}{a}(\sin \theta_2 \cos \theta_1 - \cos \theta_2 \sin \theta_1) = \sin \theta_2 - \sin \theta_1,$$

$$\text{i.e. } \frac{x}{a} \sin (\theta_2 - \theta_1) = 2 \cos \frac{\theta_1 + \theta_2}{2} \sin \frac{\theta_2 - \theta_1}{2},$$

$$\text{i.e. } \frac{2x}{a} \sin \frac{\theta_2 - \theta_1}{2} \cos \frac{\theta_2 - \theta_1}{2} = 2 \cos \frac{\theta_1 + \theta_2}{2} \sin \frac{\theta_2 - \theta_1}{2}$$

$$\text{i.e. } \frac{x}{a} = \frac{\cos \frac{1}{2}(\theta_1 + \theta_2)}{\cos \frac{1}{2}(\theta_2 - \theta_1)},$$

$$\therefore x = \pm 2a \cos \tfrac{1}{2}(\theta_1 + \theta_2),$$

since $\theta_2 \sim \theta_1 = 120°$ or $240°$.

Similarly $y = \pm 2b \sin \frac{1}{2}(\theta_1 + \theta_2)$.

The mid-point of PQ is given by

$$\begin{aligned} x &= \tfrac{1}{2}(a \cos \theta_1 + a \cos \theta_2) = a \cos \tfrac{1}{2}(\theta_1 + \theta_2) \cos \tfrac{1}{2}(\theta_1 - \theta_2) \\ &= \pm \tfrac{1}{2} a \cos \tfrac{1}{2}(\theta_1 + \theta_2) \\ y &= \tfrac{1}{2}(b \sin \theta_1 + b \sin \theta_2) = b \sin \tfrac{1}{2}(\theta_1 + \theta_2) \cos \tfrac{1}{2}(\theta_1 - \theta_2) \\ &= \pm \tfrac{1}{2} b \sin \tfrac{1}{2}(\theta_1 + \theta_2). \end{aligned}$$

The centroid of the triangle PTQ is the point dividing the join of the mid-point of PQ to T in the ratio 1 : 2, and, if its co-ordinates are $(\bar{x}, \bar{y})$, then

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$$\bar{x} = \frac{2[\pm \frac{1}{2} a \cos \frac{1}{2}(\theta_1 + \theta_2)] + [\pm 2a \cos \frac{1}{2}(\theta_1 + \theta_2)]}{2 + 1}$$

$$= \pm \frac{3a}{3} \cos \tfrac{1}{2}(\theta_1 + \theta_2) = \pm a \cos \tfrac{1}{2}(\theta_1 + \theta_2).$$

Similarly $\bar{y} = \pm b \sin \frac{1}{2}(\theta_1 + \theta_2)$.

Since the centroid is of the form $(a \cos \theta, b \sin \theta)$ it must lie on the given ellipse.

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EXAMPLE (H.S.C.). The normal at P, a point on the ellipse

$$x^2/a^2 + y^2/b^2 = 1,$$

meets the major axis at G and the ordinate at P meets this axis at M. Prove that $GM = (b^2/a) \cos \theta$, where θ is the eccentric angle of P.

Prove also that the locus of the mid-point of GP is a second ellipse concentric with the first, and find the eccentricity of the second ellipse if $a = 4$, $b = \sqrt{15}$.

P is the point $(a \cos \theta, b \sin \theta)$ on the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose centre is C the origin.

The normal at P to the ellipse has the equation

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad \ldots\ldots(1).$$

At G, $y = 0$ in (1) and

$$\frac{ax}{\cos\theta} = a^2 - b^2$$

$$\therefore\ x = \frac{a^2 - b^2}{a}\cos\theta$$

$$\text{i.e. } CG = \frac{a^2 - b^2}{a}\cos\theta.$$

Now $CM = a\cos\theta$,

$$\therefore\ MG = CM - CG = a\cos\theta - \frac{a^2 - b^2}{a}\cos\theta$$

$$= a\cos\theta - a\cos\theta + \frac{b^2}{a}\cos\theta$$

$$= \frac{b^2}{a}\cos\theta.$$

Let $(\bar{x}, \bar{y})$ be the mid-point of GP.

$$\therefore\ \bar{x} = \tfrac{1}{2}\left\{a\cos\theta + \frac{a^2 - b^2}{a}\cos\theta\right\}$$

$$= \tfrac{1}{2}\left\{\frac{2a^2 - b^2}{a}\right\}\cos\theta \qquad \ldots\ldots(2)$$

$$\bar{y} = \tfrac{1}{2}b\sin\theta \qquad \ldots\ldots(3)$$

From (2), $$\frac{\bar{x}}{\left(\dfrac{2a^2 - b^2}{2a}\right)} = \cos\theta.$$

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From (3), $$\bar{y}/\tfrac{1}{2}b = \sin\theta.$$

Squaring and adding these

$$\frac{\bar{x}^2}{\left(\dfrac{2a^2 - b^2}{2a}\right)^2} + \frac{\bar{y}^2}{\left(\dfrac{b}{2}\right)^2} = \cos^2\theta + \sin^2\theta = 1.$$

Thus the equation of the required locus (changing to running co-ordinates) is

$$\frac{x^2}{\left(\dfrac{2a^2 - b^2}{2a}\right)^2} + \frac{y^2}{\left(\dfrac{b}{2}\right)^2} = 1,$$

which is an ellipse having the same centre C as the original ellipse.

When $a = 4$ and $b = \sqrt{15}$ the equation of the locus becomes

$$\frac{x^2}{\left(\dfrac{32 - 15}{8}\right)^2} + \frac{y^2}{\dfrac{15}{4}} = 1,$$

$$\text{i.e. } \frac{x^2}{289/64} + \frac{y^2}{15/4} = 1.$$

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If e be the eccentricity of this ellipse

$$\frac{15}{4} = \frac{289}{64}(1 - e^2), \qquad [b^2 = a^2(1 - e^2)]$$

$$\therefore 1 - e^2 = \frac{15}{4} \times \frac{64}{289} = \frac{240}{289},$$

$$\therefore e^2 = 1 - \frac{240}{289} = \frac{49}{289},$$

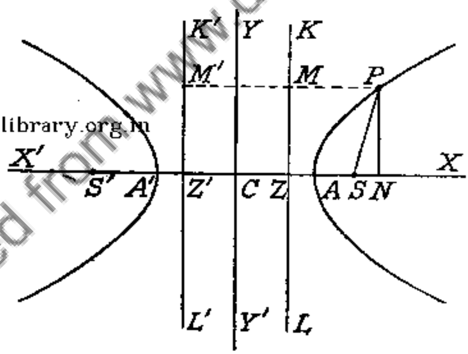
$$\therefore e = \frac{7}{17}.$$

The Hyperbola

Definition. The hyperbola is a conic whose eccentricity is greater than unity, i.e. the locus of a point that moves in a plane so that its distance from a fixed point (focus) in the plane bears a constant ratio (eccentricity), greater than unity, to its distance from a fixed straight line (directrix) in the plane.

Theorem. *To find, with the most convenient choice of origin and axes, the simplest (canonical) form of the equation of a hyperbola.*

The notation in this theorem is used throughout the work on the hyperbola.



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In the diagram, S is the focus, KL the directrix, SZ is the perpendicular from S on KL, and ZS produced is chosen as the x-axis. A and A' are points dividing SZ internally and externally respectively in the ratio $e : 1$, where e is the eccentricity of the hyperbola, and hence A and A' lie on the hyperbola.

AA' is taken to be of length $2a$, and C the mid-point of AA' is chosen as the origin of co-ordinates with CY perpendicular to CS as the y-axis.

$P \equiv (x, y)$ is any point on the hyperbola with the above choice of origin and axes, and PN is its ordinate. S' and Z' are the images of S and Z respectively in the axis CY, and $K'L'$ is the parallel to KL through Z'. PM and PM' are the perpendiculars from P on KL and $K'L'$ respectively.

By construction, $SA = e \, . \, AZ$ .... (1)

$SA' = e \, . \, A'Z$ .... (2)

(1) + (2) gives $SA + SA' = e(AZ + A'Z)$,

i.e. $(CS - CA) + (CS + CA') = e \, . \, AA'$,

i.e. $CS - a + CS + a = 2ae$, $\therefore$ $2CS = 2ae$,

i.e. $CS = ae$, and $S \equiv (ae, 0)$.

(2) − (1) gives

$$SA' - SA = e(A'Z - AZ),$$

$$\text{i.e. } AA' = e[(A'C + CZ) - (CA - CZ)],$$

$$\text{i.e. } 2a = e[a + CZ - a + CZ],$$

$$= 2e \, . \, CZ,$$

$$\therefore CZ = a/e, \text{ and } Z \equiv (a/e, 0).$$

Using the distance between two points,

$$PS^2 = (x - ae)^2 + y^2 \text{ .... (3).}$$

From the diagram, $PM = NZ = CN - CZ$

$= x - a/e$,

$\therefore$ by definition, $SP = e \, . \, PM = ex - a$,

i.e. $SP^2 = (ex - a)^2$ .... (4).

From (3) and (4), $(x - ae)^2 + y^2 = (ex - a)^2$,

i.e. $x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2$,

$\therefore$ $x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$,

$$\text{i.e. } \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1.$$

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$e^2 - 1$ is positive, since $e^2 > 1$, therefore, using $b^2 = a^2(e^2 - 1)$ in the above result, the required equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

NOTE. This equation is the standard equation of the general hyperbola and should always be used as its equation if choice be permitted.

Simple Properties of the Hyperbola $x^2/\;^2 - y^2/b^2 = 1$. Using the equation in the form $y^2/b^2 = x^2/a^2 - 1$, it can be seen that, for y to be real $x^2 > a^2$, i.e. there is no portion of the curve between $x = \pm a$, and the curve must therefore consist of two branches.

If $x^2 > a^2$ there are two equal and opposite values of y for every value of x and the curve will be symmetrical about CX. Similarly it is symmetrical about the y-axis.

From the symmetry of the curve $K'L'$ will be a second directrix with S' as its corresponding focus.

As $x \to \pm \infty$, $y \to \pm \infty$ and therefore the hyperbola extends towards infinity in both directions.

The line AA' is known as the *transverse axis* and if B and B' be points on YY' such that $CB = CB' = b$, then BB' is the *conjugate axis* of the hyperbola.

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Theorem. *To find the length of the semi-latus rectum of the hyperbola*

$$x^2/a^2 - y^2/b^2 = 1.$$

The latus rectum is a chord through the focus S parallel to the directrix and therefore parallel to the y-axis, and hence can be obtained by using $x = ae$ in the equation $x^2/a^2 - y^2/b^2 = 1$, giving

$$e^2 - \frac{y^2}{b^2} = 1,$$

$$\text{i.e. } \frac{y^2}{b^2} = e^2 - 1$$

$$= \frac{b^2}{a^2}, \text{ since } b^2 = a^2(e^2 - 1),$$

$$\therefore y^2 = \frac{b^4}{a^2}, \text{ for the extremities of the latus rectum,}$$

$$\text{i.e. } y = \pm\frac{b^2}{a}.$$

Thus the length of the semi-latus rectum is b^2/a.

Theorem. *To prove that, if $P \equiv (x_1, y_1)$ be any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, then $SP = ex_1 - a$, $S'P = ex_1 + a$,* and *$S'P - SP = 2a$.*

Using the previous diagram with $P \equiv (x_1, y_1)$,

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$$PM = NZ = CN - CZ = x_1 - a/e,$$

$$\therefore SP = e \,.\, PM = ex_1 - a.$$

Also $$PM' = NZ' = CN + CZ' = x_1 + a/e,$$

$$\therefore S'P = e \,.\, PM' = ex_1 + a.$$

Hence, $$S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a.$$

Comparison of the Hyperbola and Ellipse. The standard equation

$$x^2/a^2 - y^2/b^2 = 1$$

for the hyperbola is the same as the standard equation

$$x^2/a^2 + y^2/b^2 = 1$$

for the ellipse with b^2 replaced by $-b^2$. Thus certain properties of the hyperbola can be deduced from those of the ellipse by replacing b^2 by $-b^2$ in the corresponding result for the ellipse, wherever necessary.

The results for the hyperbola $x^2/a^2 - y^2/b^2 = 1$ thus obtained, using standard notation, are as follows:

(*a*) The equations of the tangents of slope m are given by

$$y = mx \pm \sqrt{(a^2m^2 - b^2)}.$$

(*b*) The equation of the tangent at (x_1, y_1) is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1.$

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(*c*) The equation of the normal at (x_1, y_1) is

$$\frac{x - x_1}{x_1/a^2} + \frac{y - y_1}{y_1/b^2} = 0.$$

(*d*) The line $lx + my = n$ touches the hyperbola if

$$a^2l^2 - b^2m^2 = n^2.$$

(*e*) The equation of the director circle (i.e. the locus of points of intersection of perpendicular tangents) is $x^2 + y^2 = a^2 - b^2$.

Definition. An *asymptote* to any curve is a line that cuts the curve at two coincident points at infinity (i.e. a tangent to the curve at infinity).

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad \text{.................(1).}$$

Replacing x by $1/y$ it becomes

$$a/y^2 + b/y + c = 0,$$
$$\text{i.e. } a + by + cy^2 = 0 \qquad \text{.................(2).}$$

The two values of y obtained from equation (2) will both be zero if $a = 0$ and $b = 0$ simultaneously. But when $y = 0$ the value of x is infinity. Thus the conditions that the two roots of equation (1) shall both be infinite are that $a = 0$; $b = 0$ simultaneously.

Theorem. *To find the equations of the asymptotes of the hyperbola*

$$x^2/a^2 - y^2/b^2 = 1.$$

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NOTE. The hyperbola is the only conic that has asymptotes.

Let the equation of an asymptote of the hyperbola

$$x^2/a^2 - y^2/b^2 = 1 \qquad \text{.................. (1)}$$

be

$$y = mx + c \qquad \text{.................... (2).}$$

Substituting for y from (2) in (1) to obtain the abscissae of the points of intersection of the line and the hyperbola,

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1,$$

$$\text{i.e. } b^2x^2 - a^2(m^2x^2 + 2cmx + c^2) = a^2b^2,$$

$$\text{i.e. } x^2(b^2 - a^2m^2) - 2a^2mcx - (a^2c^2 + a^2b^2) = 0 \qquad \text{...... (3)}$$

If the line (2) is to be an asymptote of the hyperbola (1) the roots of equation (3) in x must both be infinite, and the conditions for this are

$$b^2 - a^2m^2 = 0 \qquad \text{..................... (4)}$$
$$a^2mc = 0 \qquad \text{..................... (5)}$$

From (4), $m^2 = b^2/a^2$, therefore $m = \pm b/a$, and from (5), $c = 0$, since $m \neq 0$ and $a \neq 0$.

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Thus the equations of the asymptotes are $y = \pm bx/a$, which in the combined form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

NOTE. The equations of the asymptotes in the combined form differ by a constant from the equation of the hyperbola.

Theorem. *To prove that the point $x = a \sec \varphi$, $y = b \tan \varphi$ lies on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ for all values of φ.*

Substituting $x = a \sec \varphi$, $y = b \tan \varphi$ in the equation

$$x^2/a^2 - y^2/b^2 = 1,$$

it becomes $\sec^2 \varphi - \tan^2 \varphi = 1$, which is true for all values of φ. Hence $(a \sec \varphi, b \tan \varphi)$ lies on the hyperbola for all values of φ, and one parametric equation of a hyperbola is $x = a \sec \varphi$, $y = b \tan \varphi$.

Definition. If $b = a$ in the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the hyperbola is said to be *rectangular*, since its asymptotes

$$x^2/a^2 - y^2/a^2 = 0, \text{ i.e. } x^2 - y^2 = 0)$$

will be at right angles, and in this case the equation of the hyperbola will be $x^2 - y^2 = a^2$.

Theorem. *To find the equation of a rectangular hyperbola referred to its asymptotes as axes.*

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Take the equation of the hyperbola as

$$x^2 - y^2 = a^2 \quad \text{.....................(1)},$$

referred to its own axes as axes of reference.

The asymptotes will be at right angles and will have as their equations

$$x + y = 0 \quad \text{.....................(2)},$$

and

$$x - y = 0 \quad \text{.....................(3)}.$$

The perpendicular from the point (p, q) on to the lines (3) and (4) will be given by

$$X = \frac{p + q}{\sqrt{2}}, \text{ and } Y = \frac{p - q}{\sqrt{2}},$$

respectively (neglecting signs), and the product of these perpendiculars is given by $XY = \frac{1}{2}(p^2 - q^2)$.

If the point (p, q) lie on the hyperbola (1), $p^2 - q^2 = a^2$, and therefore $XY = \frac{1}{2}a^2$.

But (X, Y) will be the co-ordinates of the point on the hyperbola referred to the asymptotes as axes of reference.

Hence the required equation in running co-ordinates is $xy = \frac{1}{2}a^2$.

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Theorem. *To find the equation of the tangent to the rectangular hyperbola $xy = c^2$ at the point (x_1, y_1).*

Let the points (x_1, y_1), (x_2, y_2) lie on the hyperbola

$$xy = c^2 \qquad \ldots\ldots(1)$$

$$\therefore\ x_1y_1 = c^2, \text{ i.e. } x_1 = c^2/y_1 \qquad \ldots\ldots(2),$$

$$\text{and } x_2y_2 = c^2, \text{ i.e. } x_2 = c^2/y_2 \qquad \ldots\ldots(3).$$

(2) − (3) gives $\quad x_1 - x_2 = c^2(y_2 - y_1)/y_1y_2 \qquad \ldots\ldots(4).$

The equation of the line joining two points is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \qquad \ldots\ldots(5).$$

Combining (4) and (5), the line joining (x_1, y_1), (x_2, y_2) has as its equation

$$x - x_1 = -\frac{c^2(y - y_1)}{y_1y_2}.$$

Using $y_2 = y_1$ in this, the equation of the tangent at (x_1, y_1) is

$$\begin{aligned} x - x_1 &= -c^2(y - y_1)/y_1^2 \\ &= \frac{-x_1y_1(y - y_1)}{y_1^2} \quad \text{(using (2))} \\ &= -x_1(y - y_1)/y_1 \end{aligned}$$

$$\begin{aligned} \therefore\ y_1(x - x_1) + x_1(y - y_1) &= 0, \\ \text{i.e. } xy_1 + yx_1 &= 2x_1y_1 \\ &= 2c^2 \qquad \text{(using (2)).} \end{aligned}$$

Hence the required equation is $xy_1 + yx_1 = 2c^2$. www.dbraulibrary.org.in

NOTE. This result can be readily obtained by calculus.

EXAMPLE. Find the points in which the line $5y = 3x - 5$ cuts the hyperbola $4x^2 - 25y^2 = 15$, and the equations of the tangents and normals to the hyperbola at these points.

From the equation of the line

$$y = \tfrac{1}{5}(3x - 5) \qquad \ldots\ldots(1)$$

Substituting for y from (1) in $4x^2 - 25y^2 = 15$, for the points of intersection of the line and the hyperbola

$$\begin{aligned} 4x^2 - (3x - 5)^2 &= 15, \\ \text{i.e. } 4x^2 - 9x^2 + 30x - 25 &= 15, \\ \text{i.e. } 0 &= 5x^2 - 30x + 40, \\ \text{i.e. } x^2 - 6x + 8 &= 0, \\ \therefore\ (x - 2)(x - 4) &= 0, \\ \text{i.e. } x &= 2 \text{ or } 4. \end{aligned}$$

Using these in (1),

$$\text{when } x = 2, \quad y = \tfrac{1}{5},$$
$$\text{and when } x = 4, \quad y = \tfrac{7}{5}.$$

Thus the required points of intersection are $(2, \tfrac{1}{5})$, $(4, \tfrac{7}{5})$.

The equation of the tangent at (x_1, y_1) to the given hyperbola is

$$4xx_1 - 25yy_1 = 15.$$

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The equation of the tangent at $(2, \frac{1}{5})$ will be

$$8x - 5y = 15, \qquad (x_1 = 2,\ y_1 = \tfrac{1}{5}).$$

and the equation of the tangent at $(4, \frac{7}{5})$ will be

$$16x - 35y = 15, \qquad (x_1 = 4,\ y_1 = \tfrac{7}{5}).$$

The slope of the tangent at $(2, \frac{1}{5})$ is $\frac{8}{5}$, therefore the slope of the normal is $-\frac{5}{8}$, and the equation of the normal at $(2, \frac{1}{5})$ is

$$y - \tfrac{1}{5} = -\tfrac{5}{8}(x - 2),$$
$$\text{i.e. } 40y - 8 = -25x + 50,$$
$$\text{i.e. } 25x + 40y = 58.$$

The slope of the tangent at $(4, \frac{7}{5})$ is $\frac{16}{35}$, therefore the slope of the normal is $-\frac{35}{16}$, and the equation of the normal at this point is

$$y - \tfrac{7}{5} = -\frac{35}{16}(x - 4),$$
$$\text{i.e. } 80y - 112 = -175x + 700,$$
$$\text{i.e. } 175x + 80y = 812.$$

EXAMPLE. Find the equations of the tangents from the point (4, 4) to the hyperbola $9x^2 - 9y^2 = 16$.

The equation of the hyperbola can be written

$$\frac{x^2}{16/9} - \frac{y^2}{16/9} = 1 \ldots\ldots\ldots\ldots (1).$$

The equations of the tangents of slope m to the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

are given by $y = mx \pm \sqrt{(a^2m^2 - b^2)}$.

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Hence the equations of the tangents of slope m to the hyperbola (1) are given by

$$y = mx \pm \sqrt{\left(\frac{16}{9}m^2 - \frac{16}{9}\right)}.$$

If the point (4, 4) lies on this tangent,

$$4 = 4m \pm \sqrt{\left(\frac{16}{9}m^2 - \frac{16}{9}\right)},$$

$$\therefore (4 - 4m) = \pm\sqrt{\left(\frac{16}{9}m^2 - \frac{16}{9}\right)}.$$

Squaring $$16 - 32m + 16m^2 = \frac{16}{9}m^2 - \frac{16}{9},$$
$$\text{i.e. } 9(1 - 2m + m^2) = m^2 - 1,$$
$$\therefore 9 - 18m + 9m^2 = m^2 - 1,$$
$$\text{i.e. } 8m^2 - 18m + 10 = 0,$$
$$\text{i.e. } 4m^2 - 9m + 5 = 0,$$
$$\text{i.e. } (4m - 5)(m - 1) = 0,$$
$$\therefore m = \tfrac{5}{4} \text{ or } 1.$$

When $m = \frac{5}{4}$ the equation of the tangent is

$$y - 4 = \tfrac{5}{4}(x - 4),$$
$$\text{i.e. } 4y - 16 = 5x - 20,$$
$$\text{i.e. } 4y = 5x - 4.$$

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When $m = 1$ the equation of the tangent is

$$y - 4 = 1(x - 4)$$
$$\text{i.e. } y = x \qquad \text{(an asymptote).}$$

Example (H.S.C.). The circle $2x^2 + 2y^2 = a^2 + b^2$ and the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ intersect in the first quadrant in the point P. The line joining the point P to the origin meets the hyperbola $xy = c^2$ at Q and R. Show that the tangent at P to the ellipse and the tangents at Q and R to the hyperbola are all parallel.

The equation of the circle can be written

$$x^2 + y^2 = \tfrac{1}{2}(a^2 + b^2) \ldots\ldots\ldots\ldots (1),$$

and the equation of the ellipse can be put in the form

$$\frac{b^2}{a^2}x^2 + y^2 = b^2 \ldots\ldots\ldots\ldots (2).$$

Taking (1) − (2) for the points of intersection of the circle and ellipse

$$x^2(1 - b^2/a^2) = \tfrac{1}{2}(a^2 - b^2),$$
$$\therefore x^2/a^2 = \tfrac{1}{2}, \qquad (a^2 + b^2)$$
$$\text{i.e. } x^2 = \frac{a^2}{2}, \text{ and } x = \pm\frac{a}{\sqrt{2}}.$$

Similarly
$$y = \pm\frac{b}{\sqrt{2}}.$$

Since P is in the first quadrant it is the point $(a/\sqrt{2}, b/\sqrt{2})$.

The equation of the tangent at $(a/\sqrt{2}, b/\sqrt{2})$ to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

is
$$b^2x \times a/\sqrt{2} + a^2y \times b/\sqrt{2} = a^2b^2,$$
$$\text{i.e. } bx + ay = ab\sqrt{2},$$

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and its slope is $-b/a$.

The line joining P to the origin O will have the equation

$$y = \frac{b/\sqrt{2}}{a/\sqrt{2}}x$$
$$\text{i.e. } y = bx/a \ldots\ldots\ldots\ldots (3).$$

Where this meets the hyperbola

$$xy = c^2 \ldots\ldots\ldots\ldots (4),$$

substituting from (3) in (4) for y,

$$bx^2/a = c^2, \quad \therefore x^2 = ac^2/b.$$
$$\text{i.e. } x = \pm c\sqrt{(a/b)},$$

and using this in (3), $y = \pm c\sqrt{(b/a)}$.

The equation of the tangent at (x_1, y_1) to the hyperbola $xy = c^2$ is

$$xy_1 + yx_1 = 2c^2.$$

Therefore the tangents at Q and R, where $Q \equiv [c\sqrt{(a/b)}, c\sqrt{b/a}]$, $R \equiv [-c\sqrt{a/b}, -c\sqrt{b/a}]$ are

$$c\left(\sqrt{\frac{b}{a}}\right)x + c\left(\sqrt{\frac{a}{b}}\right)y = 2c^2 \ldots\ldots\ldots (5),$$

$$-c\left(\sqrt{\frac{b}{a}}\right)x - c\left(\sqrt{\frac{a}{b}}\right)y = 2c^2 \ldots\ldots\ldots (6)$$

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The slopes of the tangents (5) and (6) are each

$$-c\sqrt{(b/a)} \div c\sqrt{(a/b)} = -b/a.$$

Hence the required tangents are all parallel.

EXAMPLE. Any straight line cuts the rectangular hyperbola $xy = c^2$ at A and B, and the asymptotes at C and D. Prove that the mid-point of AB bisects CD.

Let the given line be

$$lx + my = n \dots\dots\dots\dots (1)$$

The asymptotes are the two axes OX and OY.

Let the line (1) cut OX in C and OY in D.

When $y = 0$ in (1), i.e. at C, $lx = n$,

$$\text{i.e. } x = n/l,$$
$$\therefore C \equiv (n/l, 0).$$

Similarly $D \equiv (0, n/m)$.

The hyperbola can be written

$$y = c^2/x \dots\dots\dots\dots (2).$$

Substituting this in (1) for the points of intersection A and B of the line (1) and the hyperbola,

$$lx + mc^2/x = n,$$
$$\text{i.e. } lx^2 + mc^2 = nx,$$
$$\therefore lx^2 - nx + mc^2 = 0 \dots\dots\dots\dots (3)$$

If $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, x_1 and x_2 will be the roots of the equation (3) and $x_1 + x_2 = n/l$.

But the mid-point of AB will have x-co-ordinate of $\frac{1}{2}(x_1 + x_2)$, i.e. $n/2l$, and using this in (2) its ordinate is $n/2m$.

Therefore the mid-point of AB is $(n/2l, n/2m)$.

Now the mid-point of CD is $(n/2l, n/2m)$, hence the middle point of AB bisects CD.

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EXAMPLES IX

1. Prove that the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is $y + tx = 2at + at^3$.

The normals at the points P, Q of a parabola meet at R and the tangents at P, Q meet at S. If RS be parallel to the axis of the parabola, show that the chord PQ passes through the focus.

2. Prove that the equation of the tangent at the point (x_1, y_1) on the parabola $y^2 = 4ax$ is $yy_1 = 2ax + 2ax_1$.

The normal at the point $P(x_1, y_1)$ to the parabola $y^2 = 8x$ meets the x-axis at G. Find the co-ordinates of G, and prove that, if the perpendicular from P to the x-axis meets it at N, then ON is equal to the length of the tangent from the origin O to the circle whose centre is G and whose radius is GP.

3. If the tangents to the parabola $y^2 = 4ax$ at the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ meet at (x_0, y_0), show that $a^2(t_1 - t_2)^2 = y_0^2 - 4ax_0$.

The chord of fixed gradient m through a variable point P on the parabola meets the curve again in Q, and the chord of gradient $2m$ through P meets the curve again in R. Show that the locus of the intersection of the tangents at Q and R is a curve whose equation is of the form $y^2 - 4ax = k$, and state the value of k in terms of a and m.

4. Prove that the equation of the tangent to the parabola $y^2 = 4ax$, at the point (x_1, y_1) on it, is $yy_1 = 2a(x + x_1)$.

This parabola and the circle $x^2 + y^2 = 32a^2$ meet at the point P in the first quadrant. The tangent to the parabola at P meets the y-axis at T and the normal to the parabola at P meets this axis at R. The foot of the perpendicular from P to the x-axis is N. Prove that $OR = 3 \cdot ON = 6 \cdot OT$, where O is the origin.

5. (*a*) A line is drawn parallel to the axis of a parabola, intersecting the parabola in P and the directrix in Q. Show that, if S is the focus, the normal at P is parallel to QS.

(*b*) If the mid-point of a chord UV of the parabola $y^2 = 4ax$ lies on the line $y = x$, prove that the point of intersection of the tangents at U and V lie on the curve $y^2 = 2a(x + y)$.

6. Find the co-ordinates of the point of intersection of the tangents at the points $(ap^2, 2ap)$, $(aq^2, 2aq)$ on the parabola $y^2 = 4ax$.

The tangents at the points P and Q on a parabola meet at T and the normals at P and Q meet at N. If the tangents are perpendicular to each other show that TN is parallel to the axis of the parabola. Show also that in this case, if P and Q vary, the locus of N is a parabola.

7. A circle passes through the focus of the parabola $y^2 = 4ax$. The curves meet at right angles at one of their points of intersection. Prove that the radius of the circle is at least $3a\sqrt{3}/4$.

8. A straight line parallel to the y-axis cuts the parabola $y^2 = 4ax$ at P, and the parabola $y^2 = 4bx$ at P_1. A straight line parallel to the x-axis cuts the first parabola in Q and the second in Q_1. Prove that the tangents to the respective parabolas at P and P_1 meet on the x-axis, and those at Q and Q_1 meet on the y-axis.

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Show also that the tangents at the points, other than the origin, where a line through the origin meets the parabolas, are parallel.

9. Prove that the equation of the circle which has the points (x_1, y_1), (x_2, y_2) as extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Find the equation of the circle which has as diameter that focal chord of the parabola $y^2 = 4ax$ which passes through the point $(a/m^2, 2a/m)$. Show that the common chord of any two circles having focal chords as diameters passes through the vertex of the parabola. (It may be assumed that tangents to a parabola at the extremities of a focal chord are perpendicular.)

10. Find the equation of the chord through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ of the parabola $y^2 = 4ax$, and show that it cuts the directrix where $y = 2a(t_1t_2 - 1)/(t_1 + t_2)$.

Deduce the equation of the tangent at $(at^2, 2at)$, and prove that the tangents at the ends of a focal chord intersect at right angles on the directrix.

11. Find the equation of the tangent at the point $(at^2, 2at)$ to the parabola $y^2 = 4ax$, and show that, if the tangents at (x_1, y_1) and (x_2, y_2) meet at (x_3, y_3), then $x_3^2 = x_1x_2$, $y_3 = \frac{1}{2}(y_1 + y_2)$.

The straight line through the vertex of the parabola $y^2 = 4ax$ perpendicular to the tangent at $P(4a, 4a)$ meets this tangent at Q. Find the equation of the other tangent from Q.

12. Obtain the equation of the tangent to the parabola $y^2 = 4ax$ which is parallel to the line $2x + y = 0$, and find the co-ordinates of the point of contact.

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13. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

Two perpendicular tangents to the parabola meet at P and intersect the tangent at the vertex in L, M. Show that $LM = PS$, where S is the focus of the parabola.

14. A circle is drawn with its centre at the focus S of the parabola $y^2 = 4ax$ to touch the parabola at the vertex. A radius SQ of the circle meets the parabola at a point $P(a \cot^2 \theta, 2a \cot \theta)$. Show that the co-ordinates of Q are $(2a \cos^2 \theta, 2a \sin \theta \cos \theta)$.

If the tangent at the point P of the parabola intersects the tangent at the vertex in the point R, prove that the line RQ is perpendicular to the line SP.

15. Find the points of intersection of the parabola $y^2 = 4x$ and the circle $4(x^2 + y^2) - 25x + y + 3 = 0$.

Show that the curves touch, and find the equation of the common tangent at their point of contact.

Show, also, that this tangent and the chord joining the other two intersections are equally inclined to the axis of the parabola.

16. P, Q are the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$ on the parabola $x = at^2$, $y = 2at$. Show that the equation of the chord PQ is

$$2x - (p + q)y + 2apq = 0.$$

If O is the origin and the chords OP, OQ are perpendicular, prove that the chord PQ cuts the axis of x in the same point for all possible positions of P and Q.

17. Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

If the chord PQ is normal at P and makes an angle θ with the axis of the parabola, show that its length is $4a \sec \theta \operatorname{cosec}^2 \theta$.

18. Find the equations of the tangent and normal at the point P, whose co-ordinates are (18, 12) on the parabola $y^2 = 8x$.

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Verify that the foot of the perpendicular from the focus S whose co-ordinates are (2, 0) to the tangent at P lies on the tangent at the origin; and that, if the normal at P meets the axis of the parabola at G, then $SG = SP$.

19. Find the equations of each of the tangents drawn from the point (27, 8) to the ellipse $x^2 + 9y^2 = 9$.

20. Show that the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (x_1, y_1) on it is $xx_1/a^2 + yy_1/b^2 = 1$, and deduce that the line $lx + my = n$ touches the ellipse if $a^2l^2 + b^2m^2 = n^2$.

Find the equations of the tangent and normal to the ellipse $x^2 + 3y^2 = 2a^2$ at the point $(a, a/\sqrt{3})$. If the tangent meets the x-axis at P and the normal meets the y-axis at Q, show that PQ touches the ellipse.

21. Show that the equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2.$$

If the above normal meet the x-axis at G, find the co-ordinates of Q, the mid-point of PG.

Show that, as P varies on the ellipse, the locus of Q is an ellipse of eccentricity $2e\sqrt{(e^2 + 3)}/(e^2 + 1)$, where e is the eccentricity of the given ellipse.

22. Prove that the equation of the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(a \cos \theta, b \sin \theta)$ is $x \cos \theta/a + y \sin \theta/b = 1$.

The tangent at the point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ meets the parabola $y^2 = 4ax$ at the points Q, R and is such that the mid-point of QR lies on

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the line $y + 2a = 0$. Prove that the product of the perpendiculars from $(a, 0)$ and $(-a, 0)$ on to the tangent is $b^2/2$.

23. The points A and B are the extremities of the major and minor axes of an ellipse and the point P is a variable point on the ellipse. Prove that the locus of the orthocentre of the triangle PAB is a similar ellipse with its major axis perpendicular to that of the original ellipse.

24. A point P is taken on the circle $x^2 + y^2 = a^2$, and the ordinate NP meets the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the point Q between N and P. If the angle NOP is θ, where O is the origin, find the equation of the tangent to the ellipse at Q.

Another point P_1 is taken on the circle such that angle POP_1 has a constant value α, and Q_1 is the point on the ellipse where the ordinate N_1P_1 meets it, Q_1 lying between N_1 and P_1. Prove that the tangents at Q and Q_1 meet at a point on the ellipse $x^2/a^2 + y^2/b^2 = \sec^2 \frac{1}{2}\alpha$.

25. From the equation of the tangent deduce that the equation of the normal at the point $P(a \cos\theta, b \sin\theta)$ of an ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - \quad .$$

The normal at P on the given ellipse meets the axes at G and g, and R is the other vertex of the rectangle $OGRg$, where O is the origin. Prove that the locus of the mid-point of QR, where Q is the point on the ellipse diametrically opposite P, is an ellipse of area one-quarter that of the original ellipse.

26. If the point P on the ellipse $x^2/a^2 + y^2/b^2 = 1$ has the eccentric angle θ, find the equations of the lines PA, PA' joining P to the ends A, A' of the major axis.

The lines through P perpendicular to PA, PA' meet the major axis in K, K'; show that the length KK' is constant.

27. Find the equations of the tangents to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ with a given slope m. Prove that the locus of the point, such that the product of the slopes of the tangents from the point to the ellipse is constant, is a co-axial hyperbola.

28. Obtain the equation of the tangent at the point (x_1, y_1) on the hyperbola $xy = c^2$.

The tangent at any point P on a rectangular hyperbola meets the asymptotes at T and T' and the normal at P meets the transverse axis of the hyperbola at G. Prove that T, T', G, and the centre of the hyperbola lie on a circle of which P is the centre.

29. The tangents to a rectangular hyperbola at P and Q meet one of the asymptotes of the hyperbola in A_1 and B_1 respectively, and meet the other asymptote in A_2 and B_2 respectively. Prove that the lines A_1B_2 and A_2B_1 are parallel to PQ and equidistant from it.

30. Points L, M lie on each of the asymptotes of the hyperbola

$$x^2/a^2 - y^2/b^2 = 1.$$

Show that their co-ordinates can be expressed in the form (at, bt), $(at', -bt')$.

If the mid-point T of LM lies on the curve, show that $tt' = 1$, and that LM is then the tangent at T. Show also that in this case $CL \cdot CM =$ constant, and $CT^2 - TL^2 =$ constant, C being the centre of the hyperbola.

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CHAPTER X

Calculus

Limiting Values, Differentiation, Maxima and Minima, and Exponentials

Theory of Limits. The result of replacing x by $(2 + h)$ in the fraction $(3 - x)/(2 + x)$ is $(1 - h)/(4 + h)$, and as h becomes smaller and smaller the value of this fraction becomes closer and closer to $\frac{1}{4}$.

In this case it is said that, as $h \to 0$ (i.e. h approaches zero) the value of $(1 - h)/(4 + h) \to \frac{1}{4}$. But as $h \to 0$ it is seen that $x \to 2$, and therefore, as $x \to 2$, the value of $(3 - x)/(2 + x) \to \frac{1}{4}$.

By 'the *limiting value* of $f(x)$ as x approaches a' is meant the value that $f(x)$ approaches as $x \to a$.

Hence, the above result can be written: the limiting value (or limit) of $(3 - x)/(2 + x)$ as $x \to 2$ is equal to $\frac{1}{4}$, and this is written more briefly as,

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$$\underset{x \to 2}{\text{Lt}} \frac{3 - x}{2 + x} = \frac{1}{4},$$

where Lt represents 'the limit of', and $x \to 2$ represents 'as x approaches 2'.

It is to be noted that, in the case of the fraction $(3 - x)/(2 + x)$, if x be given the value 2 the result is $\frac{1}{4}$, which is the value previously obtained in the case of the limit.

Next replace x by $(2 + h)$ in the fraction $(x^2 - 4)/(x - 2)$ obtaining the result $(4h + h^2)/h = 4 + h$.

If h be allowed to approach zero in this result the value of the expression approaches 4. Hence, it follows that,

$$\underset{x \to 2}{\text{Lt}} \frac{x^2 - 4}{x - 2} = 4.$$

If x be put equal to 2 in the fraction $(x^2 - 4)/(x - 2)$ the result is $0/0$, which is an indeterminate quantity. But if the fraction $(x^2 - 4)/(x - 2)$ be reduced to its lowest terms, viz. $(x + 2)$, and x be then given the value 2, the result is 4 as obtained for the limit.

This can be shown to be true in all types of examples taken, and suggests the procedure, which is generally carried out, as follows: reduce the function, of which the limit is required, to its lowest terms,

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and then insert the value which x approaches in order to obtain the value of the required limit.

EXAMPLE. Find the values of the following limits.

$$\text{(i)} \quad \underset{x\to 3}{\text{Lt}} \frac{x^2 - 9}{x - 3},$$

$$\text{(ii)} \quad \underset{x\to -2}{\text{Lt}} \frac{x^3 + 8}{x + 2},$$

$$\text{(iii)} \quad \underset{x\to 0}{\text{Lt}} \frac{x^3 + 2x^2 + x}{x^2 + 3x},$$

$$\text{(iv)} \quad \underset{x\to \infty}{\text{Lt}} \frac{(1 + x)(2 + x) \cdot x}{x^3 + x}.$$

NOTE. In general, when dealing with algebraic functions, and when x approaches a finite value, to find the limit, first reduce the given function to its *lowest* terms, and then insert in the result the value that x approaches. As long as the indeterminate quantity $0/0$ (or ∞/∞) is not obtained, it follows that the result is the required value.

If $0/0$ or ∞/∞ is obtained the function has not been reduced sufficiently.

$$\text{(i)} \quad \underset{x\to 3}{\text{Lt}} \frac{x^2 - 9}{x - 3} = \underset{x\to 3}{\text{Lt}} \; (x + 3) = 6.$$

$$\text{(ii)} \quad \underset{x\to -2}{\text{Lt}} \frac{x^3 + 8}{x + 2} = \underset{x\to -2}{\text{Lt}} \; (x^2 - 2x + 4) = 4 + 4 + 4 = 12.$$

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$$\text{(iii)} \quad \underset{x\to 0}{\text{Lt}} \frac{x^3 + 2x^2 + x}{x^2 + 3x} = \underset{x\to 0}{\text{Lt}} \frac{x^2 + 2x + 1}{x + 3} = \tfrac{1}{3}.$$

(iv) This is a special example, and, usually, when $x \to \infty$ the numerator and denominator are divided by x^n, where n is the highest power of x present on expansion of both numerator and denominator.

Dividing numerator and denominator by x^3

$$\underset{x\to \infty}{\text{Lt}} \frac{(1 + x)(2 + x)x}{x^3 + x} = \underset{x\to \infty}{\text{Lt}} \frac{(1/x + 1)(2/x + 1) \cdot 1}{1 + 1/x^2} = 1.$$

$$(1/x \to 0 \text{ as } x \to \infty)$$

NOTE. In general, it will be found, in the limits dealt with in differentiation, that the variable approaches zero.

Theorems on Limits. The following theorems on limits will be used extensively but no proofs are offered.

(i) The limit of the sum of a finite number of functions is equal to the sum of their limits.

(ii) The limit of the product of a finite number of functions is equal to the product of their limits.

(iii) The limit of the quotient of two functions is equal to the quotient of their limits.

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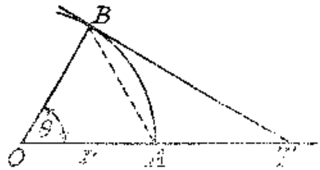
Theorem. *If* θ *be an angle in radians, then*

$$\operatorname*{Lt}_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\quad \textit{and}\quad \operatorname*{Lt}_{\theta\to 0}\frac{\tan\theta}{\theta}=1.$$

AB is an arc of a circle, centre O and radius r, subtending an angle $\theta\ (<\frac{1}{2}\pi)$ radians at O. BT is the tangent at B meeting OA produced at T.



The area of $\triangle AOB=\frac{1}{2}r^2\sin\theta$; area of sector $AOB=\frac{1}{2}r^2\theta$; area of $\triangle OBT=\frac{1}{2}OB\times BT=\frac{1}{2}r\,.\,r\tan\theta=\frac{1}{2}r^2\tan\theta$.

By inspection of the diagram,

$$\text{area }\triangle AOB<\text{area sector }AOB<\text{area }\triangle OBT,$$
$$\therefore\ \tfrac{1}{2}r^2\sin\theta<\tfrac{1}{2}r^2\theta<\tfrac{1}{2}r^2\tan\theta,$$
$$\text{i.e. }\sin\theta<\theta<\tan\theta\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Dividing through the inequality (1) by $\sin\theta$, which is positive since θ is acute,

$$1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}.$$

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Now $\cos\theta\to 1$ as $\theta\to 0$, and thus $\dfrac{1}{\cos\theta}\to 1$ as $\theta\to 0$.

Hence, $\theta/\sin\theta$ lies between 1 and a quantity which approaches unity as $\theta\to 0$,

$$\therefore\ \operatorname*{Lt}_{\theta\to 0}\frac{\theta}{\sin\theta}=1,$$

$$\text{i.e. }\operatorname*{Lt}_{\theta\to 0}\frac{1}{\sin\theta/\theta}=1,\quad\text{i.e. }\frac{\operatorname*{Lt}_{\theta\to 0}1}{\operatorname*{Lt}_{\theta\to 0}\sin\theta/\theta}=1,$$
(Lt. of quotient)

$$\therefore\ \frac{1}{\operatorname*{Lt}_{\theta\to 0}\sin\theta/\theta}=1,\quad\therefore\quad\operatorname*{Lt}_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Dividing through the inequality (1) by $\tan\theta$,

$$\cos\theta<\frac{\theta}{\tan\theta}<1.$$

Now, $\cos\theta\to 1$ as $\theta\to 0$, therefore $\theta/\tan\theta$ lies between unity and a quantity that approaches unity as $\theta\to 0$.

Hence, $$\operatorname*{Lt}_{\theta\to 0}\theta/\tan\theta=1.$$

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From this result, as previously,

$$\underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = 1$$

These two results show that, if θ be very small and measured in radians, then tan θ can be replaced by θ radians, and also sin θ can be replaced by θ radians. The smaller the value of θ the more accurate will be the results. The tables show that there is a fair measure of agreement between θ, sin θ, and tan θ for angles up to about $5° = 0{\cdot}08727$ radians.

EXAMPLE (L.U.). Show that, if $0 < x < \frac{1}{2}\pi$, $\tan x > x > \sin x$, and deduce that $\sin x/x \to 1$ as $x \to 0$.

Find the limits of

(i) $(\frac{1}{2}\pi - x)\tan x$, as $x \to \frac{1}{2}\pi$

(ii) $(\cos x - \cos 2x)/x^2$ as $x \to 0$.

The first part of the question is covered in the previous theorem.

(i) $$\underset{x \to \frac{1}{2}\pi}{\text{Lt}} (\tfrac{1}{2}\pi - x)\tan x = \underset{h \to 0}{\text{Lt}} [\tfrac{1}{2}\pi - (\tfrac{1}{2}\pi + h)] \tan(\tfrac{1}{2}\pi + h)$$

$$= \underset{h \to 0}{\text{Lt}} (-h)(-\cot h)$$

$[\tan(\frac{1}{2}\pi + \theta) = -\cot\theta]$

$$= \underset{h \to 0}{\text{Lt}} \frac{h}{\tan h} = 1 \text{ (by previous theorem).}$$

(ii) $$\underset{x \to 0}{\text{Lt}} \frac{\cos x - \cos 2x}{x^2} = \underset{x \to 0}{\text{Lt}} \frac{2\sin\frac{3}{2}x \sin\frac{1}{2}x}{x^2}$$

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$$= \underset{x \to 0}{\text{Lt}} \frac{2\sin\frac{3}{2}x}{\frac{3}{2}x} \cdot \frac{\sin\frac{1}{2}x}{\frac{1}{2}x} \times \tfrac{3}{4}$$

$$= \underset{x \to 0}{\text{Lt}} \tfrac{3}{2} \frac{\sin\frac{3}{2}x}{\frac{3}{2}x} \cdot \underset{x \to 0}{\text{Lt}} \frac{\sin\frac{1}{2}x}{\frac{1}{2}x} \text{ (Lt. of product)}$$

$$= \tfrac{3}{2}. \qquad \left(\text{since } \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} = 1\right)$$

Differential Calculus. Calculus is the science of small quantities and is a very important branch of mathematics.

When a train is said to be moving at 60 miles per hour this does not mean that it will cover 60 miles in the next hour or that it covered 60 miles in the previous hour, unless it is travelling at a steady (uniform) speed, as, generally, its speed fluctuates throughout its journey.

Now a speed of 60 miles per hour is equivalent to 88 feet per second, and we can say that it is more likely to cover 88 feet in the next second of motion than to cover 60 miles in the next hour, since its speed has much less chance of varying in the next second than in the following hour.

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It would be more accurate to say that in the next one-tenth second it would cover 8·8 feet, and more accurate still to say that it would cover 0·88 feet in the next one-hundredth of a second.

Thus the smaller the interval of time considered the more accurately will the quotient of the distance traversed to the time taken be a measure of the actual speed at the particular moment under consideration.

Consider a particle that has moved through a distance s units in t seconds. If δs be a small increase in the distance traversed in a small increment δt of time (δs is a symbol denoting a small increase in s and is *not* the product of δ and s, and similarly for δt, etc.), there will not be very much variation in the speed v during that short interval of time, and the smaller the value of δt the less chance there is of a change in the speed.

Thus it can be said that the value of v is the value that $\delta s/\delta t$ approaches as $\delta t \to 0$, i.e.

$$v = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t}.$$

The quantity

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t}$$

is usually denoted by the symbol

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$$\frac{ds}{dt},$$

and is known as the *first differential coefficient of s with respect to t*, or more briefly as the *differential (or derivative) of s with respect to t*.

It is to be noted that *average speed* means the total distance travelled divided by the total time taken.

Also the *velocity* of a particle moving on a certain path at a particular moment is the speed at that instant considered tangential to its path, and therefore has both magnitude and direction and is a *vector quantity*, whilst speed is a *scalar quantity*.

The quantity

$$\frac{ds}{dt}$$

is merely a symbol representing

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t},$$

and the ds and dt are not detachable in normal circumstances, i.e.

$$\frac{ds}{dt}$$

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cannot be taken as a fraction with ds as the numerator and dt as the denominator.

From the value of

$$\frac{ds}{dt}\left(\text{i.e. } \underset{\delta t \to 0}{\text{Lt}} \frac{\delta s}{\delta t}\right),$$

it can be seen that it is the rate of change of distance with respect to time.

If f represents the acceleration of the particle under consideration at time t, it is known that f represents the rate of change of velocity (v) with respect to time t, and therefore with the previous notation

$$f = \underset{\delta t \to 0}{\text{Lt}} \frac{\delta v}{\delta t},$$

where δv is a small increase in the velocity v due to a small increase of δt in t.

The quantity

$$\underset{\delta t \to 0}{\text{Lt}} \frac{\delta v}{\delta t}$$

is denoted by the symbol

$$\frac{dv}{dt},$$

and is known as the first differential coefficient (or derivative) of v with respect to t.

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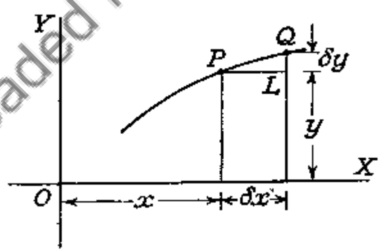
Acceleration has magnitude and direction and is a vector quantity subject to the laws of vectors.



Next consider the graph of any curve with points P and Q very close together on the curve, where $P \equiv (x, y)$, and

$$Q \equiv (x + \delta x, y + \delta y),$$

δy being a small increase in y due to a small increase of δx in x. (δy and δx are enlarged in the diagram, or otherwise they could not be seen).

PL is the perpendicular from P on the ordinate at Q, and therefore $QL = \delta y$, and $PL = \delta x$.

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The slope of the chord PQ is

$$\frac{QL}{PL} = \frac{\delta y}{\delta x},$$

and as $\delta x \to 0$, Q moves up into coincidence with P, and the chord PQ produced becomes the tangent at P.

Hence, the slope of the tangent at P is the value that

$$\frac{\delta y}{\delta x}$$

approaches as $\delta x \to 0$, i.e. the slope of the tangent at P is

$$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

This quantity

$$\operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x}$$

is denoted by the symbol

$$\frac{dy}{dx}$$

and is known as the first differential coefficient of y with respect to x.

Hence, the slope of the tangent at P is

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$$\frac{dy}{dx} = \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x}.$$

From the slope of the tangent

$$\left(\frac{dy}{dx}\right)$$

on a graph, which represents the rate of change of y with respect to x, it can be seen that the rate of change will be positive if y increases as x increases, and negative if y decreases as x increases.

From the preceding explanation, it can be seen that the definition of the first differential coefficient (or derivative) of p with respect to q, where p is some function of q, will be

$$\operatorname*{Lt}_{\delta q \to 0} \frac{\delta p}{\delta q},$$

and will be denoted by

$$\frac{dp}{dq}.$$

(δp is a small increase in p due to a small increase of δq in q.)

The determination and the application of derivatives is known as *differential calculus.*

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Theorem. *If $y = f(x)$, to prove that*

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

$$y = f(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If y increase by δy as x increases by a small amount δx, the new value $y + \delta y$ will correspond to $x + \delta x$, i.e. the point $(x + \delta x, y + \delta y)$ must satisfy the equation (1),

$$\therefore y + \delta y = f(x + \delta x) \ldots\ldots\ldots\ldots\ldots (2).$$

(2) − (1) gives $\delta y = f(x + \delta x) - f(x)$,

$$\therefore \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

i.e. $$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This result for

$$\frac{dy}{dx}, \text{ viz. } \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x},$$

is known as the *derived definition* of

$$\frac{dy}{dx},$$

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and is used in preference to the ordinary definition for finding

$$\frac{dy}{dx}$$

in particular examples, since it saves a certain number of steps in the working.

Note. To find a differential coefficient from *first principles* means to obtain its value with either the use of the ordinary definition or the derived definition, and not employing any theorems or results in calculus that may be proved later. (Results in the other branches of mathematics, such as arithmetic, algebra, trigonometry, can be used.)

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Example. Find the derivatives with respect to x, of the following functions, from first principles: (i) x, (ii) x^2, (iii) x^3, (iv) $2x^2 + x + 2$, (v) $1/(2x + 1)$, (vi) $\sqrt{(x - 1)}$.

(i) Using the derived definition,

$$\frac{d}{dx}(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{(x + \delta x) - x}{\delta x}$$

($f(x) = x$ in previous formula)

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\delta x}{\delta x} = \underset{\delta x \to 0}{\text{Lt}}\ 1 = 1.$$

NOTE. This result could also be obtained by considering the graph of $y = x$, which is a straight line of slope unity, and, since the slope $= \dfrac{dy}{dx}$,
$$\frac{dy}{dx} = 1.$$

(ii) Let
$$y = x^2 \quad\ldots\ldots(1),$$
and y increase by δy as x increases by a small amount δx. Then, when the value of x is $x + \delta x$, the new value of y is $y + \delta y$, and it follows that
$$\begin{aligned} y + \delta y &= (x + \delta x)^2 \\ &= x^2 + 2x \,.\, \delta x + (\delta x)^2 \quad\ldots\ldots(2). \end{aligned}$$
(2) − (1) gives
$$\delta y = 2x \,.\, \delta x + (\delta x)^2,$$
$$\therefore \frac{\delta y}{\delta x} = 2x + \delta x,$$
$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} (2x + \delta x) = 2x,$$
$$\therefore \frac{d}{dx}(x^2) = 2x.$$

(iii) Using the derived definition,
$$\begin{aligned} \frac{d}{dx}(x^3) &= \underset{\delta x \to 0}{\text{Lt}} \frac{(x + \delta x)^3 - x^3}{\delta x} \\ &= \underset{\delta x \to 0}{\text{Lt}} \frac{x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 - x^3}{\delta x} \\ &= \underset{\delta x \to 0}{\text{Lt}} \frac{3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3}{\delta x} \\ &= \underset{\delta x \to 0}{\text{Lt}} [3x^2 + 3x \,.\, \delta x + (\delta x)^2] \\ &= 3x^2. \end{aligned}$$

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(iv) Let
$$y = 2x^2 + x + 2 \quad\ldots\ldots(1),$$
and let y increase by δy as x increases by a small amount δx,
$$\begin{aligned} \therefore y + \delta y &= 2(x + \delta x)^2 + (x + \delta x) + 2 \\ &= 2[x^2 + 2x \,.\, \delta x + (\delta x)^2] + x + \delta x + 2 \\ &= 2x^2 + 4x \,.\, \delta x + 2(\delta x)^2 + x + \delta x + 2 \quad\ldots\ldots(2) \end{aligned}$$
(2) − (1) gives
$$\delta y = 4x \,.\, \delta x + 2(\delta x)^2 + \delta x,$$
$$\therefore \frac{\delta y}{\delta x} = 4x + 2\delta x + 1.$$
$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} (4x + 2\delta x + 1) = 4x + 1,$$
$$\therefore \frac{d}{dx}(2x^2 + x + 2) = 4x + 1.$$

(v) Using the derived definition
$$\frac{d}{dx}\left(\frac{1}{2x + 1}\right) = \underset{\delta x \to 0}{\text{Lt}} \frac{\dfrac{1}{2(x + \delta x) + 1} - \dfrac{1}{2x + 1}}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{(2x+1)-(2x+2\delta x+1)}{\delta x[2(x+\delta x)+1][2x+1]}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-2.\delta x}{\delta x[2(x+\delta x)+1][2x+1]}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{-2}{[2(x+\delta x)+1][2x+1]} = \frac{-2}{(2x+1)^2}.$$

(vi) Let $$y = \sqrt{(x-1)} \quad \ldots\ldots\ldots\ldots(1),$$

and let y increase by δy, as x increases by a small amount δx,

$$\therefore y + \delta y = \sqrt{[(x+\delta x)-1]} \quad \ldots\ldots\ldots\ldots(2).$$

(2) − (1) gives

$$\delta y = \sqrt{[(x+\delta x)-1]} - \sqrt{(x-1)}$$

$$= \frac{[\sqrt{[(x+\delta x)-1]} - \sqrt{(x-1)}][\sqrt{[(x+\delta x)-1]} + \sqrt{(x-1)}]}{\sqrt{[(x+\delta x)-1]} + \sqrt{(x-1)}}$$

$$= \frac{(x+\delta x-1)-(x-1)}{\sqrt{(x+\delta x-1)}+\sqrt{(x-1)}} = \frac{\delta x}{\sqrt{(x+\delta x-1)}+\sqrt{(x-1)}}.$$

$$\therefore \frac{\delta y}{\delta x} = \frac{1}{\sqrt{(x+\delta x-1)}+\sqrt{(x-1)}},$$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{1}{\sqrt{(x+\delta x-1)}+\sqrt{(x-1)}}$$

$$= \frac{1}{2\sqrt{(x-1)}}.$$

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i.e. $$\frac{d}{dx}[\sqrt{(x-1)}] = \frac{1}{2\sqrt{(x-1)}}.$$

Notice the special method of using conjugate surds in (vi) to avoid the use of the binomial theorem.

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NOTE. The results of (i), (ii), (iii) are

$$\frac{d}{dx}(x^1) = 1 = 1 \cdot x^0 = 1 \cdot x^{1-1}.$$

$$\frac{d}{dx}(x^2) = 2x = 2 \cdot x^{2-1}.$$

$$\frac{d}{dx}(x^3) = 3x^2 = 3x^{3-1}.$$

The new forms of these results show that they all obey the general result

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

where n is any constant, and this leads to the following general theorem.

Theorem. *To prove that*

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Using the derived definition, and the binomial theorem,

$$\frac{d}{dx}(x^n) = \underset{\delta x \to 0}{\text{Lt}} \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{\left\{x^n + nx^{n-1}\,\delta x + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \ldots\right\} - x^n}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \frac{nx^{n-1}\,\delta x + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \ldots}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \left\{nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}(\delta x) + \text{higher powers of } \delta x\right\}$$

$$= nx^{n-1}.$$

Note. Before this result can be used the term in x involved must first be put in the form x^n. x is chosen as the independent variable in proofs, but any other variable could be used.

Example. Find the derivatives of the following with respect to the variable contained in each.

(i) x^5, (ii) $\frac{1}{\sqrt[3]{s}}$, (iii) $z^{\frac{1}{3}}$, (iv) $\frac{1}{t^{3\cdot1}}$, (v) $\sqrt{y^3}$.

(i) $\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$.

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(ii) $\frac{d}{ds}\left(\frac{1}{\sqrt[3]{s}}\right) = \frac{d}{ds}(s)^{-\frac{1}{3}} = -\frac{1}{3}s^{-4/3} = \frac{-1}{3\sqrt[3]{s^4}}$.

(iii) $\frac{d}{dz}(z^{\frac{1}{3}}) = \frac{1}{3}z^{\frac{1}{3}-1} = \frac{1}{3}z^{-\frac{2}{3}}$.

(iv) $\frac{d}{dt}\left(\frac{1}{t^{3\cdot1}}\right) = \frac{d}{dt}(t^{-3\cdot1}) = -3\cdot1t^{-4\cdot1} = \frac{-3\cdot1}{t^{4\cdot1}}$.

(v) $\frac{d}{dy}(\sqrt{y^3}) = \frac{d}{dy}(y^{3/2}) = \frac{3}{2}y^{\frac{1}{2}} = \frac{3}{2}\sqrt{y}$.

Theorem. *To find the derivative of C with respect to x, where C is any constant.*

Let $y = C$ ....(1),
and y increase by δy as x increases by a small amount δx.

$\therefore y + \delta y = C$ ....(2).
(C unchanged by increase in x)

(2) − (1) gives $\delta y = 0$, $\therefore \frac{\delta y}{\delta x} = 0$,

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} (0) = 0,$$

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$$\therefore \frac{d}{dx}(C) = 0.$$

NOTE. This result could also have been obtained by considering the graph of $y = C$, which is a straight line parallel to OX and has therefore zero slope. But the slope must equal

$$\frac{dy}{dx}$$

at the point under consideration, and thus, in this case,

$$\frac{dy}{dx} = 0.$$

E.g. $$\frac{d}{dx}(2^3) = 0, \quad \frac{d}{dx}\left(\frac{1}{\sqrt[3]{2}}\right) = 0, \text{ etc.}$$

Theorem. *To find the differential of $a . f(x)$ with respect to x, where a is a constant.*

Using the derived definition,

$$\frac{d}{dx}[af(x)] = \underset{\delta x \to 0}{\text{Lt}} \frac{af(x+\delta x) - af(x)}{\delta x}$$

$$= \underset{\delta x \to 0}{\text{Lt}}\, a\left\{\frac{f(x+\delta x) - f(x)}{\delta x}\right\}$$

$$= \underset{\delta x \to 0}{\text{Lt}}\, a \,.\, \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$$

(Lt. of product)

$$= a \underset{\delta x \to 0}{\text{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= a\frac{d}{dx}[f(x)].$$

EXAMPLE. Find the derivatives of the following functions with respect to the variable contained in each:

$$\text{(i) } 5x^{3.5}, \quad \text{(ii) } \frac{2{\cdot}4}{s^{3.4}}, \quad \text{(iii) } \sqrt[3]{(4t)}, \quad \text{(iv) } \frac{1}{\sqrt{(2y^5)}}.$$

NOTE. Each function must first be put in the form $a \times (\text{variable})^n$, where a and n are constants, before the previous theorems can be applied.

(i) $$\frac{d}{dx}(5x^{3.5}) = 5\frac{d}{dx}(x^{3.5}) = 5 \times 3{\cdot}5x^{2.5} = 17{\cdot}5x^{2.5}.$$

(ii) $$\frac{d}{ds}\left(\frac{2{\cdot}4}{s^{3.4}}\right) = 2{\cdot}4\frac{d}{ds}(s^{-3.4}) = 2{\cdot}4(-3{\cdot}4)s^{-4.4} = \frac{-8{\cdot}16}{s^{4.4}}.$$

(iii) $$\frac{d}{dt}(\sqrt[3]{4t}) = \frac{d}{dt}(\sqrt[3]{4} \,.\, \sqrt[3]{t}) = \sqrt[3]{4}\frac{d}{dt}(t^{\frac{1}{3}}) = \sqrt[3]{4}\,(\tfrac{1}{3}t^{-\frac{2}{3}})$$

$$= \tfrac{1}{3}\sqrt[3]{4} \,.\, \frac{1}{\sqrt[3]{t^2}} = \tfrac{1}{3}\sqrt[3]{\frac{4}{t^2}}.$$

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(iv) $\frac{d}{dy}\left[\frac{1}{\sqrt{(2y^5)}}\right] = \frac{d}{dy}\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{y^5}}\right)$

$$= \frac{1}{\sqrt{2}}\frac{d}{dy}(y^{-5/2}) = \frac{1}{\sqrt{2}}\left(\frac{-5}{2}\right)y^{-7/2}$$

$$= \frac{-5}{2\sqrt{2}}\cdot\frac{1}{\sqrt{y^7}} = \frac{-5}{2\sqrt{(2y^7)}}.$$

Theorem. *To prove that the differential of the sum of a finite number of functions of x with respect to x is equal to the sum of their differentials.*

Let $$y = u + v - w + \ldots \qquad (1),$$

where y, u, v, w, etc., are functions of x, and let $y, u, v, w, \ldots$ increase by $\delta y, \delta u, \delta v, \delta w, \ldots$ respectively as x increases by a small amount δx.

Then $y + \delta y = (u + \delta u) + (v + \delta v) - (w + \delta w) + \ldots \qquad (2)$

(2) − (1) gives, $\delta y = \delta u + \delta v - \delta w + \ldots$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} - \frac{\delta w}{\delta x} + \ldots$$

$$\therefore \frac{dy}{dx} = \operatorname*{Lt}_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$= \operatorname*{Lt}_{\delta x\to 0}\left\{\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} - \frac{\delta w}{\delta x} + \ldots\right\}$$

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$$= \operatorname*{Lt}_{\delta x\to 0}\frac{\delta u}{\delta x} + \operatorname*{Lt}_{\delta x\to 0}\frac{\delta v}{\delta x} - \operatorname*{Lt}_{\delta x\to 0}\frac{\delta w}{\delta x} + \ldots$$

(Lt. of sum)

$$= \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \ldots,$$

i.e. $\frac{d}{dx}(u + v - w + \ldots) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx} + \ldots$

NOTE. Before this result can be used, the given function of x must be first expressed entirely of the sum of terms of the form Ax^n, where possible.

EXAMPLE. Find the derivatives, with respect to x, of the following functions, after simplifying where necessary:

(i) $3x^3 + 2x^3 - 3x + 4$, (ii) $\frac{x^8 + x^2}{x^5}$, (iii) $(x^2 + 2)^3$,

(iv) $x^3(2^3 - x^{-2})$, (v) $\frac{x^4 - 4}{x^2 + 2}$.

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(i) $\dfrac{d}{dx}(3x^3 + 2x^2 - 3x + 4)$

$$= \frac{d}{dx}(3x^3) + \frac{d}{dx}(2x^2) - \frac{d}{dx}(3x) + \frac{d}{dx}(4)$$
$$= 3 \times 3x^2 + 2 \times 2x - 3 + 0$$
$$= 9x^2 + 4x - 3.$$

(ii) $\dfrac{d}{dx}\left(\dfrac{x^8 + x^2}{x^5}\right) = \dfrac{d}{dx}(x^3 + x^{-3}) = \dfrac{d}{dx}(x^3) + \dfrac{d}{dx}(x^{-3})$

$$= 3x^2 + (-3)x^{-4} = 3x^2 - 3/x^4.$$

(iii) $\dfrac{d}{dx}[(x^2 + 2)^3] = \dfrac{d}{dx}(x^6 + 6x^4 + 12x^2 + 8)$

$$= \frac{d}{dx}(x^6) + \frac{d}{dx}(6x^4) + \frac{d}{dx}(12x^2) + \frac{d}{dx}(8)$$
$$= 6x^5 + 24x^3 + 24x + 0 = 6x(x^4 + 4x^2 + 4).$$

(iv) $\dfrac{d}{dx}[x^3(2^3 - x^{-2})] = \dfrac{d}{dx}(8x^3 - x) = 8 \times 3x^2 - 1 = 24x^2 - 1.$

(v) $\dfrac{d}{dx}\left(\dfrac{x^4 - 4}{x^2 + 2}\right) = \dfrac{d}{dx}(x^2 - 2) = 2x.$

Tangents to a Curve. The slope of the tangent at the point (x, y) of a curve will be given by dy/dx, and thus the slope of the tangent at the point (x_1, y_1) is $(dy/dx)_1$, which denotes the value of dy/dx when x is replaced by x_1 and y by y_1. www.dbraulibrary.org.in

Using the equation $y - y_1 = m(x - x_1)$ for the equation of a line through the point (x_1, y_1) of slope m, the equation of the tangent at this point is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1),$$

and since the slope of the normal at (x_1, y_1) is

$$-1\bigg/\left(\frac{dy}{dx}\right)_1,$$

the equation of the normal at (x_1, y_1) will be

$$y - y_1 = \frac{-1}{\left(\dfrac{dy}{dx}\right)_1}(x - x_1).$$

Example. Find the equations of the tangent and normal at the point (1, 4) of the curve $y = x + 3/x$.

Now $\qquad \dfrac{dy}{dx} = \dfrac{d}{dx}(x + 3x^{-1}) = 1 + 3(-1)x^{-2}$

$$= 1 - 3/x^2,$$

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therefore slope of tangent $\left(\frac{dy}{dx}\right)_{x=1}$ at the point (1, 4)

$$= 1 - 3/x^2, \text{ where } x = 1$$
$$= 1 - 3 = -2.$$

The slope of the normal at (1, 4) $= \dfrac{-1}{\text{slope of tangent}} = +\dfrac{1}{2}$.

Equation of tangent at (1, 4) is

$$y - 4 = -2(x - 1), \text{ i.e. } 2x + y = 6.$$

Equation of normal at (1, 4) is

$$y - 4 = \tfrac{1}{2}(x - 1), \quad \text{i.e. } 2y - 8 = x - 1, \quad \text{i.e. } 2y = x + 7.$$

Theorem. *To find the derivative of sin x with respect to x, where x is in* **radian measure.**

Using the derived definition,

$$\frac{d}{dx}(\sin x) = \operatorname*{Lt}_{\delta x \to 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$= \operatorname*{Lt}_{\delta x \to 0} \frac{2\cos(x + \frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\delta x}$$

$$= \operatorname*{Lt}_{\delta x \to 0} \cos(x + \tfrac{1}{2}\delta x) \cdot \frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}$$

(dividing numerator and denominator by 2)

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$$= \operatorname*{Lt}_{\delta x \to 0} \cos(x + \tfrac{1}{2}\delta x) \cdot \operatorname*{Lt}_{\delta x \to 0} \frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

(Lt. of product)

But $\operatorname*{Lt}_{\delta x \to 0} \cos(x + \frac{1}{2}\delta x) = \cos x$, and $\operatorname*{Lt}_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$,

(θ in *radians*)

$$\therefore \operatorname*{Lt}_{\delta x \to 0} \frac{\sin\frac{1}{2}\delta x}{\frac{1}{2}\delta x} = 1.$$

Hence, $$\frac{d}{dx}(\sin x) = \cos x.$$

Theorem. *To find the derivative of cos x with respect to x, where x is in* **radian measure.**

Let $$y = \cos x \text{.......................(1)},$$

and y increase by δy as x increases by a small amount δx,

$$\therefore y + \delta y = \cos(x + \delta x) \text{...............(2)}.$$

(2) − (1) gives $$\delta y = \cos(x + \delta x) - \cos x$$
$$= -[\cos x - \cos(x + \delta x)]$$
$$= -2\sin(x + \tfrac{1}{2}\delta x)\sin\tfrac{1}{2}\delta x,$$

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$$\therefore \frac{\delta y}{\delta x} = \frac{-2 \sin (x + \frac{1}{2}\delta x) \sin \frac{1}{2}\delta x}{\delta x}$$

$$= -\sin (x + \tfrac{1}{2}\delta x) \cdot \frac{\sin \frac{1}{2}\delta x}{\frac{1}{2}\delta x},$$

$$\therefore \frac{dy}{dx} = \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \operatorname*{Lt}_{\delta x \to 0} \left\{ -\sin(x + \tfrac{1}{2}\delta x) \cdot \frac{\sin \frac{1}{2}\delta x}{\frac{1}{2}\delta x} \right\}$$

$$= \operatorname*{Lt}_{\delta x \to 0} -\sin (x + \tfrac{1}{2}\delta x) \cdot \operatorname*{Lt}_{\delta x \to 0} \frac{\sin \frac{1}{2}\delta x}{\frac{1}{2}\delta x}.$$

(Lt. of product)

Now, as before, $\operatorname*{Lt}_{\delta x \to 0} \frac{\sin \frac{1}{2}\delta x}{\frac{1}{2}\delta x} = 1,$

and $\operatorname*{Lt}_{\delta x \to 0} [-\sin (x + \frac{1}{2}\delta x)] = -\sin x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\cos x)$$

$$= -\sin x.$$

Function of a Function Theorem. This states that, if y be some function of z where z is some function of x, then,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

(E.g. $y = z^{\frac{1}{2}}$ where $z = x^3 - 3x + 2$.)

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Let y and z increase by δy and δz respectively as x increases by a small amount δx.

Then, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$ (ordinary algebra).

$$\therefore \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \operatorname*{Lt}_{\delta x \to 0} \left(\frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x} \right)$$

$$= \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta z} \cdot \operatorname*{Lt}_{\delta x \to 0} \frac{\delta z}{\delta x}.$$

(Lt. of product)

Since z is a function of x it follows that $\delta z \to 0$ as $\delta x \to 0$.

$$\therefore \operatorname*{Lt}_{\delta x \to 0} \frac{\delta y}{\delta x} = \operatorname*{Lt}_{\delta z \to 0} \frac{\delta y}{\delta z} \cdot \operatorname*{Lt}_{\delta x \to 0} \frac{\delta z}{\delta x}$$

$$\text{i.e. } \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

This result is used to a large extent when differentiating complicated functions of x. The procedure is to let z equal some suitable function of x which can easily be differentiated.

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Then, if y be the original function of x, it can be obtained in terms of z and dy/dz found, and by multiplying this result by the value of dz/dx the required value of dy/dx is obtained.

Example. Find the x-derivatives (i.e. the derivatives with respect to x) of:

(i) $\sqrt{(x^2 + a^2)}$, (ii) $\sqrt[3]{(3x^3 - 2)}$, (iii) $\sin^2 x$,

(iv) y^2 (y is some function of x), (v) $\dfrac{1}{\sqrt{(2x^2 - 3x + 4)}}$.

(i) Let $y = \sqrt{(x^2 + a^2)}$, and $z = x^2 + a^2$ ....(1),

$\therefore y = z^{\frac{1}{2}}$ ....(2).

From (2), $$\frac{dy}{dz} = \tfrac{1}{2}z^{-1/2} = \frac{1}{2\sqrt{z}} = \frac{1}{2\sqrt{(x^2 + a^2)}}.$$

From (1), $$\frac{dz}{dx} = 2x.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{2\sqrt{(x^2 + a^2)}} \times 2x = \frac{x}{\sqrt{(x^2 + a^2)}},$$

i.e. $$\frac{d}{dx}[\sqrt{(x^2 + a^2)}] = \frac{x}{\sqrt{(x^2 + a^2)}}.$$

This is a result worth memorising.

(ii) Let $y = \sqrt[3]{(3x^3 - 2)}$, and $z = 3x^3 - 2$ ....(1),

$\therefore y = z^{1/3}$ ....(2).

From (2), $$\frac{dy}{dz} = \tfrac{1}{3}z^{-2/3} = \frac{1}{3\sqrt[3]{z^2}} = \frac{1}{3\sqrt[3]{(3x^3 - 2)^2}}.$$

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From (1), $$\frac{dz}{dx} = 9x^2.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{3\sqrt[3]{(3x^3 - 2)^2}}\cdot 9x^2 = \frac{3x^2}{\sqrt[3]{(3x^3 - 2)^2}}.$$

(iii) Let $y = \sin^2 x$, and $z = \sin x$ ....(1).

$\therefore y = z^2$ ....(2

From (2), $dy/dz = 2z = 2\sin x$.

From (1), $dz/dx = \cos x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\sin^2 x) = \frac{dy}{dz}\cdot\frac{dz}{dx} = 2\sin x\cos x = \sin 2x.$$

(iv) Using the function of a function theorem,

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

(v) Let $y = \dfrac{1}{\sqrt{(2x^2 - 3x + 4)}}$, and $z = 2x^2 - 3x + 4$ ......(1),

$\therefore y = \dfrac{1}{z^{1/2}} = z^{-1/2}$ ....(2).

From (2), $$\frac{dy}{dz} = -\tfrac{1}{2}z^{-3/2} = -\frac{1}{2\sqrt{(2x^2 - 3x + 4)^3}}.$$

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From (1), $$\frac{dz}{dx} = 4x - 3.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{-(4x-3)}{2\sqrt{(2x^2 - 3x + 4)^3}}.$$

Theorem. *To prove that*

$$\frac{d}{dx}[f(ax+b)] = a\frac{d}{dz}[f(z)],$$

where $z = ax + b$.

Let $$y = f(ax+b), \text{ and } z = ax + b \ldots\ldots\ldots\ldots(1)$$
$$\therefore y = f(z) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (2), $$\frac{dy}{dz} = \frac{d}{dz}[f(z)].$$

From (1) $$\frac{dz}{dx} = a.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = a \cdot \frac{d}{dz}[f(z)].$$

Corollary (i). If $f(ax+b) = (ax+b)^n$, and $z = ax + b$, then,

$$\frac{d}{dx}[f(ax+b)] = a\frac{d}{dz}(z^n) = anz^{n-1}$$

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i.e. $$\frac{d}{dx}[(ax+b)^n] = an(ax+b)^{n-1}.$$

Corollary (ii). If $f(ax+b) = \sin(ax+b)$, and $z = ax + b$,

$$\frac{d}{dx}[\sin(ax+b)] = a\frac{d}{dz}(\sin z) = a\cos z = a\cos(ax+b).$$

Similarly, $$\frac{d}{dx}[\cos(ax+b)] = -a\sin(ax+b).$$

Example. Find the derivatives of the following functions with respect to the variable contained in each.

(i) $(2t+3)^5$, (ii) $\frac{1}{(5x-2)^2}$, (iii) $\sqrt[3]{(2y+1)}$,

(iv) $\sin 3\theta$, (v) $\cos(3 - 2\varphi)$, (vi) $(1-z)^{-1}$.

(i) $\frac{d}{dt}(2t+3)^5 = 2 \times 5(2t+3)^4 = 10(2t+3)^4.$

(ii) $$\frac{d}{dx}\left\{\frac{1}{(5x-2)^2}\right\} = \frac{d}{dx}[(5x-2)^{-2}] = 5(-2)(5x-2)^{-3}$$

$$= \frac{-10}{(5x-2)^3}$$

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(iii) $\dfrac{d}{dy}[\sqrt[3]{(2y+1)}] = \dfrac{d}{dy}[(2y+1)^{\frac{1}{3}}] = \frac{1}{3}(2)(2y+1)^{-\frac{2}{3}}$

$$= \frac{2}{3\sqrt[3]{(2y+1)^2}}.$$

(iv) $\dfrac{d}{d\theta}(\sin 3\theta) = 3\cos 3\theta.$

(v) $\dfrac{d}{d\varphi}[\cos(3-2\varphi)] = (-2)[-\sin(3-2\varphi)] = 2\sin(3-2\varphi).$

(vi) $\dfrac{d}{dz}[(1-z)^{-1}] = (-1)(-1)(1-z)^{-2} = (1-z)^{-2}.$

Theorem. *If u and v be functions of x, to find the differential coefficient of the product uv with respect to x.*

Let $$y = uv \quad \ldots\ldots (1)$$

and let y, u, v increase by δy, δu, δv respectively as x increases by a small amount δx, then,

$$y + \delta y = (u + \delta u)(v + \delta v)$$
$$= uv + u\,.\,\delta v + v\,.\,\delta u + (\delta u)(\delta v) \quad \ldots\ldots (2)$$

(2) − (1) gives $\delta y = u\delta v + v\delta u + (\delta u)(\delta v).$

$$\therefore \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\,.\,\delta v$$

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$$= u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta x}\cdot\delta x.$$

Hence,

$$\frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Lt}}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta x}\cdot\delta x\right)$$

$$= \underset{\delta x\to 0}{\text{Lt}}\, u\frac{\delta v}{\delta x} + \underset{\delta x\to 0}{\text{Lt}}\, v\frac{\delta u}{\delta x} + \underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta x}\cdot\delta x \quad \text{(Lt. of sum)}$$

$$= \underset{\delta x\to 0}{\text{Lt}}\, u\,.\,\underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta v}{\delta x} + \underset{\delta x\to 0}{\text{Lt}}\, v\,.\,\underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta u}{\delta x}$$
$$+ \underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta u}{\delta x}\cdot\underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta v}{\delta x}\cdot\underset{\delta x\to 0}{\text{Lt}}\,\delta x \quad \text{(Lt. of product)}$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx}\cdot\frac{dv}{dx}\cdot\underset{\delta x\to 0}{\text{Lt}}\,\delta x$$

i.e. $$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}, \quad \text{since } \underset{\delta x\to 0}{\text{Lt}}\,\delta x = 0.$$

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EXAMPLE. Find the derivatives of the following functions with respect to the variable contained in each.

(i) $x\sqrt{(2x+1)}$, (ii) $(2t+3)^2(t+2)^2$, (iii) $(\theta^2+1)\sin\theta$,
(iv) $2t^\circ \cos 3t^\circ$, (v) $(2y-3)\sqrt[3]{(y^3-5)}$.

All the functions are products and the result

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

will be used.

(i)
$$\begin{aligned}\frac{d}{dx}[x\sqrt{(2x+1)}] &= \frac{d}{dx}[x(2x+1)^{\frac{1}{2}}] \\ &= x\frac{d}{dx}[(2x+1)^{\frac{1}{2}}] + (2x+1)^{\frac{1}{2}}\frac{d}{dx}(x) \\ &= x(2x+1)^{-\frac{1}{2}} + (2x+1)^{\frac{1}{2}} \\ &= \frac{x}{\sqrt{(2x+1)}} + \sqrt{(2x+1)} \\ &= \frac{x+(2x+1)}{\sqrt{(2x+1)}} = \frac{3x+1}{\sqrt{(2x+1)}}.\end{aligned}$$

(ii)
$$\begin{aligned}\frac{d}{dt}[(2t+3)^2(t+2)^2] &= (2t+3)^2\frac{d}{dt}[(t+2)^2] \\ &\qquad + (t+2)^2\frac{d}{dt}[(2t+3)^2] \\ &= (2t+3)^2 \times 2(t+2) + \\ &\qquad (t+2)^2 \times 2 \times 2(2t+3) \\ &= 2(t+2)(2t+3)[(2t+3)+2(t+2)] \\ &= 2(t+2)(2t+3)(4t+7).\end{aligned}$$

(iii)
$$\begin{aligned}\frac{d}{d\theta}[(\theta^2+1)\sin\theta] &= (\theta^2+1)\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(\theta^2) \\ &= (\theta^2+1)\cos\theta + 2\theta\sin\theta.\end{aligned}$$

(iv)
$$\frac{d}{dt}(2t^\circ\cos 3t^\circ) = \frac{d}{dt}\left\{2t \times \frac{\pi}{180} . \cos\frac{3t\times\pi}{180}\right\}$$

(angle must be converted to **radians**)

$$\begin{aligned}&= \frac{d}{dt}\left\{\frac{\pi t}{90}\cos\frac{\pi t}{60}\right\} \\ &= \frac{\pi t}{90}\cdot\frac{d}{dt}\left\{\cos\frac{\pi t}{60}\right\} + \cos\frac{\pi t}{60}\cdot\frac{d}{dt}\left(\frac{\pi t}{90}\right) \\ &= \frac{\pi t}{90}\left(-\frac{\pi}{60}\sin\frac{\pi t}{60}\right) + \frac{\pi}{90}\cos\frac{\pi t}{60} \\ &= \frac{\pi}{90}\left\{\cos\frac{\pi t}{60} - \frac{\pi t}{60}\sin\frac{\pi t}{60}\right\}.\end{aligned}$$

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(v) Using the function of a function theorem,

$$\frac{d}{dy}[\sqrt[3]{(y^3 - 5)}] = \frac{d}{dz}(z)^{\frac{1}{3}}\frac{dz}{dy}, \quad \text{where } z = y^3 - 5$$

$$= \frac{1}{3z^{\frac{2}{3}}} \cdot 3y^2 = \frac{y^2}{\sqrt[3]{(y^3 - 5)^2}}.$$

$$\therefore \frac{d}{dy}[(2y - 3)(\sqrt[3]{(y^3 - 5)}] = (2y - 3)\frac{d}{dy}[\sqrt[3]{(y^3 - 5)}] + \sqrt[3]{(y^3 - 5)}\,\frac{d}{dy}(2y - 3)$$

$$= (2y - 3)\frac{y^2}{\sqrt[3]{(y^3 - 5)^2}} + 2\sqrt[3]{(y^3 - 5)}$$

$$= \frac{(2y^3 - 3y^2) + (2y^3 - 10)}{\sqrt[3]{(y^3 - 5)^2}}$$

$$= \frac{4y^3 - 3y^2 - 10}{\sqrt[3]{(y^3 - 5)^2}}.$$

NOTE. All results should be simplified as far as possible.

Theorem. *To find the derivative of the quotient u/v with respect to x, where u and v are functions of x.*

Let $y = u/v$ ....(1),

and y, u, v increase by small amounts δy, δu, δv respectively as x increases by a small amount δx,

$$\therefore y + \delta y = \frac{u + \delta u}{v + \delta v} \quad \text{.................(2).}$$

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(2) – (1) gives, $\delta y = \dfrac{u + \delta u}{v + \delta v} - \dfrac{u}{v} = \dfrac{uv + v\delta u - u\delta v - uv}{v(v + \delta v)}$

$$= \frac{v\delta u - u\delta v}{v(v + \delta v)},$$

$$\therefore \frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)},$$

$$\therefore \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}$$

$$= \frac{\underset{\delta x \to 0}{\text{Lt}}\left(v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}\right)}{\underset{\delta x \to 0}{\text{Lt}}\ v(v + \delta v)} \qquad \text{(Lt. of quotient)}$$

$$= \frac{\underset{\delta x \to 0}{\text{Lt}}\ v\dfrac{\delta u}{\delta x} - \underset{\delta x \to 0}{\text{Lt}}\ u\dfrac{\delta v}{\delta x}}{\underset{\delta v \to 0}{\text{Lt}}\ v(v + \delta v)},$$

(Lt. of sum and $\delta v \to 0$ as $\delta x \to 0$)

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i.e. $$\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE. Find the derivatives of the following functions with respect to the variable contained in each:

(i) $\frac{2t-3}{2t+1}$, (ii) $\frac{\cos x}{x}$, (iii) $\sqrt{\frac{2y+1}{2y-1}}$,

(iv) $\tan\theta$, (v) $\frac{(2z-5)^2}{1-z}$, (vi) $\frac{\sin 3\varphi}{\sqrt{(\varphi+1)}}$.

NOTE. All angles are in *radians* unless otherwise stated and the result

$$\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

will be used throughout.

(i) $$\frac{d}{dt}\left\{\frac{2t-3}{2t+1}\right\} = \frac{(2t+1)\frac{d}{dt}(2t-3) - (2t-3)\frac{d}{dt}(2t+1)}{(2t+1)^2}$$

$$= \frac{2(2t+1) - 2(2t-3)}{(2t+1)^2} = \frac{8}{(2t+1)^2}.$$

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(ii) $$\frac{d}{dx}\left(\frac{\cos x}{x}\right) = \frac{x\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(x)}{x^2}$$

$$= \frac{-x\sin x - \cos x}{x^2} = \frac{-(x\sin x + \cos x)}{x^2}.$$

(iii) $$\frac{d}{dy}\left\{\sqrt{\frac{2y+1}{2y-1}}\right\} = \frac{d}{dy}\left\{\frac{(2y+1)^{\frac{1}{2}}}{(2y-1)^{\frac{1}{2}}}\right\}$$

$$= \frac{(2y-1)^{\frac{1}{2}}\cdot\frac{d}{dy}[(2y+1)^{\frac{1}{2}}] - (2y+1)^{\frac{1}{2}}\frac{d}{dy}[(2y-1)^{\frac{1}{2}}]}{2y-1}$$

$$= \frac{(2y-1)^{\frac{1}{2}} \times 2 \times \frac{1}{2}(2y+1)^{-\frac{1}{2}} - (2y+1)^{\frac{1}{2}} \times 2 \times \frac{1}{2}(2y-1)^{-\frac{1}{2}}}{2y-1}$$

$$= \frac{(2y-1) - (2y+1)}{(2y-1)^{3/2}(2y+1)^{1/2}}.$$

$$= \frac{-2}{(2y-1)^{3/2}(2y+1)^{1/2}}.$$

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(iv) $\frac{d}{d\theta}\tan\theta) = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta\frac{d}{d\theta}(\sin\theta) - \sin\theta\frac{d}{d\theta}(\cos\theta)}{\cos^2\theta}$

$$= \frac{\cos\theta(\cos\theta) - \sin\theta(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$= \sec^2\theta.$$

(v) $\frac{d}{dz}\left\{\frac{(2z-5)^2}{1-z}\right\} = \frac{(1-z)\times 4(2z-5) - (2z-5)^2(-1)}{(1-z)^2}$

$$= \frac{(2z-5)(4-4z+2z-5)}{(1-z)^2}$$

$$= \frac{-(2z-5)(2z+1)}{(1-z)^2}.$$

(vi) $\frac{d}{d\varphi}\left\{\frac{\sin 3\varphi}{\sqrt{(\varphi+1)}}\right\} = \frac{\sqrt{(\varphi+1)}\times 3\cos 3\varphi - \sin 3\varphi \times \frac{1}{2\sqrt{(\varphi+1)}}}{(\varphi+1)}$

$$= \frac{6(\varphi+1)\cos 3\varphi - \sin 3\varphi}{2(\varphi+1)^{3/2}}.$$

EXAMPLE (L.U.). Gas is being pumped into a spherical balloon at the rate of 12 cubic feet per minute.

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Find the rate at which the radius of the sphere is increasing at the instant when its length is 6 feet.

Also show that at this instant the area of the balloon is increasing at the rate of 4 square feet per minute.

Let V cubic feet be the volume and A the surface area in square feet when the radius is x feet.

Then $V = \frac{4}{3}\pi x^3,$

$$\therefore \frac{dV}{dt} = \frac{d}{dt}(\tfrac{4}{3}\pi x^3) = \frac{d}{dx}(\tfrac{4}{3}\pi x^3)\cdot\frac{dx}{dt}$$

$$= 4\pi x^2\frac{dx}{dt}.$$

When $x = 6$ and $\frac{dV}{dt} = 12,$

$$12 = 4\pi \times 36\frac{dx}{dt},$$

$$\therefore \frac{dx}{dt} = \frac{12}{4\pi\times 36} = \frac{1}{12\pi} \text{ feet per minute.}$$

Also, $A = 4\pi x^2,$

$$\therefore \frac{dA}{dt} = \frac{d}{dt}(4\pi x^2) = \frac{d}{dx}(4\pi x^2)\cdot\frac{dx}{dt} = 8\pi x\frac{dx}{dt}.$$

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When $x = 6$, $\dfrac{dx}{dt} = \dfrac{1}{12\pi}$.

$$\therefore \frac{dA}{dt} = 8\pi \times 6 \times \frac{1}{12\pi} = 4 \text{ square feet per minute.}$$

EXAMPLE (L.U.). Find the equation of the tangent to the parabola $y^2 + 6x = 0$ at the point $(-\frac{3}{8}, -\frac{3}{2})$. Find also the co-ordinates of the point of contact of the tangent which is perpendicular to the tangent at the point $(-6, 6)$.

It can be seen, by inspection, that the point $(-\frac{3}{8}, -\frac{3}{2})$ lies on the parabola

$$y^2 + 6x = 0 \qquad \text{.......................(1).}$$

Differentiating (1) with respect to x,

$$\frac{d}{dx}(y^2) + 6 = 0,$$

$$\text{i.e. } \frac{d}{dy}(y^2) \cdot \frac{dy}{dx} + 6 = 0,$$

$$\therefore 2y\frac{dy}{dx} + 6 = 0. \qquad \therefore \frac{dy}{dx} = \frac{-3}{y}.$$

When $y = -\frac{3}{2}$,

$$\frac{dy}{dx} = -3(-\tfrac{2}{3}) = +2.$$

Hence, the equation of the tangent at $(-\frac{3}{8}, -\frac{3}{2})$ is

$$y + \tfrac{3}{2} = 2(x + \tfrac{3}{8}),$$
$$\text{i.e. } 4y + 6 = 8x + 3$$
$$\text{i.e. } 8x - 4y = 3.$$

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The slope of the tangent at the point $(-6, 6)$ is $-3/y$, where $y = 6$, i.e. slope $= -\frac{3}{6} = -\frac{1}{2}$.

Therefore the slope of the tangent perpendicular to this is 2, which is the slope of the previous tangent. Hence, the required point of contact is $(-\frac{3}{8}, -\frac{3}{2})$.

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EXAMPLE (L.U.). If the co-ordinates of any point P on a certain curve are $(at^2, 2at)$, where t is variable and a constant, find the equations of the tangent and normal at P.

Find the co-ordinates of the point Q, where the normal at P meets the curve again, and find also for what values of t the angle POQ is a right angle, O being the origin.

The equation of the curve is given by

$$x = at^2 \qquad \text{.......................(1),}$$
$$y = 2at \qquad \text{.......................(2),}$$

(known as the *parametric* equation of the curve, which is a parabola $y^2 = 4ax$ obtained by eliminating the *parameter* t between equations (1) and (2)).

From (1), $\dfrac{dx}{dt} = 2at$, and from (2) $\dfrac{dy}{dt} = 2a$.

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \quad \text{(see next theorem}$$

$$= \frac{2a}{2at} = \frac{1}{t}$$

therefore slope of tangent is $1/t$, and hence the slope of the normal is $-t$.

Thus, the equation of the tangent at the point $(at^2, 2at)$ is

$$y - 2at = (1/t)(x - at^2),$$

$$\text{i.e. } ty - 2at^2 = x - at^2,$$

$$\text{i.e. } ty - x = at^2.$$

Also, the equation of the normal at the point is

$$y - 2at = -t(x - at^2),$$

$$\text{i.e. } tx + y = 2at + at^3 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let the normal meet the curve again in the point $(aT^2, 2aT)$. Then this point must satisfy the equation (3),

$$\therefore atT^2 + 2aT = 2at + at^3,$$

$$\text{i.e. } tT^2 + 2T - (2t + t^3) = 0, \qquad \text{(quadratic in } T).$$

$$\text{i.e. } (T - t)[tT + (2 + t^2)] = 0,$$

$$\therefore T = t, \text{ or } T = -(2 + t^2)/t.$$

But $T \neq t$, therefore the normal cuts the curve again in the point where $T = -(2 + t^2)/t$, i.e. at the point

$$\left\{\frac{a(2 + t^2)^2}{t^2}, \frac{-2a(2 + t^2)}{t}\right\}.$$

The slope of $OP = \dfrac{y\text{-co-ordinate of } P}{x\text{-co-ordinate of } P} = \dfrac{2at}{at^2} = \dfrac{2}{t}$.

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Similarly, the slope of $OQ = \dfrac{2}{T} = \dfrac{2t}{-(2+t^2)}$.

OP and OQ are at right angles if the product of their slopes equals -1,

$$\text{i.e. if } \frac{2}{t} \times \frac{2t}{-(2 + t^2)} = -1,$$

$$\text{i.e. if } \frac{4}{2 + t^2} = 1, \text{ giving } t^2 + 2 = 4,$$

$$\text{i.e. } t^2 = 2, \text{ i.e. if } t = \pm\sqrt{2}.$$

Theorem. *To prove that, if y be some function of x, then*

$$\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}.$$

Let δy be a small increase in y corresponding to a small increase of δx in x.

Then
$$\frac{\delta x}{\delta y} = \frac{1}{\delta y/\delta x}$$

(dividing numerator and denominator by δx),

$$\therefore \operatorname*{Lt}_{\delta y \to 0} \frac{\delta x}{\delta y} = \operatorname*{Lt}_{\delta y \to 0} \frac{1}{\delta y/\delta x}$$

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$$= \underset{\delta y \to 0}{\text{Lt}} 1 \Big/ \underset{\delta y \to 0}{\text{Lt}} \frac{\delta y}{\delta x}$$

(Lt. of quotient)

$$= 1 \Big/ \underset{\delta y \to 0}{\text{Lt}} \frac{\delta y}{\delta x},$$

$$= 1 \Big/ \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} \quad (\delta x \to 0 \text{ as } \delta y \to 0),$$

$$\therefore \frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}.$$

Exponentials and Napierian Logarithms. The quantity e^x, defined as being

$$\underset{n \to \infty}{\text{Lt}} (1 + x/n)^n$$

is known as the *exponential function*, and with this definition

$$e = \underset{n \to \infty}{\text{Lt}} (1 + 1/n)^n.$$

This quantity e, whose value will be shown later to be 2·71828 . . ., is the base for *Napierian* (*or natural*) *logarithms*, and is a very important quantity in advanced mathematics.

Theorem. *To find a series in ascending powers of x for* e^x.

Expanding by the binomial theorem

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$$(1 + x/n)^n = 1 + n(x/n) + \frac{n(n-1)}{2!}(x/n)^2 + \frac{n(n-1)(n-2)}{3!}(x/n)^3 + \dots$$

$$= 1 + x + \frac{1(1 - 1/n)}{2!}x^2 + \frac{1(1 - 1/n)(1 - 2/n)}{3!}x^3 + \dots$$

$$\therefore \underset{n \to \infty}{\text{Lt}} (1 + x/n)^n = \underset{n \to \infty}{\text{Lt}} \left\{1 + x + \frac{(1 - 1/n)}{2!}x^2 + \frac{(1 - 1/n)(1 - 2/n)}{3!}x^3 + \dots\right\}$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

since $1/n$, $2/n$, etc., each $\to 0$ when $n \to \infty$.

But $\underset{n \to \infty}{\text{Lt}} (1 + x/n)^n = e^x$,

$$\therefore e^x = 1 + x + x^2/2! + x^3/3! + \dots \to \infty.$$

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This series is known as the exponential series, and, by replacing x in it by $(-x)$, it can be seen that

$$e^{-x} = 1 - x + x^2/2! - x^3/3! + \ldots$$

Using $x = 1$ in the series for e^x,

$$e = 1 + 1 + 1/2! + 1/3! + 1/4! + \ldots$$

and by the arithmetical working shown it can be seen that the value of e to five decimal places is 2·71828.

$$\begin{aligned} 1 + 1 &= 2{\cdot}000000 \\ 1/2! &= 0{\cdot}500000 \\ 1/3! &= 0{\cdot}166667 \\ 1/4! = 1/4 \times 1/3! &= 0{\cdot}041667 \\ 1/5! = 1/5 \times 1/4! &= 0{\cdot}008333 \\ 1/6! = 1/6 \times 1/5! &= 0{\cdot}001389 \\ 1/7! = 1/7 \times 1/6! &= 0{\cdot}000198 \\ 1/8! = 1/8 \times 1/7! &= 0{\cdot}000025 \\ 1/9! = 1/9 \times 1/8! &= 0{\cdot}000003 \\ & \overline{2{\cdot}718282} \end{aligned}$$

Theorem. *To find the derivative of e^x with respect to x.*

Using the previous theorem,

$$e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$$

$$\begin{aligned} \therefore \frac{d}{dx}(e^x) &= \frac{d}{dx}\left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right\} \\ &= 0 + 1 + 2x/2! + 3x^2/3! + 4x^3/4! + \ldots \\ &= 1 + x + x^2/2! + x^3/3! + \ldots \\ &= e^x. \end{aligned}$$

$$\therefore \frac{d}{dx}(e^x) = e^x.$$

Employing the function of a function theorem, where a and b are constants,

$$\begin{aligned} \frac{d}{dx}(e^{ax+b}) &= \frac{d}{dx}(e^z), \quad \text{where } z = ax + b \\ &= \frac{d}{dz}(e^z)\,\frac{dz}{dx} \\ &= e^z \times a = ae^{ax+b}. \end{aligned}$$

Thus $\dfrac{d}{du}\left(\dfrac{1}{e^{2u}}\right) = \dfrac{d}{du}(e^{-2u}) = -2e^{-2u} = \dfrac{-2}{e^{2u}}$.

$$\frac{d}{d\theta}\left(\sqrt{\frac{1}{e^\theta}}\right) = \frac{d}{d\theta}[(e^{-\theta})^{\frac{1}{2}}] = \frac{d}{d\theta}(e^{-\frac{1}{2}\theta}) = -\tfrac{1}{2}e^{-\theta/2}.$$

$$\frac{d}{dx}(xe^{2x+3}) = x\frac{d}{dx}(e^{2x+3}) + e^{2x+3}\frac{d}{dx}(x)$$

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$$= x \times 2e^{2x+3} + e^{2x+3} \times 1$$
$$= (2x+1)e^{2x+3}.$$

$$\frac{d}{dy}\left(\frac{y^2}{e^{y+2}}\right) = \frac{e^{y+2}\frac{d}{dy}(y^2) - y^2\frac{d}{dy}(e^{y+2})}{(e^{y+2})^2}$$
$$= \frac{e^{y+2} \times 2y - y^2e^{y+2}}{(e^{y+2})^2} = \frac{y(2-y)}{e^{y+2}}.$$

$$\frac{d}{dt}(e^{t^2-2t+3}) = \frac{d}{dt}(e^z),$$

(where $z = t^2 - 2t + 3$ and $\therefore \frac{dz}{dt} = 2t - 2$)

$$= \frac{d}{dz}(e^z) \cdot \frac{dz}{dt}$$
$$= e^z(2t-2) = 2(t-1)e^{t^2-2t+3}.$$

Theorem. *To find* $\frac{d}{dx}(\log_e x)$, *and* $\frac{d}{dx}[\log_e(ax+b)]$, *where a and b are constants.*

Let $y = \log_e x$, (y is the Napierian logarithm of x)

$$\therefore x = e^y \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiating this with respect to y,

$$\frac{dx}{dy} = \frac{d}{dy}(e^y)$$

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$$= e^y = x. \qquad \text{(from (1))}$$

It has been shown earlier that

$$\frac{dx}{dy} = \frac{1}{dy/dx}, \quad \therefore \quad \frac{1}{dy/dx} = x,$$

and $\frac{dy}{dx} = \frac{1}{x}$,

i.e. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$.

Using the function of a function theorem,

$$\frac{d}{dx}[\log_e(ax+b)] = \frac{d}{dx}(\log_e z),$$

(where $z = ax + b$ and $\therefore \frac{dz}{dx} = a$)

$$= \frac{d}{dz}(\log_e z) \cdot \frac{dz}{dx}$$
$$= \frac{1}{z} \times a = \frac{a}{ax+b}.$$

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EXAMPLE. Find the derivatives of the following functions with respect to x:
(i) $2\log_e(2x+3)$, (ii) $x^2\log_e(4-x)$, (iii) $(1/x)\log_e x$, (iv) $\log_e(x^3-3x^2+6)$, (v) $\log_e(\sin 2x)$.

(i) $$\frac{d}{dx}[2\log_e(2x+3)] = 2\times\frac{2}{2x+3} = \frac{4}{2x+3}.$$

(ii) $$\frac{d}{dx}[x^2\log_e(4-x)] = x^2\cdot\frac{d}{dx}[\log_e(4-x)] + \log_e(4-x)\times\frac{d}{dx}(x^2)$$
$$= x^2\times(-1)/(4-x) + [\log_e(4-x)]\times 2x$$
$$= x\left\{2\log_e(4-x) - \frac{x}{4-x}\right\}.$$

(iii) $$\frac{d}{dx}[(1/x)\log_e x] = \frac{x\frac{d}{dx}(\log_e x) - (\log_e x)\cdot\frac{d}{dx}(x)}{x^2}$$
$$= \frac{x\times 1/x - (\log_e x)\times 1}{x^2} = \frac{1-\log_e x}{x^2}.$$

(iv) Let $$z = x^3-3x^2+6,$$
$$\therefore dz/dx = 3x^2-6x,$$
and $$\frac{d}{dx}[\log_e(x^3-3x^2+6)] = \frac{d}{dx}(\log_e z)$$

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$$= \frac{d}{dz}(\log_e z)\cdot\frac{dz}{dx}$$
$$= 1/z\times(3x^2-6x)$$
$$= \frac{3x(x-2)}{x^3-3x^2+6}.$$

(v) Let $$z = \sin 2x,$$
$$\therefore dz/dx = 2\cos 2x,$$
and $$\frac{d}{dx}[\log_e(\sin 2x)] = \frac{d}{dx}(\log_e z)$$
$$= \frac{d}{dz}(\log_e z)\cdot\frac{dz}{dx}$$
$$= 1/z\times 2\cos 2x$$
$$= \frac{2\cos 2x}{\sin 2x}$$
$$= 2\cot 2x.$$

Second Differential Coefficients, etc. If the result of differentiating $f(x)$ with respect to x be $\varphi(x)$, then the result of differentiating $\varphi(x)$ with respect to x is known as the *second differential coefficient* (or *derivative*) of $f(x)$ with respect to x.

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If $y = f(x)$, the first differential coefficient of y with respect to x is represented by

$$\frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \text{ or } f'(x).$$

The second derivative of y with respect to x will be the differential of

$$\frac{dy}{dx}$$

with respect to x, and is represented by

$$\frac{d}{dx}\left(\frac{dy}{dx}\right),$$

or more briefly

$$\frac{d^2y}{dx^2},$$

other symbols when $y = f(x)$ being

$$\frac{d^2}{dx^2}[f(x)], f''(x).$$

Similarly, the derivative of

$$\frac{d^2y}{dx^2}$$

with respect to x is known as the *third differential coefficient* of y with respect to x, and is denoted by any one of the symbols

$$\frac{d^3y}{dx^3}, \quad \frac{d^3}{dx^3}[f(x)], f'''(x),$$

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and so on.

EXAMPLE. Find the first and second derivatives of the following functions with respect to t, and find their values when $t = 4$:

$$\text{(i) } (2t - 3)^3, \quad \text{(ii) } 4t^{3/2} - 3t + 2t^{1/2}, \quad \text{(iii) } \frac{t^3 - 1}{t}.$$

(i) Let $\qquad y = (2t - 3)^3$

$$\therefore \frac{dy}{dt} = 2 \times 3(2t - 3)^2 = 6(2t - 3)^2,$$

and $\qquad \dfrac{d^2y}{dt^2} = 6 \times 2 \times 2(2t - 3) = 24(2t - 3).$

Therefore when $t = 4$,

$$\frac{dy}{dt} = 6 \times 5^2 = 150, \quad \frac{d^2y}{dt^2} = 24 \times 5 = 120.$$

(ii) Let $\qquad y = 4t^{3/2} - 3t + 2t^{1/2},$

$$\therefore \frac{dy}{dt} = 6t^{1/2} - 3 + t^{-1/2},$$

$$\frac{d^2y}{dt^2} = 3t^{-1/2} - \tfrac{1}{2}t^{-3/2}.$$

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When $t = 4$,

$$\frac{dy}{dt} = 6 \times 2 - 3 + \tfrac{1}{2} = \tfrac{19}{2}, \text{ and } \frac{d^2y}{dt^2} = \tfrac{3}{2} - \tfrac{1}{2} \times \tfrac{1}{8} = \tfrac{3}{2} - \tfrac{1}{16} = \tfrac{23}{16}.$$

(iii) Let $\qquad y = \dfrac{t^2 - 1}{t} = t - t^{-1},$

$$\therefore \frac{dy}{dt} = 1 + t^{-2}, \text{ and } \frac{d^2y}{dt^2} = -2t^{-3}.$$

When $t = 4$,

$$\frac{dy}{dt} = 1 + \frac{1}{16} = \frac{17}{16}, \text{ and } \frac{d^2y}{dt^2} = \frac{-2}{64} = \frac{-1}{32}.$$

Example (L.U.). The radial stress p in a rotating flywheel at a distance r from the centre is given by $p = A + Br^{-2} + Kr^2$, where A, B, K are constants.

Prove that $\qquad r\dfrac{d^2p}{dr^2} + 3\dfrac{dp}{dr} = 8Kr.$

If $p = 0$, when $r = a$, and when $r = b$, find A and B in terms of a, b, K.

$$p = A + Br^{-2} + Kr^2,$$

$$\therefore \frac{dp}{dr} = -2Br^{-3} + 2Kr,$$

$$\frac{d^2p}{dr^2} = 6Br^{-4} + 2K,$$

$$r\frac{d^2p}{dr^2} + 3\frac{dp}{dr} = (6Br^{-3} + 2Kr) + 3(-2Br^{-3} + 2Kr)$$

$$= 8Kr.$$

Since $p = 0$ when $r = a$,

$$0 = A + \frac{B}{a^2} + Ka^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since $p = 0$ when $r = b$,

$$0 = A + \frac{B}{b^2} + Kb^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

(1) − (2) gives $\qquad 0 = B(1/a^2 - 1/b^2) + K(a^2 - b^2),$

$$\therefore B\left(\frac{a^2 - b^2}{a^2b^2}\right) = K(a^2 - b^2),$$

$$\therefore B = Ka^2b^2. \qquad (a^2 \neq b^2)$$

Using this in (1),

$$0 = A + Kb^2 + Ka^2,$$

$$\therefore A = -K(a^2 + b^2).$$

Example. The distance s feet moved in a straight line by a particle in time t seconds is given by the equation

$$s = t \cos \frac{\pi}{10}t.$$

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Find the acceleration after five seconds.

$$s = t\cos\frac{\pi}{10}t \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating (1) with respect to t,

$$\text{velocity } v = \frac{ds}{dt} = t\left\{-\frac{\pi}{10}\sin\frac{\pi}{10}t\right\} + \cos\frac{\pi}{10}t$$

(differential of product)

$$= -\frac{\pi}{10}t\sin\frac{\pi}{10}t + \cos\frac{\pi}{10}t.$$

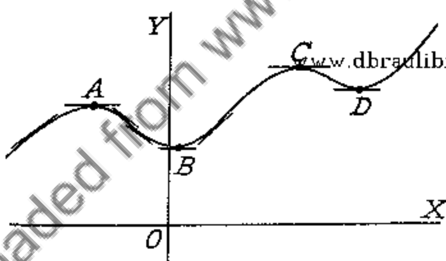
$$\text{Acceleration} = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -\frac{\pi}{10}\left\{t \times \frac{\pi}{10}\cos\frac{\pi}{10}t + \sin\frac{\pi}{10}t\right\} - \frac{\pi}{10}\sin\frac{\pi}{10}t$$

$$= -\frac{\pi^2}{100}t\cos\frac{\pi}{10}t - \frac{2\pi}{10}\sin\frac{\pi}{10}t.$$

Therefore when $t = 5$,

$$\text{acceleration} = -\frac{\pi^2}{100} \times 5\cos\frac{\pi}{2} - \frac{\pi}{5}\sin\frac{\pi}{2}$$

$$= 0 - \frac{\pi}{5} = -\frac{\pi}{5} \text{ feet per second, per second.}$$



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Maxima, Minima, etc. In the graph shown, the points A and C on the curve are known as maximum points, and the points B and D are minimum points.

A *maximum point* on a curve is a point at which the ordinate is greater than any other ordinate in its immediate vicinity on either side of it.

A *minimum point* on a curve is a point at which the ordinate is algebraically less than any ordinate in its immediate vicinity on either side of it.

The two types of points are classed together under the single name of *turning points*, due to the fact that the curve turns at this type of point.

From the graph shown, it is clear that the tangent at any turning point is parallel to OX, and therefore has zero slope.

But

$$\frac{dy}{dx}$$

is the slope of the tangent at any point (x, y) of the curve, and hence for a turning point at (x, y) the value of

$$\frac{dy}{dx} \text{ must be zero.}$$

Thus, for the curve $y = f(x)$, the equation in x given by

$$\frac{dy}{dx} = 0,$$

will give the values of x at which there are turning points, and the corresponding values of y can then be found from the equation $y = f(x)$.

EXAMPLE. Find the turning points on the curve $y = x^3 - x^2 - x$.

From the equation $y = x^3 - x^2 - x$,

$$\frac{dy}{dx} = 3x^2 - 2x - 1 = (3x + 1)(x - 1).$$

For turning points $\frac{dy}{dx} = 0$,

i.e. $(3x + 1)(x - 1) = 0$, i.e. $x = -\frac{1}{3}$, or 1.

When $x = 1$, $y = -1$, and when $x = -\frac{1}{3}$,

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$$y = -\tfrac{1}{27} - \tfrac{1}{9} + \tfrac{1}{3} = \tfrac{5}{27}.$$

Hence, the required turning points are $(1, -1)$, $(-\frac{1}{3}, \frac{5}{27})$.

As can be seen from the graph, a maximum value does not necessarily mean the greatest ordinate on the curve, nor does a minimum value necessarily mean the least ordinate, but in practical questions it is generally found that the maximum value is the greatest value and a minimum value is generally the least value.

In certain cases it is necessary to distinguish between maximum and minimum values, and a further condition is required for this purpose.

From the graph it is clear that the slope of the tangent immediately to the left of a maximum point is positive and immediately to the right of the maximum point it is negative. Hence dy/dx changes from positive through zero to negative in passing through a maximum point with x increasing. Thus in passing through a maximum point with x increasing the rate of change of dy/dx is negative. But the rate of change of dy/dx is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Hence for a *maximum point* $\frac{d^2y}{dx^2}$ *is negative.*

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Immediately to the left of a minimum point, it is seen from the graph that the value of dy/dx is negative, and immediately to the right of the minimum point dy/dx is positive. Thus, by a similar argument to the previous one, it follows that, *for a minimum point, d^2y/dx^2 is positive.*

The procedure when finding the maximum and minimum values of a given function $f(x)$ is to equate $f(x)$ to y, and then find dy/dx and d^2y/dx^2.

The result for dy/dx is then equated to zero, thus obtaining an equation in x from which values of x are obtained which give the turning points on the curve $y = f(x)$, and, if required, the corresponding values of y are determined by using $y = f(x)$.

To distinguish between maxima and minima, these values of x will be inserted in the result obtained for d^2y/dx^2, and those giving a negative value of d^2y/dx^2 will give maximum values of the function, and those giving positive values of d^2y/dx^2 will determine minimum values of the function.

NOTE. In certain practical questions dealing with mensuration, etc., the quantity for which a maximum or minimum value is required will first be found in terms of two variables. From data in the problems a relationship between these two variables will be obtained, and this must be used in conjunction with the first equation to eliminate one of the variables in the first equation. The normal procedure is then employed.

In practical questions the distinction between a maximum and a minimum value can be generally ascertained from the nature of the question without recourse to the conditions that d^2y/dx^2 shall be negative or positive respectively.

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EXAMPLE (L.U.). If $y = 3x^4 - 20x^3 + 48x^2 - 48x - 2$, find the values of x for which $dy/dx = 0$, and for which of these values of x, if any, y is a maximum or minimum.

$$y = 3x^4 - 20x^3 + 48x^2 - 48x - 2.$$

$$\therefore \frac{dy}{dx} = 12x^3 - 60x^2 + 96x - 48 \quad \text{.............. (1)},$$

$$\frac{d^2y}{dx^2} = 36x^2 - 120x + 96 \quad \text{.................... (2)}.$$

From (1), $\frac{dy}{dx} = 0$ if $12x^3 - 60x^2 + 96x - 48 = 0$,

$$\text{i.e. if } x^3 - 5x^2 + 8x - 4 = 0 \quad \text{.............. (3)}.$$

By inspection, $x = 1$ is a solution of (3).

Equation (3) can be written,

$$(x^3 - x^2) - (4x^2 - 8x + 4) = 0$$
$$\text{i.e. } x^2(x-1) - 4(x-1)^2 = 0$$
$$\therefore (x-1)(x^2 - 4x + 4) = 0$$
$$\text{i.e. } (x-1)(x-2)^2 = 0$$
$$\therefore x = 1 \text{ or } 2 \text{ (twice)}.$$

When $x = 1$, $\quad \dfrac{d^2y}{dx^2} = 36 - 120 + 96 = +12,$

therefore y has a minimum value when $x = 1$.

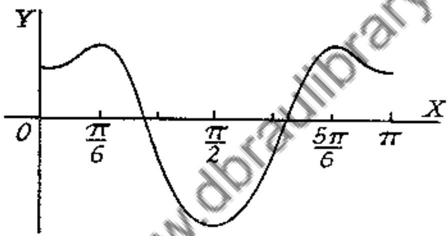
When $x = 2$, $\quad \dfrac{d^2y}{dx^2} = 144 - 240 + 96 = 0,$

therefore there is neither a maximum nor a minimum point when $x = 2$ (the special point is known as a *point of inflexion*).

EXAMPLE (L.U.). From first principles obtain the differential coefficient of $\cos x$ with respect to x.

Find the maximum and minimum values of $2\cos 2x - \cos 4x$ in the range $0 \leqslant x \leqslant \pi$.

Sketch the curve $y = 2\cos 2x - \cos 4x$ from $x = 0$ to $x = \pi$.



The first part of the question has been proved as a theorem.

Let $\qquad y = 2\cos 2x - \cos 4x,$

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$$\therefore \frac{dy}{dx} = -4\sin 2x + 4\sin 4x,$$

$$\frac{d^2y}{dx^2} = -8\cos 2x + 16\cos 4x.$$

For a turning point $\dfrac{dy}{dx} = 0,$

$$\text{i.e. } -4\sin 2x + 4\sin 4x = 0,$$
$$\text{i.e. } \sin 4x - \sin 2x = 0,$$
$$\text{i.e. } 2\cos 3x \sin x = 0,$$
$$\therefore \cos 3x = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$\text{or } \sin x = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1), $\quad 3x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2},$ etc.

From (2) $\qquad x = 0, \pi, 2\pi,$ etc.

Therefore in the range $0 \leqslant x \leqslant \pi$, there are turning points if $x = 0$, $\frac{1}{6}\pi$, $\frac{1}{2}\pi$, $\frac{5}{6}\pi$, π.

When $x = 0$ or π, $\dfrac{d^2y}{dx^2} = -8 + 16 = +8$, and $y = 2 - 1 = 1$.

When $x = \frac{1}{2}\pi$, $\dfrac{d^2y}{dx^2} = +8 + 16 = +24$, and $y = -2 - 1 = -3$.

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When $x = \frac{1}{6}\pi$, $\dfrac{d^2y}{dx^2} = -4 - 8 = -12$, and $y = 1 + \frac{1}{2} = \frac{3}{2}$.

When $x = \frac{5}{6}\pi$, $\dfrac{d^2y}{dx^2} = +4 - 8 = -4$, and $y = 1 + \frac{1}{2} = \frac{3}{2}$.

Therefore y has a maximum value of $\frac{3}{2}$ and minimum values of 1 and -3.

From the data obtained the sketch opposite is drawn.

EXAMPLE (L.U.). An open cylindrical can is to be made to contain one litre (1,000ccs.). If the radius be r cm., find the area of metal required and calculate for what value of r this is a minimum.

Prove that the minimum area is about 440 square cm.

Let h cm. be the height of the can.

Then, the volume $= \pi r^2 h = 1{,}000$,

$$\therefore h = \frac{1{,}000}{\pi r^2} \quad \ldots\ldots\ldots\ldots(1).$$

If A square cm. be the total surface area,

$$A = 2\pi rh + \pi r^2 \quad \ldots\ldots\ldots\ldots(2).$$

Using (1) in (2), $$A = \frac{2{,}000}{r} + \pi r^2 \quad \ldots\ldots\ldots\ldots(3).$$

From (3), $$\frac{dA}{dr} = -\frac{2{,}000}{r^2} + 2\pi r,$$

and $$\frac{d^2A}{dr^2} = \frac{4{,}000}{r^3} + 2\pi.$$

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Now A has a turning value when $dA/dr = 0$, i.e. when,

$$-\frac{2{,}000}{r^2} + 2\pi r = 0$$

$$\text{i.e. } r^3 = \frac{1{,}000}{\pi}, \quad \therefore r = \sqrt[3]{\frac{1{,}000}{\pi}} = \frac{10}{\sqrt[3]{\pi}} = 6{\cdot}83 \text{ cm.}$$

When $r^3 = \dfrac{1{,}000}{\pi}$,

$$\frac{d^2A}{dr^2} = 4\pi + 2\pi = +6\pi,$$

therefore for a minimum value of A, $r = 6{\cdot}83$ cm.

From (3), $$A = \frac{2{,}000 + \pi r^3}{r},$$

therefore when $r^3 = \dfrac{1{,}000}{\pi}$,

$$A = \frac{2{,}000 + 1{,}000}{\sqrt[3]{(1{,}000/\pi)}} = \frac{3{,}000}{10/\sqrt[3]{\pi}}$$
$$= 300\sqrt[3]{\pi} = 300 \times 1{\cdot}46459$$
$$= 439{\cdot}38 \text{ square cm.}$$

i.e. the minimum area is approximately 440 square cm.

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EXAMPLE. An open tank is to have a horizontal square base and vertical sides. Its volume is to be 60 cubic yards. The cost of lining the base is p/– per square yard, and the cost of lining the sides is q/– per square yard, where p and q are constants.

Prove that, when the dimensions of the tank are such that the total cost of lining it is a minimum, the cost of lining the sides will be double the cost of lining the base,

Let the square base be of side x yards and its height be y yards.

The area of the base is x^2 square yards and the cost of lining it is px^2 shillings.

The total area of the vertical sides is $4xy$ square yards and the cost of lining these sides is $4qxy$ shillings.

The total cost (C/–) of lining the vessel is given by

$$C = px^2 + 4qxy \qquad \ldots\ldots(1).$$

But the volume = 60 cubic yards, therefore $yx^2 = 60$, from which

$$y = \frac{60}{x^2}.$$

Using this in (1),

$$C = px^2 + \frac{240q}{x} = px^2 + 240qx^{-1},$$

$$\therefore \frac{dC}{dx} = 2px - 240qx^{-2}.$$

For a maximum or a minimum value of C, $\frac{dC}{dx} = 0$,

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$$\text{i.e. } 2px - \frac{240q}{x^2} = 0, \quad \left(\begin{matrix}C \text{ maximum from} \\ \text{practical considerations}\end{matrix}\right)$$

$$\text{i.e. } 2px^3 = 240q,$$

$$\therefore 2px^2 = \frac{240q}{x}.$$

But $240q/x$ shillings = cost of lining the sides and px^2 shillings = cost of lining the base.

Hence, the cost of lining the sides = twice cost of lining the base.

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EXAMPLE (L.U.). A point moves in a straight line from a fixed point O through a distance s in time t, the value of s being given by

$$s = \frac{2at}{1 + t^2}.$$

Show that the point moves through a distance a from O and then returns towards O, and that the maximum speed towards O is $\frac{1}{4}a$.

$$s = \frac{2at}{1 + t^2}$$

$$\therefore \text{speed } v = \frac{ds}{dt} = \frac{2a(1 + t^2) - 2at(2t)}{(1 + t^2)^2}$$

(Differential of a quotient)

$$= \frac{2a(1 - t^2)}{(1 + t^2)^2} \qquad \ldots\ldots (1)$$

Also, the acceleration $f = \frac{dv}{dt} = \frac{d}{dt}\left\{\frac{2a(1-t^2)}{(1+t^2)^2}\right\}$

$$= \left\{\frac{-4at(1+t^2)^2 - 2a(1-t^2)\times 4t(1+t^2)}{(1+t^2)^4}\right\}$$

$$= -4at\left\{\frac{(1+t^2)+2(1-t^2)}{(1+t^2)^3}\right\}$$

$$= -4at\left\{\frac{3-t^2}{(1+t^2)^3}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The point comes to a stop when the velocity v is zero, and, from equation (1), this occurs when

$$\frac{2a(1-t^2)}{(1+t^2)^2} = 0, \quad \text{i.e. when } (1-t^2) = 0$$
$$\text{i.e. when } t = +1. \quad (t \text{ cannot be negative}).$$

When $t = 1$, the acceleration $f = -4a[\frac{2}{8}] = -a$, i.e. f is negative taking a positive, therefore the point moves towards O.

When $t = 1$, $s = \frac{1}{2}\times 2a = a$, therefore the point moves a distance a from O and then commences to move towards O.

From (2), the speed is a maximum or a minimum when

$$\frac{dv}{dt} = 0,$$

$$\text{i.e. when } -4at\left\{\frac{3-t^2}{(1+t^2)^3}\right\} = 0,$$

$$\text{i.e. when } t[3-t^2] = 0,$$

$$\text{i.e. when } t = 0 \text{ or } t^2 = 3.$$

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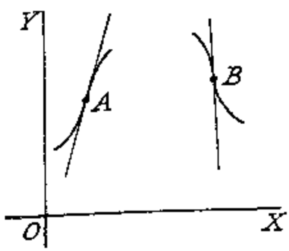
Now $t = 0$ clearly gives a minimum speed since the particle starts with zero speed. Therefore $t^2 = 3$ gives the maximum speed.

When $t^2 = 3$, $\quad v = 2a\frac{(1-3)}{(1+3)^2} = \frac{-4a}{16} = \frac{-a}{4}$.

Hence, the maximum speed is numerically $\frac{1}{4}a$.

Points of Inflexion. These are points at which the tangent crosses the curve as shown in the diagram at points A and B, i.e. the portions of the curve on either side of the point of inflexion lie on opposite sides of the tangent at the point.



Mathematically, a point of inflexion on a curve is a point at which the tangent cuts the curve in *three* coincident points.

From the diagram, it can be seen that the slope of the curve immediately to the left of A is increasing as x increases, and immediately to the right of A it is decreasing as x increases. Hence, dy/dx must have a maximum value at A, and therefore

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$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0 \text{ at } A, \quad \text{i.e. } \frac{d^2y}{dx^2} = 0 \text{ at } A$$

and also $\dfrac{d^2}{dx^2}\left(\dfrac{dy}{dx}\right) = -\text{ve at } A$, i.e. $\dfrac{d^3y}{dx^3}$ is negative at A.

By similar reasoning, there is a minimum value of the slope at B, and therefore $\dfrac{d^2y}{dx^2} = 0$ at B and $\dfrac{d^3y}{dx^3}$ is positive at B.

Hence for any point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and $\dfrac{d^3y}{dx^3}$ is not zero.

Example. Find the points of inflexion on the curve

$$y = 6x^5 - 5x^3 + 2.$$

From the given equation $\dfrac{dy}{dx} = 30x^4 - 15x^2$

$$\frac{d^2y}{dx^2} = 120x^3 - 30x,$$

$$\frac{d^3y}{dx^3} = 360x^2 - 30$$

For points of inflexion $\dfrac{d^2y}{dx^2} = 0$

i.e. $120x^3 - 30x = 0 \quad \therefore \quad x(4x^2 - 1) = 0$

www.dbraulibrary.org.in $\therefore x = 0$, or $\pm\frac{1}{2}$.

Using these in the value of

$$\frac{d^3y}{dx^3},$$

in no case is zero obtained and therefore these values of x give true points of inflexion.

(Note. It is possible to have a maximum or a minimum point when $\dfrac{d^2y}{dx^2} = 0$ if $\dfrac{dy}{dx}$ is also zero.)

When $x = 0$, $y = 2$.
When $x = \frac{1}{2}$, $y = \frac{6}{32} - \frac{5}{8} + 2 = \frac{25}{16}$.
When $x = -\frac{1}{2}$, $y = -\frac{6}{32} + \frac{5}{8} + 2 = \frac{39}{16}$.
Hence, the points of inflexion are $(0, 2)$, $(-\frac{1}{2}, \frac{39}{16})$, $(\frac{1}{2}, \frac{25}{16})$.

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EXAMPLES X

1. Differentiate x^3 with respect to x from first principles.

Draw the curve whose equation is $y = x^2(3 - x)$ and find the equation of the tangent at the point $(-1, 4)$. Show that this tangent meets the curve again at the point $(5, -50)$.

2. Prove, from first principles, that the differential coefficient of $\sin 2x$ with respect to x is $2 \cos 2x$.

If $y = \frac{1}{3}x^3 + 2x + 4x \sin x + 4 \cos x + \sin 2x$, prove that

$$\frac{dy}{dx} = (x + 2 \cos x)^2,$$

and, hence, that y increases with x for all real values of x.

3. (i) Find, from first principles, the differential coefficient of $\sqrt{x}$ with respect to x.

(ii) The distance s moved in a straight line by a particle in time t is given by $s = at^2 + bt + c$, where a, b, c are constants.

If v be the velocity of the particle after time t, show that $4a(s - c) = v^2 - b^2$.

4. Sketch the curve $ay^2 = x^3$, and show that the equation of the tangent at the point $P(a, a)$ is $3x - 2y - a = 0$.

The tangent at P meets the curve again at Q. Find the length of PQ.

5. Establish the formula for the differentiation of the quotient u/v of two functions of x with respect to x.

Differentiate (i) $\dfrac{x}{\tan x}$, (ii) $\left(\dfrac{1 + 2x}{1 + x}\right)^2$, (iii) $\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

6. If u and v are functions of x, obtain the formula for the differential coefficient of the product uv with respect to x.

Differentiate with respect to x, $(x^2 - x)^6$, $\cos(2 - 3x)$, $x^2 \sin 2x$.

7. Show that the equation of the tangent to the curve $y = x^3$ at the point (a, a^3) is $3a^2x - y - 2a^3 = 0$.

The tangent at P meets the x-axis at S, and O is the origin. Prove that $SO = 2SQ$, where Q is the foot of the ordinate of P.

8. Show that $(a - b)$ is a factor of $a^n - b^n$, where n is a positive integer, and write down the other factor. www.dbraulibrary.org.in

Hence prove that

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1},$$

when n is a positive integer.

Find the derivatives of: (i) $(x + 1/x) \cos x$, (ii) $\dfrac{a \sin x + b \cos x}{a \cos x - b \sin x}$.

Show that, as x increases, the value of the function (ii) always increases.

9. If n be a positive integer, prove that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

and deduce the differential coefficient of x^{-n}.

If $y = Ax^m + Bx^{-n}$, where A and B are constants, find values of m and n, both positive integers, so that

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 12y.$$

10. Differentiate $1/(x + a)$ with respect to x from first principles.

Find the equations of the tangents to the curve $y = 1/(x - 1) - 4/(x - 2)$ at the points where it crosses the co-ordinate axes.

Show that y increases as x increases for negative values of x.

11. (i) Differentiate $(x^4 - x^3)$ with respect to x from first principles.

(ii) Find the differential coefficients with respect to x of (a) $(1 - 3x^2)^5$, (b) $\sin(3x + \frac{3}{7}\pi)$.

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(iii) Find the x co-ordinates of the points on the curve

$$y = x^3 - 6x^2 + 9x + 1$$

at which the tangent is parallel to $y = 24x$.

12. Find, from first principles, the derivative of $\sin ax$ with respect to x. If $y = \frac{1}{2}x + \frac{1}{3}(4x^3) + 4\sin x - 4x\cos x - \frac{1}{4}\sin 2x$, prove that

$$\frac{dy}{dx} = (2x + \sin x)^2,$$

and hence that y increases as x increases for all real values of x.

13. Write down the quotient when $x^m - a^m$ is divided by $(x - a)$, m being a positive integer, and use the result to find

$$\underset{x \to a}{\text{Lt}}(x^m - a^m)/(x - a)$$

Deduce the value of

$$\underset{x \to a}{\text{Lt}}(x^{-m} - a^{-m})/(x - a),$$

when m is a positive integer.

The curve whose equation is $y = Ax^2 + Bx + C$ passes through the point $(1, 1)$. At this point the tangent to the curve is inclined at $30°$ to the x-axis. When $x = 3$, the inclination of the tangent to the x-axis is $60°$. Find the values of A, B, and C.

14. (i) Find from first principles the differential coefficient of $\cos 2x$ with respect to x.

(ii) The displacement x at time t of a moving particle is given by

$$x = a\sin 2t + b\cos 2t.$$

If v be the speed at time t, prove that $v = 2\sqrt{(a^2 + b^2 - x^2)}$.

15. A function y is such that $y = ax^2$ for $x < 2$, and $y = b + 9x - 2ax^2$ for $x > 2$. If ax^2 and $b + 9x - 2ax^2$ have the same value for $x = 2$ and also the same gradient for $x = 2$, find the values of a and b, and give a rough graph of the function y for values of x between -2 and $+5$.

16. Find, from first principles,

$$\frac{d}{dx}(\sin x).$$

Show that the curves $y = 2\sin x - \sqrt{3}$ and $y = (3x^2)/2\pi - \frac{1}{6}\pi$ touch at the point $(\frac{1}{3}\pi, 0)$, and find the intercept on the y-axis between the common tangent and the common normal at their point of contact.

17. Find the limit as X tends to x of

$$\frac{X^{15} - x^{15}}{X^4 - x^4}.$$

Differentiate $x/(x^2 + 9)$; $\sin x - x\cos x$; $\frac{1}{3}x^3 + 3/x^3$; $(1 + x)^{-2}$; $\cos 2x + 2\cos\frac{1}{2}x$.

18. Find the limit as x tends to zero of $x\cot x$.

Differentiate $\cos^2 x$ and $(x^3 - 1)/(x + 1)$.

19. Prove that, if x be measured in radians,

$$\frac{d}{dx}(\sin x) = \cos x.$$

Find (without tables) the value of

$$\frac{2\sin(30° + \theta) - 1}{\sin\theta},$$

when the angle θ is one second.

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20. Prove that the gradient of the curve $y = (3x^2 - 2)/(2x)$ is always greater than 1·5.

Find the co-ordinates of the points of this curve at which the gradient is 5·5 and find where the tangent at one of these points intersects the x-axis.

21. Plot the curve $y = x^3 - 3x + 2$ between $x = -2$ and $x = +2$, using 1 inch for the unit along each axis of co-ordinates.

Calculate the points at which the tangents have gradient 3, and draw these tangents.

22. If $y = x^2(3 - x)$, tabulate the values of

$$y, \quad \frac{dy}{dx}, \quad \frac{d^2y}{dx^2}$$

when $x = 0, 1, 2, 3$.

Draw carefully the graph of the curve from $x = -1$ to $x = 3\frac{1}{2}$, noting the peculiarities (if any) which it presents at the four points named above. (Use 1 inch as unit for both axes.)

23. Find the gradient of the curve $y = 2x/(1 + x^2)$ at any point.

Prove that the tangents at the four points whose abscissae are $\pm x_1$, $\pm x_2$, are all parallel, provided that $(x_1^2 - 1)(x_2^2 - 1) = 4$.

24. Define the differential coefficient of a function $f(x)$ with respect to x.

Find, directly from your definition, the differential coefficients of (i) $x^{3/2}$, (ii) $\sin ax$.

If the derivative of $(x^2 - a^2)/(x^2 + a^2)$ is unity when $x = 1$, find the values of a.

25. Differentiate with respect to x: (i) $\sqrt[3]{x}$; (ii) $(5x^2 + 2)/(x^2 - x)$; (iii) $\sin^2 x \cos \frac{1}{2}x$.

26. Prove that the differential coefficient of x^n is nx^{n-1}, n being a positive integer.

Show that the tangent to $y = x^3 - 3p^2x + q$ is parallel to $y = 0$ when $x = \pm p$, and that these tangents meet the curve again where $x = \mp 2p$, p and q being constants.

27. Explain, with reference to a figure, how to determine by consideration of the sign of the differential coefficient, whether a function of x is increasing or decreasing as x increases.

Prove that $$\frac{d}{dx}(\sin x \sqrt{\cos 2x}) = \frac{\cos 3x}{\sqrt{\cos 2x}},$$

and deduce that, if x lies between 0° and 45°, $\sin x \sqrt{\cos 2x} \leqslant \frac{1}{4}\sqrt{2}$.

28. Find from *first principles* the differential coefficient of $1/x$ with respect to x.

Find the gradient of each of the two curves $y^2 = 4ax$, and $xy = 4a^2\sqrt{2}$ at their point of intersection.

29. Establish the formula for the differentiation with respect to x of a quotient u/v, where u and v are functions of x.

Differentiate with respect to x (i) $(x^2 - 3x + 2)/(2x + 1)$, (ii) $\sin 3x/\sqrt{x}$.

30. Show, with reference to a diagram, that if

$$\frac{dy}{dx}$$

is positive, y increases as x increases.

Show that (i) $x - \sin x$, and (ii) $\sin x - x \cos x$ increase as x increases from 0 to $\frac{1}{2}\pi$. Deduce that, in this range, $\tan x > x > \sin x$.

Differentiate $\sin x/x$ with respect to x, and deduce that, if $0 < x < \frac{1}{2}\pi$, $1 > \sin x/x > 2/\pi$.

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31. Find the equation of the tangent to the curve $y = ax^3 + bx^2 + cx + d$ at the point (h, k).

Find the values of a, b, c, d so that the tangents to the curve at the points $(2, 2)$ and $(-2, -4)$ may be parallel to the axis of x.

Determine also the co-ordinates of the point of intersection of this curve and the tangent to it at the point $(2, 2)$.

32. (i) Find from first principles the differential coefficient with respect to x of $x \cos x$.

(ii) Prove that the function x^3 increases less rapidly than $6x^2 - 16x + 5$ in the interval $5 > x > -1$.

33. Explain, with reference to a diagram, the meaning of the statement that

$$\Delta y \text{ is approximately equal to } \frac{dy}{dx}\Delta x,$$

where y is a function of x and Δy and Δx are corresponding small increments.

Hence, find the error made in calculating the area of a triangle in which two of the sides are accurately measured as 18 feet and 25 feet, while the included angle is measured as 60°, but is $\frac{1}{2}°$ wrong.

34. Find the equation of the tangent to the curve $y = x^3$ at the point $P(t, t^3)$.

Prove that this tangent cuts the curve again at the point $Q(-2t, -8t^3)$ and find the locus of the middle point of PQ.

35. Find, from first principles, the differential coefficient of x^n with respect to x, n being a positive integer.

Find the range of values of x in which the rate of increase in the function $(x - 1)(x - 2)(x - 3)$ is less than that of the function $3x^2 - 4x + 10$.

36. Define the differential coefficient (or derivative) of a function of x with respect to x, and employ your definition to obtain the differential coefficients of $(1 + x)^{-2}$, and of $\sin 3x$ with respect to x.

If $f(x) = 1/x^2$, show that the ratio of

$$\frac{d}{dx}[f(x)] \text{ to } \frac{1}{2h}[f(x + h) - f(x - h)] \text{ is } \left(1 - \frac{h^2}{x^2}\right)^2,$$

and prove that, when $x = 2$, $h = 0{\cdot}1$, this ratio differs from unity by 1/200 nearly.

37. The equation of a curve is of the form $y = ax + bx^2 + cx^3$. Find what values a, b, c must have in order that the gradients of the curve at the points whose abscissae are -1, 2, 4 may be $\tan^{-1} 8$, $\tan^{-1}(-1)$, $\tan^{-1} 3$ respectively.

Find the value of x for which the gradient is zero, and draw a rough graph of the curve.

38. Find the equation of the tangent to the curve $y = x - x^3$ at the point $P(x_1, y_1)$.

Prove that, if the tangent cuts the curve at Q, one point of trisection of PQ lies on the y-axis, and find the equation of the locus of the other point of trisection.

39. Find the gradient at any point of the curve $y = (3x - x^3)/(1 - 3x^2)$. Show that there are only two points on the curve at which the tangent is inclined at 45° to the axes.

40. Show that, for any value of t, the point $(4t^2, 4t^3)$ lies on the curve $4y^2 = x^3$.

Show that the gradient at the point for which $t = t_1$ is $\frac{3}{2}t_1$, and find the equation of the tangent at this point. Prove that this tangent meets the curve again where $t = -\frac{1}{2}t_1$.

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41. Find the equation of the tangent at any point (x_1, y_1) of the curve $y = x^4 - 2a^2x^2 + b^3x$.

Prove that the tangent at the point where $x_1 = a$ is also the tangent at a second point of the curve and find its gradient.

Sketch the curve assuming $a = b = 1$.

42. Prove that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right).$$

If $y = (A \cos mx + B \sin mx)/x^n$, where A, B, m, n are constants, show that

$$\frac{d^2y}{dx^2} + 2\frac{n}{x}\frac{dy}{dx} + \left\{\frac{n(n-1)}{x^2} + m^2\right\} y = 0$$

43. Show that, when x is measured in radians, $\sin x < x < \tan x$, if $0 < x < \frac{1}{2}\pi$.

Find the derivative of $\sin x/x$ with regard to x, and deduce that $\sin x > 2x/\pi$ when $0 < x < \frac{1}{2}\pi$.

44. (i) Differentiate with respect to x: (a) $3x^4 - 2x^{-\frac{3}{2}} + 3$, (b) $x^2 \cos 5x$.
(ii) If $y = ax^2 + bx^{-\frac{1}{2}}$, prove that

$$2x^2\frac{d^2y}{dx^2} = x\frac{dy}{dx} + 2y.$$

45. If u and v are functions of x prove that

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Find the differential coefficients with respect to x of

$$\frac{x^2 - 2x}{(x+2)^2} \quad \text{and} \quad \sin^2(3 - 4x).$$

46. Show that the tangents to the curve $8y = (x - 2)^2$ at the points $(4, \frac{1}{2})$, $(-6, 8)$ intersect at right angles on the line $y = 2$.

If the point of intersection is T, and S is the point $(2, 2)$, show that TS is perpendicular to the line joining the points of contact of the tangents.

47. A piece of wire 10 feet long is divided into two portions, one being bent to form a square and the other bent to form a circle. Show that the sum of the areas of the square and circle is least when the side of the square is equal to the diameter of the circle.

48. A straight line AB has two ends on two fixed perpendicular lines OX, OY, and passes through a fixed point C, whose distances from the fixed lines are a and b.

Find the position of AB which makes $\triangle AOB$ of minimum area, and calculate that minimum area.

49. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of the triangle is $4a^2 \sin\theta \cos^2\theta$, and hence that the area is a maximum when the triangle is equilateral.

50. A dispatch rider is in open country at a distance six miles from the nearest point P of a straight road. He wishes to proceed as quickly as possible to a point Q on the road twenty miles from P.

If his maximum speed across country is 40 m.p.h., and along the road 50 m.p.h., find at what distance from P he should strike the road.

51. Determine a, b, c, d, so that the curve $y = ax^3 + bx^2 + cx + d$, may pass through the points $(1, 0)$, $(3, -4)$, $(4, 0)$ and have a horizontal tangent at the first of them.

Examine the turning-points to see whether they are maxima or minima, find and any points of inflexion on the curve.

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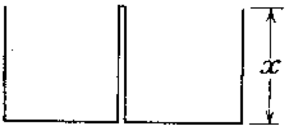
52. Find, from first principles, the differential coefficient of $1/(2 - x)$ with respect to x.

Determine the maximum and minimum values of $x + 9/(2 + x)$, and sketch a rough graph of this function.



53. A thin rectangular plate of breadth $6a$ is bent to form a double symmetrical right-angled gutter as shown in the sketch. If the depth of the gutter be x, find the capacity per unit length, and determine x for this to be a maximum.

54. A trapezium $ABCD$ has the sides AD and BC parallel, and $AB = BC = CD = a$. If the angle DAB is θ, show that the area $ABCD$ is $\frac{1}{2}a^2(2\sin\theta + \sin 2\theta)$.

Find the value of θ for which this area is a maximum.

55. Find the maximum and minimum values of the expression $3x^5 - 5x^3 + 2$, stating carefully any rules you use.

Explain why the value $x = 0$ does not give a maximum or a minimum value, and illustrate by giving a rough sketch of the graph of the expression between $x = -2$ and $x = +2$.

56. (i) Prove that
$$\frac{d}{dx}(\tan x) = \sec^2 x,$$
and that
$$\frac{d}{dx}\left(\frac{\tan x}{1 + \sin x}\right) = \frac{1 - \sin x + \sin^2 x}{\cos^2 x(1 + \sin x)}.$$

(ii) If a small object be placed u inches in front of a thin lens of focal length f inches, the image is v inches behind the lens, where $1/v + 1/u = 1/f$.

Express $(u + v)$ in terms of v and f, and hence show that the minimum distance between image and object is $4f$ inches.

57. A line AB drawn through the point (1, 8) has one end A on the x-axis and the other end B on the y-axis. If AB makes an acute angle θ with the x-axis, prove that the length of AB is $\sec\theta + 8\operatorname{cosec}\theta$, and that AB is least when $\tan\theta = 2$.

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58. Find the co-ordinates of the maximum and minimum points of the curve $y = 4 + 12x - 3x^2 - 2x^3$.

Sketch the curve and calculate its slope at the point for which $x = 2$.

59. Explain how the differential coefficient of a function may be used to find the maximum and minimum values of the function.

The total cost of a ship per hour while on a voyage is £$(4{\cdot}5 + v^3/1{,}250)$, where v is the speed of the ship in miles per hour. Find the total cost of a voyage of 2,000 miles in terms of v and find the value of v which makes this cost the least.

60. If u and v are functions of x, find an expression for the derivative of u/v in terms of the derivatives of u and v.

Prove that the greatest and least values which can be assumed by
$$\frac{3 + 2x + x^2}{1 + 2x + 3x^2},$$
for real values of x, are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

61. Find the maximum and minimum values of the expression
$$2x^3 + (a - 2x)^3,$$
explaining carefully the method used to discriminate between them, and assuming a to be a positive constant.

Illustrate by giving a graph of the expression when $a = 3$.

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62. Explain how to find the maximum and minimum values of a function of a variable quantity x.

A right circular cylinder has a given volume V. Express the total surface area S in terms of V, and the radius of the cylinder, and show that S is least when the length and diameter of the cylinder are equal.

Find this least area when $V = 64$ cubic feet.

63. A thin wire, 16 inches long, is cut into two pieces, one of which is bent into a circle of radius r, and the other into a right-angled isosceles triangle. Express the sum of the areas in terms of r, and find the value of r when this sum is a minimum.

64. Find, *from first principles*, the differential coefficient of x^{-2} with respect to x.

Calculate the maximum and minimum values of $x^3 - 15x + 7 - 12x^{-1}$ and sketch the curve $y = x^3 - 15x + 7 - 12x^{-1}$.

65. Find the maximum and minimum values of the expression

$$(x + 3a)(9a^2 - x^2),$$

assuming a to be positive.

Explain carefully how you distinguish between these two values. Draw a graph of the expression, giving sufficient to show the maximum and minimum points, and the general character of the curve.

66. Find, from first principles, the differential coefficient of $\cos 2x$ with respect to x.

Determine the maximum and minimum values of $2 - \sin x + \frac{1}{2}\cos 2x$ for values of x from 0 to 2π, and distinguish between them.

67. (i) If u and v are functions of x, prove that

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

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(ii) Differentiate $\log_e(\tan 2x)$ with respect to x, giving the answer in the simplest form.

(iii) If $y = xe^{x^2-3x}$, find $\frac{dy}{dx}$, and hence prove that y decreases as x increases from $\frac{1}{2}$ to 1.

68. (*a*) If $y = \frac{1}{2}(e^x + e^{-x})$, show that $\left(\frac{dy}{dx}\right)^2 = y^2 - 1$.

(*b*) Find $\frac{dy}{dx}$ if $y = \log_e(x^3 - 2x^2)$.

(*c*) If $y = xe^{x-x^2}$, prove that y increases as x increases from $-\frac{1}{2}$ to $+1$.

69. (*a*) Find the derivatives with respect to x of the functions

(i) $(15x^2 + 12x + 8)(1 - x)^{3/2}$, (ii) $(2x^2 + 10x + 11)e^{-2x}$.

and state in each case the value of x for which the derivative vanishes.

(*b*) The abscissa x of a point P moving along the x-axis is given in terms of the time t by the equation $x = 16(e^{-t} - e^{-2t})$. Determine the acceleration when P reaches its maximum distance from O, and the maximum speed in the subsequent motion.

70. (*a*) If $y = (A + Bx)e^{-2x}$, prove that

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

(*b*) If $y = x + e^x$, find $\frac{d^2x}{dy^2}$ in terms of x.

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71. A line drawn parallel to the y-axis meets the curve $4y = e^{2x} + e^{-2x}$ at P and the x-axis at N. Show that the projection of PN on the normal to the curve at P is of constant length.

72. (*a*) Find the derivative with respect to x of

$$6(x^3 + 1) \log_e (x + 1) - 2x^3 + 3x^2 - 6x.$$

(*b*) The abscissa x of a point P moving along the x-axis is given in terms of the time t by the equation $x = e^{-\frac{1}{2}t}(t - 3)\, t$, where $t \geqslant 0$.

Show that the ratio of the extreme distances of P from the origin is $e^{2\cdot 5} : 9$ and the accelerations towards the origin in these extreme positions are in the ratio $e^{2\cdot 5} : 1$.

73. (*a*) Differentiate with respect to x

$$\text{(i)}\ \frac{x + 2}{\sqrt{(x^2 + 4x)}}, \quad \text{(ii)}\ \frac{x + 2}{x - 2}e^{-x}.$$

(*b*) If $xy = h - 9c^2x + x^3 + k \log_e x$, where h, k, c are constants determine the values of x for which

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0.$$

74. (*a*) Show that the tangent to the parabola $y^2 = 4ax$ at the point (h, k) cuts the axis of x at $(-h, 0)$.

(*b*) The speed of signals along a certain cable is known to be directly proportional to $x^2 \log_e 1/x$, where x is a positive number less than unity.

Find the value of x corresponding to the greatest speed.

75. (*a*) Differentiate with respect to x (i) $x^3/(2 + x)$ and (ii) $(1 + \sin 2x)^2$.

(*b*) Show that $y = (A + Bx)e^{-x} + x^2 - 4x + 6$, where A and B are constants, satisfies the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2$$

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Find A and B given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

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CHAPTER XI

Calculus

Integration, Areas, Volumes, etc.

Integration. Integration is the converse of differentiation. Thus, if

$$\frac{d}{dx}[f(x)] = \varphi(x),$$

then $f(x)$ is said to be the integral of $\varphi(x)$ with respect to x.

This statement is written in the following manner,

$$\int \varphi(x)dx = f(x),$$

where the symbol $\int$ is an elongated S (from the word 'summation'), and represents 'the integral of', whilst dx represents 'with respect to x'. The quantity $\varphi(x)$ is known as the *integrand*.

Thus, since $\frac{d}{dx}(x^3) = 3x^2$, it follows that $\int 3x^2 dx = x^3$.

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Also, since $\frac{d}{dx}(x) = 1$, it follows that $\int 1 \,.\, dx = x$.

From differentiation it is known that, if C be *any* constant,

$$\frac{d}{dx}[f(x) + C] = \frac{d}{dx}[f(x)] = \varphi(x) \text{ (say)}.$$

Hence, a more general result for $\int \varphi(x)dx$ is $f(x) + C$, and the result $f(x)$ is merely a special value in which $C = 0$.

When the result is of the form $f(x) + C$, where C is *any* constant, it is known as the *general integral* and C is known as the *constant of integration*.

When the result does not contain a constant of integration it is known as the *indefinite integral*.

Unless the indefinite integral is specifically required, the general integral will always be used.

A third type of integral, known as the *definite integral*, will be dealt with later.

NOTE. C will be taken as the constant of integration throughout the remainder of this chapter.

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Theorems on Integration. From the definition of integration as the converse of differentiation, the following two theorems are valid.

(i) $\int(u + v - w + \ldots)dx = \int u\,dx + \int v\,dx - \int w\,dx + \ldots,$

where u, v, w, etc., are functions of x.

(ii) $\int af(x)dx = a\int f(x)dx,$ where a is a constant.

Theorem. *To find the value of $\int x^n\,dx$, where n is a constant and not equal to -1.*

Now, $$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n$$

(except if $n = -1$ when the result is zero)

$$\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Thus the rule for integrating x^n with respect to x is, '*increase* the index of x by unity, divide by the *new* power, and then add C the constant of integration. (The rule for differentiating x^n with respect to x is, '*decrease* the power by unity and multiply by the *old* power'.)

Combining this theorem and theorem (ii),

$$\int ax^n\,dx = a\frac{x^{n+1}}{n+1} + C.$$

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NOTE. As in differentiation, every term of the integrand after expansion, etc., must be converted to the form ax^n before integration can take place for algebraic functions which have a finite number of terms in their expansions.

EXAMPLE. Find the following integrals:

(i) $\int x^5dx,$ (ii) $\int \frac{2}{s^{2\cdot4}}\,ds,$

(iii) $\int \frac{5}{\sqrt[3]{2t}}dt,$ (iv) $\int x^2(8 - x^{-4})dx,$

(v) $\int (x^2 + 1/x^2)^2dx,$ (vi) $\int \frac{z^8 - 2z^4}{z^6}dz,$

(vii) $\int \frac{y^3 + 1}{y + 1}dy.$

Using the previous theorem the following results are obtained.

(i) $\int x^5dx = \frac{x^{5+1}}{5+1} + C = \frac{x^6}{6} + C.$

(ii) $\int \frac{2}{s^{2\cdot4}}\,ds = 2\int s^{-2\cdot4}ds = \frac{2s^{-2\cdot4+1}}{-2\cdot4+1} + C = \frac{2s^{-1\cdot4}}{-1\cdot4} + C.$

$= C - \frac{10}{7s^{1\cdot4}}.$

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(iii) $\int \frac{5}{\sqrt[3]{2t}}\,dt = \frac{5}{\sqrt[3]{2}}\int t^{-\frac{1}{3}}dt = \frac{5}{\sqrt[3]{2}}\,\frac{t^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{15}{2}\sqrt[3]{\frac{t^2}{2}} + C.$

(iv) $\int x^2(8 - x^{-4})dx = \int (8x^2 - x^{-2})dx = \int 8x^2dx - \int x^{-2}dx$

$$= 8\frac{x^3}{3} - \frac{(x^{-1})}{-1} + C = \frac{8x^3}{3} + x^{-1} + C.$$

(v) $\int (x^2 + 1/x^2)^2dx = \int (x^4 + 2 + x^{-4})dx$

$$= \frac{x^5}{5} + 2x + \frac{x^{-3}}{-3} + C$$

$$= \frac{x^5}{5} + 2x - \frac{1}{3x^3} + C.$$

(vi) $\int \frac{z^8 - 2z^4}{z^6}\,dz = \int (z^2 - 2z^{-2})dz = \frac{z^3}{3} - 2\frac{z^{-1}}{-1} + C$

$$= \frac{z^3}{3} + \frac{2}{z} + C.$$

(vii) $\int \frac{y^3 + 1}{y + 1}\,dy = \int \frac{(y + 1)(y^2 - y + 1)}{(y + 1)}\,dy = \int (y^2 - y + 1)dy$

$$= \frac{y^3}{3} - \frac{y^2}{2} + y + C.$$

NOTE. All integrands should be simplified as far as possible before integrating.

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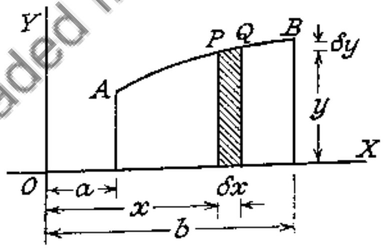
Area under a Curve and Definite Integrals. www.dbraulibrary.org.in In the diagram, AB is the portion of the curve $y = f(x)$ between $x = a$ and $x = b$.



The area between the curve AB, the axis of x and the ordinates at A and B is known as the *area under the curve AB*.

P and Q are two adjacent (*very* close together) points (x, y), $(x + \delta x,\ y + \delta y)$ on the curve AB, and S is taken to denote the area under the curve AP, whilst S_1 is used for the area under the curve AB.

The *element* of area, shaded in the diagram, is a small increase in the area S due to a small increase of δx in x, and is therefore denoted by δS.

Now, the shaded area is approximately a rectangle of length y and width δx, whose area is $y\,\delta x$, and as $\delta x \to 0$ this element of area approaches closer and closer to $y\,\delta x$.

Thus, $\delta S \to y\,\delta x$, as $\delta x \to 0$

$$\text{i.e. } \frac{\delta S}{\delta x} \to y, \text{ as } \delta x \to 0$$

$$\text{i.e. } \underset{\delta x \to 0}{\text{Lt}} \frac{\delta S}{\delta x} = y, \quad \text{i.e. } \frac{dS}{dx} = y.$$

Hence, from the definition of integration,

$$S = \int y\,dx.$$

Thus, the area under a curve can always be represented by an integral of the form

$$\int y\,dx,$$

where $y = f(x)$.

Now $$\int y\,dx = \int f(x)dx = \varphi(x) + C,$$

where $\varphi(x)$ is the indefinite integral of $f(x)$ with respect to x. Therefore $S = \varphi(x) + C$.

Since the area S commences from the ordinate at A, it follows that $S = 0$ when $x = a$.

Therefore $$0 = \varphi(a) + C,$$

$$\therefore C = -\varphi(a).$$

Hence, $$S = \varphi(x) - \varphi(a)$$

Now $S = S_1$ when $x = b$. Therefore $S_1 = \varphi(b) - \varphi(a)$.

Thus the area under the curve AB between the limits $x = a$ and $x = b$ is equal to $\varphi(b) - \varphi(a)$, where $\varphi(x)$ is the indefinite integral of $f(x)$ with respect to x.

In order to facilitate the working in any problem, this result is written

$$\Big[\varphi(x)\Big]_a^b,$$

and to show the limits for the area S_1 in the corresponding integral, it is written

$$\int_a^b f(x)dx,$$

where b is the *upper limit* and a the *lower limit* of the integral.

This integral is known as a *definite integral*, and the following statement shows the setting out for evaluating a definite integral in any particular case of the area under a curve:

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area under the curve $y = f(x)$, between $x = a$ and $x = b$

$$= \int_a^b y\,dx = \int_a^b f(x)dx = \Big[\varphi(x)\Big]_a^b = \varphi(b) - \varphi(a),$$

where $\varphi(x)$ is the indefinite integral of $f(x)$ with respect to x.

NOTE. (i) When the area under the curve considered is totally above the x-axis and $b > a$, all the ordinates are positive and δx is positive, and the resulting area will be positive ($y\,\delta x$ is positive).

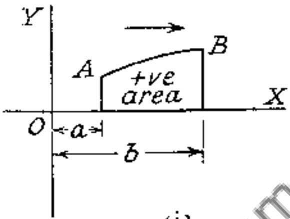
(ii) When the area under the curve is totally below OX and $b > a$, all the ordinates are negative, δx is positive (therefore $y\,\delta x$ is negative) and the resulting area will be negative.

(iii) If $b < a$ in (i), then δx is negative and the resulting area under the curve is negative.

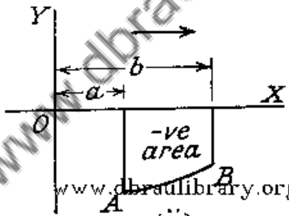
(iv) If $b < a$ in (ii), then δx is negative and the resulting area under the curve is positive.

(v) When a portion of the area is above and a portion below the x-axis, the area under the curve will be given by the algebraic sum (i.e. taking signs into account) of the areas above and below the x-axis.

These results are shown in the following diagrams.



(i)



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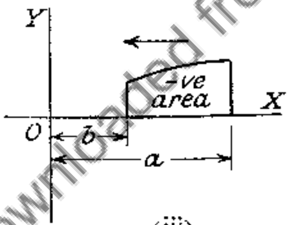
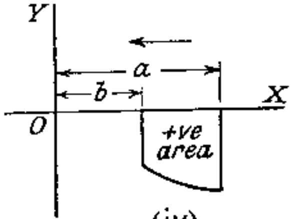
(ii)



(iii)



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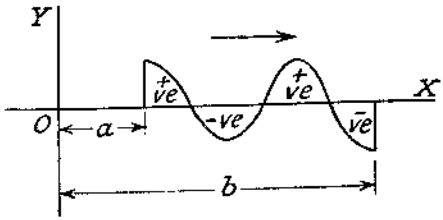

(v)

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EXAMPLE. Evaluate the following definite integrals:

(i) $\int_1^2 \frac{dx}{x^2}$, (ii) $\int_{-1}^{-2} (2 + 3t + 5t^2)dt$, (iii) $\int_0^2 (y - 1)^3 dy$

(iv) $\int_1^8 \frac{dx}{\sqrt[3]{x}}$, (v) $\int_1^2 \frac{y^2 - 1}{y^2}\, dy$.

(i) $$\int_1^2 \frac{dx}{x^2} = \int_1^2 x^{-2} \,.\, dx = \left[\frac{x^{-1}}{-1}\right]_1^2 = -\left[\frac{1}{x}\right]_1^2$$
$$= -[\tfrac{1}{2} - 1] = \tfrac{1}{2}.$$

(ii) $$\int_{-1}^{-2} (2 + 3t + 5t^2)dt = \left[2t + \tfrac{3}{2}t^2 + \tfrac{5}{3}t^3\right]_{-1}^{-2}$$
$$= (-4 + 6 - \tfrac{40}{3}) - (-2 + \tfrac{3}{2} - \tfrac{5}{3})$$
$$= -\tfrac{34}{3} + \tfrac{13}{6} = -\tfrac{55}{6}.$$

(iii) $$\int_0^2 (y - 1)^3 dy = \int_0^2 (y^3 - 3y^2 + 3y - 1)dy$$
$$= \left[\frac{y^4}{4} - \frac{3y^3}{3} + \frac{3y^2}{2} - y\right]_0^2$$
$$= (4 - 8 + 6 - 2) - (0) = 0.$$

(iv) $$\int_1^8 \frac{dx}{\sqrt[3]{x}} = \int_1^8 x^{-\frac{1}{3}}dx = \left[\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right]_1^8 = \tfrac{3}{2}\left[x^{\frac{2}{3}}\right]_1^8 = \tfrac{3}{2}[2^2 - 1^2]$$
$$= 9/2.$$

(v) $$\int_1^2 \frac{y^2 - 1}{y^2}\, dy = \int_1^2 (1 - y^{-2})dy = \left[y - \frac{y^{-1}}{-1}\right]_1^2$$

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$$= \left[y + \frac{1}{y}\right]_1^2 = (2 + \tfrac{1}{2}) - (1 + 1) = 1/2.$$

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EXAMPLE (L.U.). The gradient of a curve passing through the point (3, 1) is given by

$$\frac{dy}{dx} = x^2 - 4x + 3.$$

Find the equation of the curve, and the area enclosed by the curve, the maximum and minimum ordinates, and the x-axis.

$$\frac{dy}{dx} = x^2 - 4x + 3 \ldots\ldots\ldots\ldots\ldots (1).$$

From (1), $$y = \int (x^2 - 4x + 3)dx$$
$$= \tfrac{1}{3}x^3 - 2x^2 + 3x + C.$$

Since the curve passes through the point (3, 1), $y = 1$ when $x = 3$,

$$\therefore\ 1 = 9 - 18 + 9 + C$$
$$\therefore\ C = 1$$

therefore equation of the curve is

$$y = \frac{x^3}{3} - 2x^2 + 3x + 1.$$

Now $\frac{dy}{dx} = 0$ for turning points.

Therefore from (1) for the turning points $x^2 - 4x + 3 = 0$,

$$\text{i.e. } (x - 1)(x - 3) = 0$$

$$\text{i.e. } x = 1 \text{ or } 3.$$

Thus, the values $x = 1$ and $x = 3$ will be the limits for the area

$$\int y\,dx$$

required.

Therefore required area

$$= \int_1^3 y\,dx = \int_1^3 (\tfrac{1}{3}x^3 - 2x^2 + 3x + 1)dx$$

$$= \left[\frac{x^4}{12} - \frac{2x^3}{3} + \frac{3x^2}{2} + x\right]_1^3$$

$$= [\tfrac{27}{4} - 18 + \tfrac{27}{2} + 3] - [\tfrac{1}{12} - \tfrac{2}{3} + \tfrac{3}{2} + 1]$$

$$= \tfrac{21}{4} - \tfrac{23}{12} = \tfrac{40}{12} = \tfrac{10}{3} = 3\tfrac{1}{3} \text{ square units.}$$

Theorem. *To find $\int(ax + b)^n\,dx$, where a, b, n are constants.*

Now $\frac{d}{dx}\left\{\frac{(ax + b)^{n+1}}{a(n + 1)}\right\} = (n + 1)a\frac{(ax + b)^n}{a(n + 1)} = (ax + b)^n$

$$\therefore \int(ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C.$$

EXAMPLE. Find the following integrals:

(i) $\int(2x - 1)^5 dx$ (ii) $\int\frac{1}{(1 - 2t)^3}dt$

(iii) $\int_0^4 \sqrt{(2t + 1)}dt$ (iv) $\int\frac{1}{\sqrt[3]{(1 - 4y)}}dy$

Using the previous theorem,

(i) $\int(2x - 1)^5 dx = \frac{(2x - 1)^6}{2 \times 6} + C = \frac{(2x - 1)^6}{12} + C.$

(ii) $\int\frac{1}{(1 - 2t)^3}dt = \int(1 - 2t)^{-3}dt = \frac{(1 - 2t)^{-2}}{(-2)(-2)} + C$

$$= \frac{(1 - 2t)^{-2}}{4} + C = \frac{1}{4(1 - 2t)^2} + C.$$

(iii) $\int_0^4 \sqrt{(2t + 1)}dt = \int_0^4 (2t + 1)^{\frac{1}{2}}dt = \left[\frac{(2t + 1)^{3/2}}{2 \times \frac{3}{2}}\right]_0^4$

$$= \tfrac{1}{3}\left[(2t + 1)^{3/2}\right]_0^4 = \tfrac{1}{3}[9^{3/2} - 1^{3/2}]$$

$$= \tfrac{1}{3}[27 - 1] = \tfrac{26}{3}.$$

(iv) $\int\frac{1}{\sqrt[3]{(1 - 4y)}}dy = \int(1 - 4y)^{-1/3}dy = \frac{(1 - 4y)^{2/3}}{(-4)(\frac{2}{3})} + C$

$$= C - \tfrac{3}{8}(1 - 4y)^{2/3} = C - \tfrac{3}{8}\sqrt[3]{(1 - 4y)^2}.$$

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Theorem. *To find* (i) $\int \cos(ax+b)dx$, (ii) $\int \sin(ax+b)dx$, *where a and b are constants and x is in* radians.

(i) Now

$$\frac{d}{dx}[(1/a)\sin(ax+b)] = 1/a \,.\, a\cos(ax+b) = \cos(ax+b)$$

$$\therefore \int \cos(ax+b)dx = (1/a)\sin(ax+b) + C.$$

In the particular case when $a = 1$ and $b = 0$

$$\int \cos x\, dx = \sin x + C.$$

(ii) $$\frac{d}{dx}[(-1/a)\cos(ax+b)] = (-1/a)[-a\sin(ax+b)]$$
$$= \sin(ax+b)$$

$$\therefore \int \sin(ax+b)dx = C - (1/a)\cos(ax+b).$$

When $a = 1$ and $b = 0$

$$\int \sin x\, dx = C - \cos x.$$

NOTE. When the integrand is the product of sines or cosines, or a product of a sine and a cosine, it is necessary to convert the product into an expression consisting of first degree trigonometric functions (sines or cosines) before integration can take place. To perform this conversion, use is made of the following trigonometric formulae:

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$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta), \quad \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$
$$\sin A\cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$
$$\cos A\cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)]$$
$$\sin A\sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta, \text{ from which } \sin^3\theta = \tfrac{1}{4}(3\sin\theta - \sin 3\theta)$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta, \text{ giving } \cos^3\theta = \tfrac{1}{4}(3\cos\theta + \cos 3\theta).$$

Theorem. *To find* $I_1 = \int \cos^2\theta\, d\theta$, *and* $I_2 = \int \sin^2\theta\, d\theta$.

$$\left.\begin{aligned} I_1 &= \int \tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{2}(\theta + \tfrac{1}{2}\sin 2\theta) + C \\ I_2 &= \int \tfrac{1}{2}(1-\cos 2\theta)d\theta = \tfrac{1}{2}(\theta - \tfrac{1}{2}\sin 2\theta) + C \end{aligned}\right\}.$$

Theorem. *To find* $I_1 = \int \sin px \cos qx\, dx$, $I_2 = \int \cos px \cos qx\, dx$, $I_3 = \int \sin px \sin qx\, dx$, *where p and q are constants.*

Now,

$$\sin px\cos qx = \tfrac{1}{2}[\sin(p+q)x + \sin(p-q)x],$$

$$\therefore I_1 = \tfrac{1}{2}\int [\sin(p+q)x + \sin(p-q)x]dx$$
$$= -\tfrac{1}{2}\left\{\frac{1}{p+q}\cos(p+q)x + \frac{1}{p-q}\cos(p-q)x\right\} + C.$$

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Also,

$$\cos px \cos qx = \tfrac{1}{2}[\cos(p+q)x + \cos(p-q)x],$$

$$\therefore\ I_2 = \tfrac{1}{2}\int[\cos(p+q)x + \cos(p-q)x]dx$$

$$= \tfrac{1}{2}\left\{\frac{1}{p+q}\sin(p+q)x + \frac{1}{p-q}\sin(p-q)x\right\} + C.$$

$$\sin px \sin qx = \tfrac{1}{2}[\cos(p-q)x - \cos(p+q)x],$$

$$\therefore\ I_3 = \tfrac{1}{2}\int[\cos(p-q)x - \cos(p+q)x]dx$$

$$= \tfrac{1}{2}\left\{\frac{1}{p-q}\sin(p-q)x - \frac{1}{p+q}\sin(p+q)x\right\} + C.$$

Theorem. *To find* $I_1 = \int \sin^3 x\, dx$, *and* $I_2 = \int \cos^3 x\, dx$.

Now,

$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x),$$

$$\therefore\ I_1 = \int \tfrac{1}{4}[3\sin x - \sin 3x]dx = \tfrac{1}{4}[-3\cos x + \tfrac{1}{3}\cos 3x] + C$$

$$= \tfrac{1}{12}[\cos 3x - 9\cos x] + C.$$

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Also,

$$\cos^3 x = \tfrac{1}{4}(\cos 3x + 3\cos x),$$

$$\therefore\ I_2 = \int \tfrac{1}{4}(\cos 3x + 3\cos x)dx = \tfrac{1}{4}(\tfrac{1}{3}\sin 3x + 3\sin x) + C$$

$$= \tfrac{1}{12}(\sin 3x + 9\sin x) + C.$$

EXAMPLE. Find the following indefinite integrals:

(i) $\int \sin 2x \cos 3x\, dx$, (ii) $\int 3\sin 5x \sin 3x\, dx$,

(iii) $\int \cos 4x \cos 2x\, dx$, (iv) $\int \sin^2 3x\, dx$.

(i) $$\int \sin 2x \cos 3x\, dx = \tfrac{1}{2}\int(\sin 5x - \sin x)dx = \tfrac{1}{2}(-\tfrac{1}{5}\cos 5x + \cos x)$$

$$= \tfrac{1}{10}(5\cos x - \cos 5x).$$

(ii) $$\int 3\sin 5x \sin 3x\, dx = \tfrac{3}{2}\int(\cos 2x - \cos 8x)dx$$

$$= \tfrac{3}{2}(\tfrac{1}{2}\sin 2x - \tfrac{1}{8}\sin 8x)$$

$$= \tfrac{3}{16}(4\sin 2x - \sin 8x).$$

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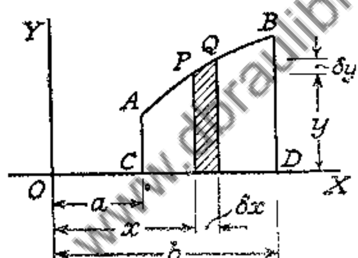
(iii) $\int \cos 4x \cos 2x\, dx = \frac{1}{2}\int (\cos 6x + \cos 2x)dx$

$$= \tfrac{1}{2}(\tfrac{1}{6}\sin 6x + \tfrac{1}{2}\sin 2x)$$
$$= \tfrac{1}{12}(\sin 6x + 3\sin 2x).$$

(iv) $\int \sin^2 3x\, dx = \frac{1}{2}\int (1 - \cos 6x)dx = \frac{1}{2}(x - \frac{1}{6}\sin 6x)$

$$= \tfrac{1}{12}(6x - \sin 6x).$$

Integration as a Summation. In the diagram AB is the portion of the curve $y = f(x)$ between $x = a$ and $x = b$. AC and BD are the ordinates at A and B and the *base* CD is divided up into small intervals by means of equidistant ordinates δx apart. The shaded interval under PQ is a specimen of these intervals, and $P \equiv (x, y)$, $Q \equiv (x + \delta x, y + \delta y)$.



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As $\delta x \to 0$ the shaded area approaches a rectangle whose area is $y\,\delta x$. **Hence**, the area under the curve AB is the sum of all such areas as $\delta x \to 0$, and is therefore denoted by

$$\underset{\delta x \to 0}{\text{Lt}} \sum_{x=a}^{x=b} y\,\delta x, \text{ where } \Sigma \text{ is the summation sign.}$$

But the area under the curve $AB = \int_a^b y\, dx.$

Thus, $$\underset{\delta x \to 0}{\text{Lt}} \sum_{x=a}^{x=b} y\,\delta x = \int_a^b y\, dx.$$

This result is very useful for evaluating the type of summation shown, by using the equivalent definite integral.

Since y is *any* function of x, it will be found that this result can be applied to volumes, centres of gravity, etc., in addition to areas.

When dealing with the application of this result, it is essential to make a suitable choice of element, which may not always be the one indicated in the above proof.

EXAMPLE. Find the area of a circle of radius r.

O is the centre of the circle and the element of area is the area between two circles of radii x and $(x + \delta x)$, each with centre O.

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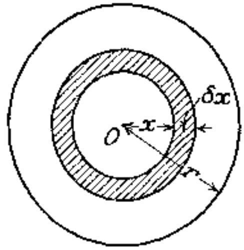
Considering this element cut and straightened out, it forms approximately a rectangle of length $2\pi x$ and width $\delta x(\delta x \to 0)$, therefore as $\delta x \to 0$, area of element $\to 2\pi x \delta x$.



The area of the circle

$$= \text{sum of all such elements}$$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x=0}^{x=r} 2\pi x \, \delta x = \int_0^r 2\pi x \, dx$$

$$= \Big[\pi x^2\Big]_0^r = \pi r^2.$$

EXAMPLE. Find the area between the parabola $y = 2 - 3x + x^2$ and the x-axis.

NOTE. In problems on areas it is advisable to have a fairly accurate diagram of the curve, especially in the case when the area between two curves is dealt with.

The area required is the shaded one in the diagram. (Left-hand figure.)

When $y = 0$, $\qquad x^2 - 3x + 2 = 0,$

$$\text{i.e. } (x - 1)(x - 2) = 0,$$

$$\therefore x = 1 \text{ or } 2,$$

therefore the area required is the area under the curve of $y = 2 - 3x + x^2$ between $x = 1$ and $x = 2$.

$$\text{i.e. } \int_1^2 y \, dx = \int_1^2 (2 - 3x + x^2)dx$$

$$= [2x - \tfrac{3}{2}x^2 + \tfrac{1}{3}x^3]_1^2$$

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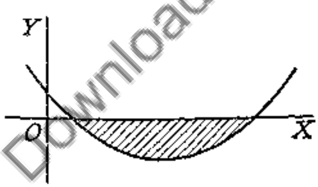
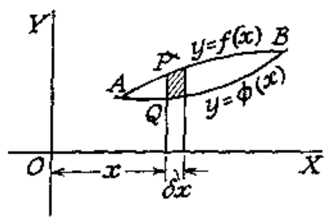
$$= (4 - 6 + \tfrac{8}{3}) - (2 - \tfrac{3}{2} + \tfrac{1}{3}) = \tfrac{2}{3} - \tfrac{5}{6}$$

$$= -\tfrac{1}{6} \text{ square units}$$

The negative sign shows that the area is below the x-axis.





Area between Two Curves. (Right-hand figure.) Consider the two curves $y = f(x)$ and $y = \varphi(x)$ intersecting in *two* points A and B such that $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$.

The area between the two curves is seen, from the diagram, to be the difference of the areas under the curves, from the ordinate at A to the ordinate at B, i.e.

$$\int_{x_1}^{x_2} f(x)dx - \int_{x_1}^{x_2} \varphi(x)dx = \int_{x_1}^{x_2} [f(x) - \varphi(x)]dx.$$

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Hence, after the x-values of the two points of intersection of the curves have been found, the area between the curves is obtained from the above formula.

The result is only proved for the first quadrant, but holds true for any quadrant.

EXAMPLE (L.U.). (i) Evaluate

$$\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} (\cos x + \sin 2x)dx$$

(ii) Find the area enclosed between the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$.

(i) $$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (\cos x + \sin 2x)dx = \Big[\sin x - \tfrac{1}{2}\cos 2x\Big]_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}$$

$$= (\sin \tfrac{1}{2}\pi - \tfrac{1}{2}\cos \pi) - [\sin(-\tfrac{1}{2}\pi) - \tfrac{1}{2}\cos(-\pi)]$$

$$= (1 + \tfrac{1}{2}) - (-1 + \tfrac{1}{2}) = 2.$$

(ii) $$y = -x^2 + 5x - 4 \quad \ldots\ldots (1)$$
$$y = x - 1 \quad \ldots\ldots (2)$$

Substituting for y from (2) in (1), the x values of the points of intersection of the parabola and line are given by

$$x - 1 = -x^2 + 5x - 4$$
$$\text{i.e. } x^2 - 4x + 3 = 0.$$
$$\therefore (x - 1)(x - 3) = 0,$$
$$\therefore x = 1 \text{ or } 3.$$

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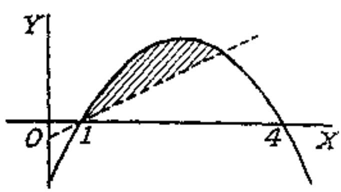
Hence, the required area (Left-hand figure.)

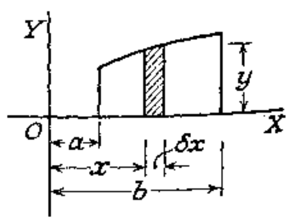
$$= \int_1^3 [(-x^2 + 5x - 4) - (x - 1)]dx$$

$$= \int_1^3 (-x^2 + 4x - 3)dx$$

$$= \Big[-\tfrac{1}{3}x^3 + 2x^2 - 3x\Big]_1^3$$

$$= (-9 + 18 - 9) - (-\tfrac{1}{3} + 2 - 3) = \tfrac{4}{3} \text{ square units.}$$





Solids of Revolution. (Right-hand figure.) If any plane closed curve be rotated through one complete revolution about a fixed line AB in its plane, the solid thus formed is known as a *solid of revolution*; AB is its *axis of revolution*, and all sections perpendicular to the axis AB will be circles or circular annuli.

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When the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is rotated through one complete revolution, about OX, the usual element of area $y\,\delta x$ under the curve will generate an elemental circular disc of radius y and thickness δx, whose volume approaches $\pi y^2\,\delta x$ as $\delta x \to 0$.

The total volume generated will be the sum of all such elements in the Lt. as $\delta x \to 0$, i.e. the total volume

$$= \operatorname*{Lt}_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2\,\delta x$$

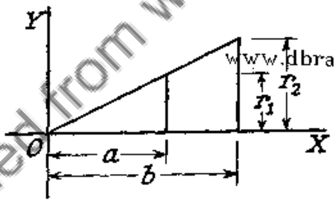
$$= \int_a^b \pi y^2\,dx.$$

NOTE. If the ordinates at A and B be c and d respectively, and the element of area chosen be $x\,\delta y$, it can similarly be shown that the volume of the solid generated by rotating the area about OY is

$$\int_c^d \pi x^2\,dy.$$

Definition. *A frustum* of a cone is the portion cut off by two planes perpendicular to the axis of the cone.

Theorem. *To find the volume of a cone of radius r and height h, and also the volume of a frustum of a cone of height h and radii of ends r_1 and r_2. (Cone means 'right circular cone' unless otherwise stated.)*



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Consider the line $y = mx$, between the limits $x = a$ and $x = b$, the ordinates at the extremities of this part of the line being r_1 and r_2. When the area under this portion of line is rotated through one complete revolution about OX, a frustum of a cone is generated, the radii of whose ends are r_1 and r_2.

The points (a, r_1), (b, r_2) must both satisfy the equation $y = mx$, and thus $r_1 = am,\ r_2 = bm$ .... (1)

If h be the height of the frustum, then $h = b - a$.

The volume of the frustum $= \int_a^b \pi y^2\,dx$

$$= \int_a^b \pi m^2 x^2\,dx = \pi m^2\left[\frac{x^3}{3}\right]_a^b = \tfrac{1}{3}\pi m^2[a^3 - b^3]$$

$$= \tfrac{1}{3}\pi m^2(b - a)(b^2 + ab + a^2) = \tfrac{1}{3}\pi h(m^2b^2 + m^2ab + m^2a^2)$$ (using (1)

$$= \tfrac{1}{3}\pi h(r_2{}^2 + (r_2 r_1 + r_1{}^2).$$

When $r_1 = 0$ and $r_2 = r$, a complete cone of radius r and height h is obtained whose volume, obtained from the previous result, is $\frac{1}{3}\pi r^2 h$.

Definition. A *zone* of a sphere is the portion of the sphere contained between two parallel planes.

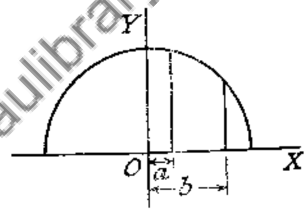
Theorem. *To find the volume of a zone of a sphere of radius r, and to deduce the volume of the sphere.*

Consider the upper half of the circle $x^2 + y^2 = r^2$, and the two ordinates $x = a$ and $x = b$.

When the area thus enclosed is rotated through one complete revolution about OX, the volume formed is a zone of a sphere.

The volume of the zone is $\int_a^b \pi y^2\, dx$

$$= \pi\int_a^b (r^2 - x^2)dx = \pi\Big[r^2x - \tfrac{1}{3}x^3\Big]_a^b$$
$$= \pi[(r^2b - \tfrac{1}{3}b^3) - (r^2a - \tfrac{1}{3}a^3)]$$
$$= \pi[r^2(b - a) - \tfrac{1}{3}(b^3 - a^3)]$$
$$= \pi(b - a)[r^2 - \tfrac{1}{3}(b^2 + ab + a^2)]$$
$$= \pi h[r^2 - \tfrac{1}{3}(b^2 + ab + a^2)],$$



where h = height of zone.

In the case of a segment of a sphere $b = r$, $a = r - h$, and using the previous result, the volume of the segment is

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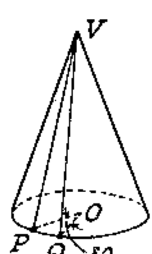
$$\pi h\left[r^2 - \frac{r^2 + ra + a^2}{3}\right]$$
$$= \tfrac{1}{3}\pi h[3r^2 - r^2 - r(r - h) - (r - h)^2]$$
$$= \frac{\pi h}{3}[3rh - h^2] = \frac{\pi h^2}{3}(3r - h).$$

When $h = 2r$ in this result, the segment becomes a full sphere of volume

$$\pi\frac{(2r)^2}{3}[3r - 2r] = \tfrac{4}{3}\pi r^3.$$



Theorem. *To find the curved surface area of a cone.*

V is the vertex and O the centre of the base of the cone, whose radius is r and slant height l.

An element VPQ of the curved surface area is chosen so that P and Q are two points close together on the circumference of the base, and the arc PQ subtends an angle $\delta\theta$ at O.

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Since P and Q are very close together, the element of surface area VPQ can be taken as triangle VPQ whose area is $\frac{1}{2}l \,.\, PQ = \frac{1}{2}l \times r\,\delta\theta$.

Hence, the curved surface area of the cone

$$= \underset{\delta\theta\to 0}{\mathrm{Lt}} \sum_{\theta=0}^{\theta=2\pi} \tfrac{1}{2}rl\,\delta\theta = \int_0^{2\pi} \tfrac{1}{2}rl\,d\theta = \tfrac{1}{2}rl\Big[\theta\Big]_0^{2\pi} = \tfrac{1}{2}rl \times 2\pi$$
$$= \pi rl.$$

Theorem. *To find the curved surface area of the frustum of a cone of slant length l_1, and radii of ends r_1, r_2 . $(r_2 > r_1)$.*

Consider the completed cone of slant length $(l_1 + l_2)$.

By similar triangles,

$$\frac{l_2}{l_1 + l_2} = \frac{r_1}{r_2},$$
$$\therefore\; l_2 r_2 = r_1 l_1 + r_1 l_2,$$
$$\therefore\; l_2(r_2 - r_1) = r_1 l_1,$$
$$\therefore\; l_2 = \frac{l_1 r_1}{r_2 - r_1},$$

therefore area of whole conical surface (curved)

$$= \pi r_2(l_1 + l_2) = \pi r_2\left(l_1 + \frac{l_1 r_1}{r_2 - r_1}\right)$$
$$= \frac{\pi r_2^2 l_1}{r_2 - r_1}.$$

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Area of curved surface of small cone $= \pi r_1 l_2$

$$= \frac{\pi r_1^2 l_1}{r_2 - r_1},$$

therefore curved surface area of frustum

$$= \frac{\pi r_2^2 l_1}{r_2 - r_1} - \frac{\pi r_1^2 l_1}{r_2 - r_1}$$
$$= \pi l_1 \frac{(r_2^2 - r_1^2)}{r_2 - r_1} = \pi l_1(r_1 + r_2).$$

Theorem. *To find the curved surface area of the zone of a sphere of radius r.*

A vertical section of the sphere is shown, with O the centre of the sphere, and the usual axes OX, OY.

The height of the zone is h, and the zone is such that the end planes of the zone are perpendicular to OY, and such that the diameters of the end faces of the zone subtend angles of 2α and 2β at O.

An element of curved surface area is chosen as being formed by two planes perpendicular to OY and such that the diameters of the end faces of the element subtend angles of 2θ and $2(\theta + \delta\theta)$ at O.

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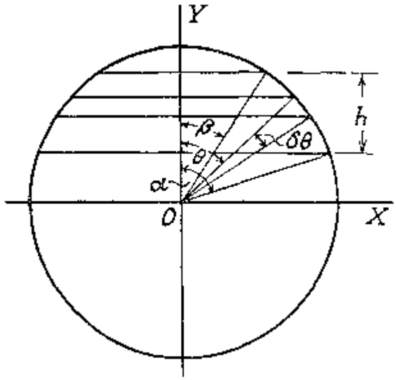

If the element be flattened out, it will be approximately a rectangle of width $r\delta\theta$ and length $2\pi r \sin\theta$, and therefore its area is $2\pi r^2 \sin\theta \,.\, \delta\theta$, in the limit as $\delta\theta \to 0$, therefore the curved surface area of the zone

$$= \int_\beta^\alpha 2\pi r^2 \sin\theta \, d\theta = -2\pi r^2 \Big[\cos\theta\Big]_\beta^\alpha$$

$$= -2\pi r^2[\cos\alpha - \cos\beta] = 2\pi r[r\cos\beta - r\cos\alpha].$$

But, $h = r\cos\beta - r\cos\alpha$. Therefore curved surface area of zone $= 2\pi rh$. This gives the important result that the *curved surface area of the zone of a sphere is equal to the curved area surface of the corresponding zone of the circumscribing cylinder of the cone.*

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NOTE. When $h = 2r$, the curved surface area of a sphere is obtained and has a value $4\pi r^2$.

EXAMPLE (L.U.). Prove that the area of the portion of a spherical surface cut off by two parallel planes is $2\pi rh$, where r is the radius of the sphere and h the distance between the two planes.

If the radius of the base of a spherical cap is $\frac{4}{5}$ the radius of the sphere, determine the ratio of the surface of the cap to the surface of the sphere.

The first part of the question is proved in the previous theorem.

The diagram on the next page (left-hand figure), shows a vertical section of the sphere, radius r, through the centre O.

AB is the diameter of the base of the cap, and OC is the perpendicular on AB (in the vertical plane), the height of the cap being h.

By geometry, C is the mid-point of AB, and AC is the radius of the base of the cap, therefore $AC = \frac{4}{5}r$.

Using Pythagoras' theorem, $OC = \frac{3}{5}r$, therefore $h = r - \frac{3}{5}r = \frac{2}{5}r$. Therefore curved surface area of the cap $= 2\pi rh = \frac{4}{5}\pi r^2$.

The curved surface of the sphere $= 4\pi r^2$, therefore required ratio $= \frac{4}{5}\pi r^2 : 4\pi r^2 = 1 : 5$.

EXAMPLE (L.U.). A frustum of a cone circumscribes a sphere. The radius of one plane end of the frustum is a and the length of the slant side is $(a + b)$. Prove that the volume of the frustum is $\frac{2}{3}\pi(a^2 + ab + b^2)\sqrt{ab}$

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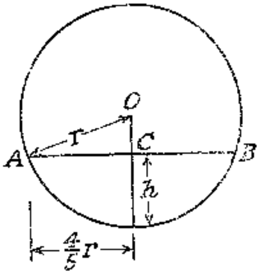
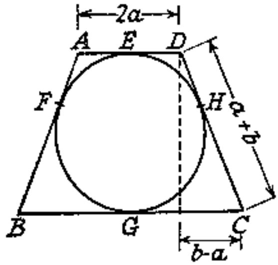
The diagram (right-hand figure) shows a section of the sphere and frustum through the centre of the sphere.





$ABCD$ is the section of the frustum touching the similar section of the sphere at E, F, G, H as shown.

Since the tangents from a point to a circle are equal ($AE = ED = a$ by symmetry), $ED = DH = a$, leaving $HC = b = CG$, therefore the radius of the base of the frustum is b.

Using Pythagoras' theorem, where h = height of frustum,

$$h^2 = (a+b)^2 - (b-a)^2 = 4ab,$$
$$\therefore\ h = 2\sqrt{ab}.$$

Now the volume of a frustum of a cone of height h and radii of ends r_1, r_2 is $\frac{1}{3}\pi h(r_1^2 + r_1r_2 + r_2^2)$, therefore the volume of the frustum in the present case $= \frac{2}{3}\pi\sqrt{ab}(a^2 + ab + b^2)$.

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EXAMPLE (L.U.). Five times the area enclosed by the curves $y = x^4$, $y = x^3$ and $x = a$, equals three times the area enclosed by the curves $y = x^3$, $y = x^2$, and the line $x = a$.

Determine a, and find the ratio of the volumes formed by the rotation of these areas about the x-axis.

Area between $y = x^4$, $y = x^3$, and $x = a$, since the curves intersect at $x = 0$ is

$$\int_0^a (x^3 - x^4)dx = \left[\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right]_0^a = \tfrac{1}{4}a^4 - \tfrac{1}{5}a^5.$$

Also, area between $y = x^3$, and $y = x^2$, and $x = a$, since the curves intersect at $x = 0$ is

$$\int_0^a (x^2 - x^3)dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^a = \tfrac{1}{3}a^3 - \tfrac{1}{4}a^4.$$

By the question,

$$5(\tfrac{1}{4}a^4 - \tfrac{1}{5}a^5) = 3(\tfrac{1}{3}a^3 - \tfrac{1}{4}a^4),$$
$$\text{i.e. } \tfrac{5}{4}a^4 - a^5 = a^3 - \tfrac{3}{4}a^4,$$
$$\text{i.e. } a^5 - 2a^4 + a^3 = 0,$$
$$\text{i.e. } a^2 - 2a + 1 = 0, \qquad (a \neq 0)$$
$$\therefore\ (a-1)^2 = 0,\ \therefore\ a = 1 \text{ (twice)}.$$

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The volume generated by the rotation of the area under the first curve $y = x^4$, from $x = 0$ to $x = a$, about OX

$$= \int_0^a \pi y^2 \, dx = \int_0^a \pi x^8 \, dx$$

The volume generated by the rotation of the area under the curve $y = x^3$, from $x = 0$ to $x = a$, about OX

$$= \int_0^a \pi y^2 \, dx = \int_0^a \pi x^6 \, dx.$$

Therefore the volume generated by the rotation of the first area (between $y = x^4$, $y = x^3$ and $x = a$) about OX = difference of the above two volumes

$$= \int_0^a \pi(x^6 - x^8)dx = \pi\left[\frac{x^7}{7} - \frac{x^9}{9}\right]_0^a = \pi\left[\frac{a^7}{7} - \frac{a^9}{9}\right]$$

$$= \pi\left[\frac{1}{7} - \frac{1}{9}\right], \text{ since } a = 1$$

$$= \frac{2\pi}{63} \text{ c. units.}$$

Similarly the volume generated by the rotation of the second area (between $y = x^3$, $y = x^2$, and $x = a$) about OX

$$= \int_0^a \pi(x^4 - x^6)dx = \pi\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^a = \pi\left[\frac{a^5}{5} - \frac{a^7}{7}\right],$$

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$$= \pi\left[\frac{1}{5} - \frac{1}{7}\right] = \frac{2\pi}{35} \text{ c. units,}$$

therefore the ratio of the two volumes $= \frac{2\pi}{63} : \frac{2\pi}{35} = 35 : 63 = 5 : 9.$

EXAMPLE (L.U.). A hole of circular section, radius r, is bored symmetrically through a solid sphere of radius R. Prove that the volume removed is $\frac{4}{3}\pi[R^3 - (R^2 - r^2)^{3/2}]$.

If exactly half the volume of the sphere remains, show that

$$(r/R)^2 = 1 - 4^{-\frac{1}{3}}.$$

Let $2x$ be the height of the hole.

From the diagram, using Pythagoras' theorem,

$$x^2 = R^2 - r^2, \ \therefore \ x = (R^2 - r^2)^{\frac{1}{2}}.$$

Using an element of the remaining volume formed by two planes perpendicular to the axis of the cylinder, and at distances y and $y + \delta y$ from O, the volume of the element

$$= \pi(R^2 - y^2)\delta y - \pi r^2 \, \delta y$$

(radius of element of sphere $= \sqrt{(R^2 - y^2)}$)

$$= \pi(R^2 - y^2 - r^2)\delta y.$$

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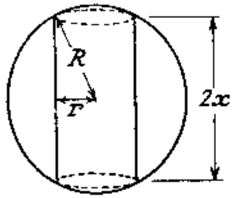
Therefore volume remaining



$$= \int_{-x}^{+x} \pi(R^2 - y^2 - r^2)dy$$

$$= 2\int_0^x \pi(R^2 - y^2 - r^2)dy \text{ (by symmetry)}$$

$$= 2\pi\left[(R^2 - r^2)y - \tfrac{1}{3}y^3\right]_0^x = 2\pi[(R^2 - r^2)x - \tfrac{1}{3}x^3]$$

$$= 2\pi x[(R^2 - r^2) - \tfrac{1}{3}x^2] = 2\pi(R^2 - r^2)^{\frac{1}{2}}[(R^2 - r^2) - \tfrac{1}{3}(R^2 - r^2)]$$

$$= \tfrac{4}{3}\pi(R^2 - r^2)^{3/2},$$

therefore volume removed $=$ volume of sphere $-$ volume remaining

$$= \tfrac{4}{3}\pi R^3 - \tfrac{4}{3}\pi(R^2 - r^2)^{3/2}$$

$$= \tfrac{4}{3}\pi[R^3 - (R^2 - r^2)^{3/2}].$$

When half the volume remains

$$\tfrac{4}{3}\pi[R^3 - (R^2 - r^2)^{3/2}] = \tfrac{2}{3}\pi R^3$$

$$\therefore 2R^3 - 2(R^2 - r^2)^{3/2} = R^3$$

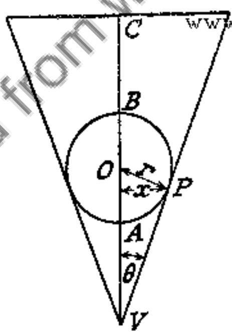
$$\text{i.e. } R^3 = 2(R^2 - r^2)^{3/2}$$

$$\text{i.e. } R^2 = 2^{2/3}(R^2 - r^2)$$

$$\therefore r^2 \,.\, 2^{2/3} = R^2(2^{2/3} - 1)$$

$$\therefore \frac{r^2}{R^2} = 1 - \frac{1}{2^{\frac{2}{3}}} = 1 - \frac{1}{4^{\frac{1}{3}}} = 1 - 4^{-\frac{1}{3}}.$$

EXAMPLE (L.U.). A spherical ball placed inside a hollow cone with its vertex downwards, rests with its highest and lowest points trisecting the axis of the cone.



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Show that the semi-vertical angle of the cone is $\sin^{-1} \frac{1}{3}$, and that the radius of the circle of contact of the cone and the sphere is $h\sqrt{2}/9$, where h is the length of the axis of the cone.

Prove also that the ratio of the volume of the ball to that of the cone is 4 : 27.

A is the lowest point and B the highest point of the sphere, and the cone has vertex V.

O is the centre of the sphere, radius r, and by the question,

$$2r = AB = \tfrac{1}{3}h, \quad \therefore r = \tfrac{1}{6}h.$$

Now $OV = OA + AV = r + \frac{1}{3}h = \frac{1}{6}h + \frac{1}{3}h = \frac{1}{2}h.$

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If θ be the semi-vertical angle of the cone (must be acute),

$$\sin\theta = \frac{r}{OV} = \frac{\frac{1}{6}h}{\frac{1}{2}h} = \tfrac{1}{3},$$

$$\therefore\ \theta = \sin^{-1}\tfrac{1}{3}.$$

Let P be a point of the circle of contact in the section of the diagram, and x be the radius of the circle of contact.

Then, $$PV = \sqrt{(OV^2 - OP^2)} = \sqrt{\left(\frac{h^2}{4} - \frac{h^2}{36}\right)}$$

$$= \frac{h}{6}\sqrt{(9-1)} = \frac{h\sqrt{8}}{6} = \frac{h\sqrt{2}}{3}.$$

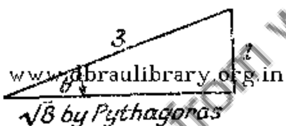
Hence, $$x = PV\sin\theta = \tfrac{1}{3}PV = \frac{h\sqrt{2}}{9}.$$

Volume of ball $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{h}{6}\right)^3 = \frac{\pi h^3}{162}.$

Radius of cone $= h\tan\theta$,

$$\therefore\ \text{volume of cone} = \tfrac{1}{3}\pi(h\tan\theta)^2 \cdot h$$
$$= \tfrac{1}{3}\pi h^3\tan^2\theta.$$

Now, $\sin\theta = \frac{1}{3}$, therefore from a right-angled triangle,



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$$\tan\theta = 1/\sqrt{8}$$

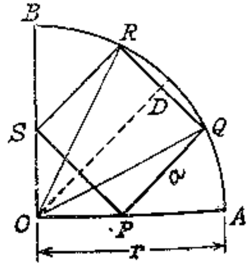
$$\therefore\ \tan^2\theta = \frac{1}{8}$$

$$\therefore\ \text{volume of cone} = \tfrac{1}{3}\pi h^3 \times \tfrac{1}{8} = \frac{\pi h^3}{24},$$

therefore ratio of volume of sphere to volume of cone

$$= \frac{\pi h^3}{162} : \frac{\pi h^3}{24} = 24 : 162 = 4 : 27.$$

Example (L.U.). OA, OB are the bounding radii of a quadrant of a circle of radius r. $PQRS$ is a square of side a inscribed in the quadrant so that P lies on OA, S on OB, Q and R on the arc of the quadrant and $OP = OS$. Show that the angle QOR is $2\tan^{-1}\frac{1}{3}$, that $a/r = \sqrt{10}/5$, and that the ratio of the area of the square to that of the quadrant is $8 : 5\pi$.



By Pythagoras, $$OP^2 + OS^2 = a^2$$
$$\text{i.e. } 2OP^2 = a^2$$
$$\therefore\ OP = a/\sqrt{2}.$$

If the perpendicular from

$$O \text{ on } SP = x,$$

then

$$x = OP\cos 45^\circ \qquad (\angle OPS = 45^\circ)$$

$$= \frac{a}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \tfrac{1}{2}a.$$

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The perpendicular OD from O on RQ bisects RQ by geometry, and

$$OD = x + a = \frac{3a}{2}.$$

$$\therefore \tan \angle QOD = \frac{QD}{OD} = \frac{\frac{1}{2}a}{3\frac{a}{2}} = \frac{1}{3}$$

$$\therefore \angle QOD = \tan^{-1} \tfrac{1}{3} \qquad \text{(acute angle).}$$
$$\therefore \angle ROQ = 2\angle QOD = 2\tan^{-1} \tfrac{1}{3}.$$

From triangle ODQ by Pythagoras,

$$OQ^2 = r^2 = DQ^2 + OD^2 = \frac{a^2}{4} + \frac{9a^2}{4} = \frac{5}{2}a^2.$$

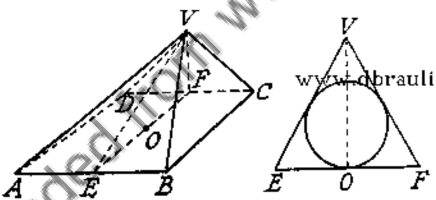
$$\therefore \frac{a^2}{r^2} = \frac{2}{5}, \therefore \frac{a}{r} = \sqrt{\frac{2}{5}} = \frac{\sqrt{10}}{5}.$$

The ratio of the area of the square to the area of the quadrant

$$= a^2 : \tfrac{1}{4}\pi r^2 = \frac{4a^2}{\pi r^2} = \frac{4}{\pi} \times \frac{2}{5} = 8 : 5\pi.$$

EXAMPLE (L.U.). A pyramid on a square base has all its edges two feet in length.

Find the radius of the inscribed sphere (to $\frac{1}{100}$ inch) and show that the sphere occupies nearly one-third of the volume of the pyramid.



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O is the centre of the base $ABCD$ of the pyramid whose vertex is V. E and F are the mid-points of AB and CD respectively, and, by symmetry the inscribed sphere will touch the sides of the triangle VEF. VAB is an equilateral triangle of side 2 feet.

$$\therefore VE = 2 \times \tfrac{1}{2}\sqrt{3} = \sqrt{3} \text{ feet.}$$

By Pythagoras, $$VO^2 = VE^2 - EO^2 \qquad (EO = 1 \text{ foot})$$
$$= 3 - 1 = 2$$
$$\therefore VO = \sqrt{2} \text{ feet.}$$

Area of triangle $VEF = \frac{1}{2}VO \times EF = \frac{1}{2} \times \sqrt{2} \times 2 = \sqrt{2}$ square feet. Semi-perimeter of triangle $VEF = VE + EO = (\sqrt{3} + 1)$ feet. Therefore radius of inscribed circle of triangle VEF = radius of inscribed sphere of the pyramid

$$= \frac{\text{area of triangle } VEF}{\text{semi-perimeter of triangle } VEF} = \frac{\sqrt{2}}{\sqrt{3}+1} = \frac{\sqrt{2}(\sqrt{3} - 1)}{3 - 1}$$

$$= \tfrac{1}{2}(\sqrt{6} - \sqrt{2}) \text{ feet} = 0{\cdot}5177 \text{ foot.}$$
$$= 6{\cdot}21 \text{ inches to } \tfrac{1}{100} \text{ inch.}$$

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The volume of the sphere

$$= \tfrac{4}{3}\pi\left\{\sqrt{2}\frac{(\sqrt{3}-1)}{2}\right\}^3 = \tfrac{4}{3}\pi \cdot \frac{2\sqrt{2}}{8}(3\sqrt{3}-9+3\sqrt{3}-1)$$

$$= \frac{\pi\sqrt{2}}{3}(6\sqrt{3}-10).$$

$$\text{The volume of the pyramid} = \tfrac{1}{3} \times \text{area of base} \times VO$$
$$= \tfrac{1}{3} \times 4 \times \sqrt{2} = \tfrac{4}{3}\sqrt{2}$$

$$\therefore \frac{\text{volume of sphere}}{\text{volume of pyramid}} = \frac{\pi\sqrt{2}}{3}(6\sqrt{3}-10) \times \frac{3}{4\sqrt{2}}$$
$$= \tfrac{1}{4}\pi(6\sqrt{3}-10)$$
$$= \tfrac{1}{4}\pi(10\cdot3926 - 10) = \tfrac{1}{4}\pi \times 0\cdot3926$$
$$= 0\cdot308, \text{ which is roughly } \tfrac{1}{3}.$$

Centres of Gravity, Centroids, and Centres of Mass. Since the weight of a particle is proportional to its mass, the centre of gravity and centre of mass of *any* body will be coincident.

The definition for the *centroid* of a body is the centre of gravity of the body, assuming the body to be uniform.

In all that follows the bodies considered will be taken to be uniform, and, where the centre of gravity lies in the plane XOY, its co-ordinates will be taken to be $(\bar{x}, \bar{y})$, and will be the same as the centroid since only uniform bodies are considered.

When the x-axis is a line of symmetry, the centre of gravity will lie on the x-axis, and only the value of $\bar{x}$ will then be required.

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It is shown in books on applied mathematics that if w_1, w_2, w_3, . . . are the weights of particles of a plane lamina lying in the plane XOY, and the positions of these particles are (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., respectively, then

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots} = \frac{\text{total moment about } OY}{\text{total weight}}$$

$$\bar{y} = \frac{w_1y_1 + w_2y_2 + w_3y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots} = \frac{\text{total moment about } OX}{\text{total weight}}.$$

If the body be continuous, and δw be the weight of a particle situated at (x, y), these results become

$$\bar{x} = \frac{\underset{\delta w \to 0}{\text{Lt}} \Sigma x\,\delta w}{\underset{\delta w \to 0}{\text{Lt}} \Sigma \delta w} = \frac{\int x\,dw}{\int dw}.$$

$$\bar{y} = \frac{\underset{\delta w \to 0}{\text{Lt}} \Sigma y\,\delta w}{\underset{\delta w \to 0}{\text{Lt}} \Sigma \delta w} = \frac{\int y\,dw}{\int dw}.$$

NOTE. The choice of the element of weight is entirely dependent upon the type of body considered.

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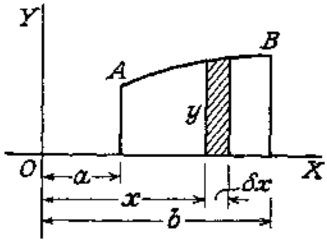
In all that follows w will be taken as the weight per unit (length, area, or volume) of matter.

Theorem. *To find the centre of gravity (C.G.) of the lamina formed by the area under a curve $y = f(x)$.*

The diagram shows a portion AB of the curve $y = f(x)$ between $x = a$ and $x = b$, with the usual choice of element.



The element of area is $y\,\delta x$ and its weight $wy\,\delta x$.
Its moment about OY is

$$wy\,\delta x\,.\,x = wxy\,\delta x,$$

$$\therefore\ \bar{x} = \frac{\text{total moment about } OY}{\text{total weight}}$$

$$= \frac{\int_a^b wxy\,dx}{\int_a^b wy\,dx} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx}$$

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Since the C.G. of the element is at a distance $\frac{1}{2}y$ from OX, its moment about OX is $\frac{1}{2}wy^2\,\delta x$.

Hence, $$\bar{y} = \frac{\text{total moment about } OX}{\text{total weight}} = \frac{\int_a^b \frac{1}{2}wy^2\,dx}{\int_a^b wy\,dx}$$

$$= \frac{\int_a^b \frac{1}{2}y^2\,dx}{\int_a^b y\,dx}.$$

Theorem. *To find the centre of gravity (or centroid) of the uniform solid formed by rotating the area under the curve $y = f(x)$ between $x = a$ and $x = b$ through one complete revolution about OX.*

Using the diagram of the previous theorem, when the element of area under the curve is rotated about OX, it generates an elemental disc, whose volume can be taken as $\pi y^2\,\delta x$ and has its C.G. on OX.

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Its moment about $OY = w\pi y^2\,\delta x \times x = w\pi xy^2\,\delta x$.

Hence,
$$\bar{x} = \frac{\int_a^b w\pi xy^2\,dx}{\int_a^b w\pi y^2\,dx} = \frac{\int_a^b xy^2\,dx}{\int y^2\,dx}$$

From symmetry, the C.G. lies on OX (at a distance $\bar{x}$ from 0).

NOTE. w can only be cancelled in the two integrals since the body is uniform and w therefore constant.

EXAMPLE (L.U.). (i) A plane area is bounded by the x-axis, the line $x = a$, and the arc of the curve $y = x^3/a^2$ from $x = 0$ to $x = a$. Show that the co-ordinates of the centroid of the area are $(4a/5, 2a/7)$.

(ii) The part of the curve $y = 3(x^2 - 1)$ from $x = 1$ to $x = 3$ is rotated about the y-axis. Find the volume so generated.

(i) Let $(\bar{x}, \bar{y})$ be the centroid of the area.

Then
$$\bar{x} = \frac{\int_0^a xy\,dx}{\int_0^a y\,dx} = \frac{\int_0^a \frac{x^4}{a^2}\,dx}{\int_0^a \frac{x^3}{a^2}\,dx} = \frac{\frac{1}{a^2}\left[\frac{x^5}{5}\right]_0^a}{\frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^a}$$
$$= \frac{a^5/5}{a^4/4} = \frac{4a}{5}.$$

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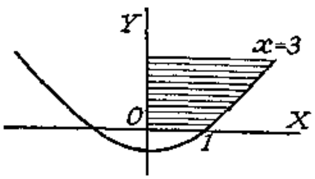
$$\bar{y} = \frac{\int_0^a \frac{1}{2}y^2\,dx}{\int_0^a y\,dx} = \frac{\int_0^a \frac{x^6}{2a^4}\,dx}{\int_0^a \frac{x^3}{a^2}\,dx} = \frac{\frac{1}{2a^4}\left[\frac{x^7}{7}\right]_0^a}{\frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^a}$$
$$= \frac{1}{2a^2} \times \frac{a^7/7}{a^4/4} = \frac{2a}{7},$$

therefore the centroid is the point $(4a/5, 2a/7)$.

(ii) In this problem it is implied that the area shown shaded in the diagram is revolved about OY.

$y = 3(x^2 - 1)$, therefore when $x = 1$, $y = 0$, and when $x = 3$, $y = 24$

Hence, volume required



$$= \int_0^{24} \pi x^2\,dy$$
$$= \pi\int_0^{24} (1 + \tfrac{1}{3}y)dy = \pi\left[y + \tfrac{1}{6}y^2\right]_0^{24}$$
$$= \pi[24 + 96] = 120\pi \text{ c. units.}$$

Theorem. *To find the C.G. of a conical shell (made of thin sheet material) excluding the base.*

The C.G. must lie on the axis by symmetry, and, using the elemental triangle, as when finding the curved surface area of a cone, the

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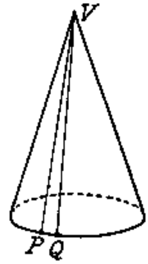
C.G. must lie one-third the way up the slant length from the base in the case of all the elemental triangles (VPQ).



It will thus lie on a section one-third the way up the slant length from the base and by similar triangles it must divide the height in the ratio of 1 : 2 from the base.

Hence, the C.G. is on the axis of the cone and divides it in the ratio 1 : 2 *from the base.*

Theorem. *To find the C.G. of the curved surface area of the zone of a sphere. (Thin-sheet material.)*

Since the join of the centres of the ends of the zone is an axis of symmetry, the C.G. must lie along this line (axis of zone).

Considering two elements of the area formed by pairs of circles parallel to the end circles, and of the same heights, their areas and therefore their weights will be equal, and, if they are equally placed on either side of the mid-point of the axis, their combined C.G. will be at the mid-point of the axis.

The whole area can be thus subdivided into equal pairs of elements, and, therefore, the C.G. of the whole surface is at *the mid-point of the axis.*

Theorem. *To find the C.G. of a solid right circular cone of radius* r *and height* h.

The cone is generated by the revolution of the line $y = mx$, between $x = 0$ and $x = h$, through one complete revolution about OX, where $m = r/h$.

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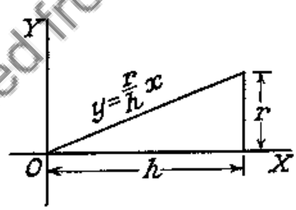

By symmetry, the C.G. must lie on OX, and, if $\bar{x}$ be the distance of the C.G. from O, then

$$\bar{x} = \frac{\int_0^h xy^2\,dx}{\int_0^h y^2\,dx} = \frac{\int_0^h m^2x^3\,dx}{\int_0^h m^2x^2\,dx} = \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx}$$

$$= \left[\frac{x^4}{4}\right]_0^h \div \left[\frac{x^3}{3}\right]_0^h = \tfrac{1}{4}h^4 \div \tfrac{1}{3}h^3 = \tfrac{3}{4}h.$$

Thus the C.G. of a solid cone *divides its axis in the ratio* 1 : 3 *from the base.*

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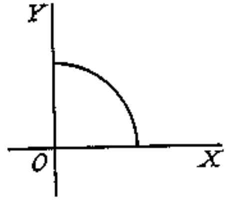
Theorem. *To find the C.G. of a solid hemisphere of radius r.*

A hemisphere of radius r is generated by the revolution about OX, through one complete turn, of the quadrant of the circle $x^2 + y^2 = r^2$, between $x = 0$, $y = r$, and $x = r$, $y = 0$ (i.e. lying in the first quadrant).



By symmetry, the C.G. lies on OX, and, if this C.G. be at a distance $\bar{x}$ from O

$$\bar{x} = \frac{\int_0^r xy^2\,dx}{\int_0^r y^2\,dx} = \frac{\int_0^r x(r^2 - x^2)dx}{\int_0^r (r^2 - x^2)dx} = \frac{\int_0^r (r^2x - x^3)dx}{\int_0^r (r^2 - x^2)dx}$$

$$= \frac{\left[\tfrac{1}{2}r^2x^2 - \tfrac{1}{4}x^4\right]_0^r}{\left[r^2x - \tfrac{1}{3}x^3\right]_0^r} = \frac{\tfrac{1}{2}r^4 - \tfrac{1}{4}r^4}{r^3 - \tfrac{1}{3}r^3} = \tfrac{1}{4}r^4 \div \tfrac{2}{3}r^3$$

$$= \tfrac{3}{8}r$$

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Thus the C.G. of a solid hemisphere is on the radius perpendicular to the plane base and *divides this radius in the ratio* 3 : 5 *from the base.*

Differential Equations. Any equation involving

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \text{ etc.,}$$

in addition to x and y, is known as a differential equation in x and y.

The only types to be considered here will be of the form that require straightforward integration, viz.,

$$\text{(i)}\ \frac{dy}{dx} = f(x), \quad \text{(ii)}\ \frac{d^2y}{dx^2} = F(x).$$

Great care must be taken to introduce a constant of integration for each integration that takes place, and this constant (or constants) can generally be determined from data given in the question (sometimes known as boundary conditions).

EXAMPLE (L.U.). For a certain curve,

$$\frac{d^2y}{dx^2} = 6x - 4$$

and y has a minimum value 5 when $x = 1$.

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Find the equation of the curve and the maximum value of y.

$$\frac{d^2y}{dx^2} = 6x - 4, \therefore \frac{dy}{dx} = \int (6x - 4)dx + C_1,$$

$$\text{i.e. } \frac{dy}{dx} = 3x^2 - 4x + C_1,$$

where C_1 is a constant of integration.

Since y has a minimum value when $x = 1$,

$$\frac{dy}{dx} = 0 \text{ when } x = 1,$$

$$\therefore 0 = 3 - 4 + C_1, \therefore C_1 = 1.$$

Hence, $$\frac{dy}{dx} = 3x^2 - 4x + 1 \ldots\ldots\ldots\ldots (1),$$

$$\therefore y = \int (3x^2 - 4x + 1)dx + C_2,$$

(C_2 is a constant)

$$= x^3 - 2x^2 + x + C_2.$$

Now $y = 5$ when $x = 1$

$$\therefore 5 = 1 - 2 + 1 + C_2, \therefore C_2 = 5,$$

$$\therefore y = x^3 - 2x^2 + x + 5.$$

From (1), $\frac{dy}{dx} = 0$ when $3x^2 - 4x + 1 = 0$,

i.e. when $(3x - 1)(x - 1) = 0$,

i.e. when $x = \frac{1}{3}$ or 1.

Now $x = 1$ gives a minimum value of y, and when $x = 1/3$,

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$$\frac{d^2y}{dx^2} = 2 - 4 = -2,$$

therefore $x = \frac{1}{3}$ gives a maximum value of y, which is

$$(\tfrac{1}{3})^3 - 2(\tfrac{1}{3})^2 + \tfrac{1}{3} + 5 = \tfrac{1}{27} - \tfrac{2}{9} + \tfrac{1}{3} + 5 = 5\tfrac{4}{27}.$$

EXAMPLE. The acceleration of a point moving in a straight line is $(4 + 2t^2)$ feet at time t seconds after the start of the motion. Find the velocity v at time 2 seconds, given that $v = 3$ ft./sec. when $t = 0$.

Find also its displacement s at time 2 seconds given that $s = 0$ when $t = 0$.

The acceleration of the moving point $= \frac{dv}{dt}$,

$$\therefore \frac{dv}{dt} = 4 + 2t^2.$$

Hence, $$v = \int (4 + 2t^2)dt + C_1$$

(C_1 constant of integration and also C_2)

$$= 4t + \tfrac{2}{3}t^2 + C_1$$

Now $v = 3$ when $t = 0$, therefore $C_1 = 3$

$$\therefore v = 4t + \tfrac{2}{3}t^3 + 3.$$

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When $t = 2$, $v = 8 + \frac{16}{3} + 3 = 16\frac{1}{3}$ ft./sec.

$$\text{Now,} \qquad v = \frac{ds}{dt}, \quad \therefore \quad \frac{ds}{dt} = 4t + \tfrac{2}{3}t^3 + 3$$

$$\therefore \; s = \int (4t + \tfrac{2}{3}t^3 + 3)dt = 2t^2 + \tfrac{1}{6}t^4 + 3t + C_2.$$

Now $s = 0$ when $t = 0$, therefore $C_2 = 0$.
Thus, when $t = 2$, $s = 8 + \frac{8}{3} + 6 = 16\frac{2}{3}$ feet.

EXAMPLE. A cantilever is of length l feet with a load of w lb. per ft. length. The deflection y at a distance x from the fixed end is given by

$$\frac{d^2y}{dx^2} = \frac{w}{2EI}(l - x)^2.$$

Solve the equation and find the greatest deflection.

From knowledge of the cantilever (clamped horizontally at one end) the following conditions are obtained:

$$y = 0 \text{ when } x = 0, \quad \text{and} \quad \frac{dy}{dx} = 0, \text{ when } x = 0.$$

$$\text{Now } \frac{d^2y}{dx^2} = \frac{w}{2EI}(l - x)^2, \quad \therefore \frac{dy}{dx} = \frac{w}{2EI}\int (l - x)^2 dx,$$

$$\text{i.e. } \frac{dy}{dx} = \frac{w}{2EI}\,\frac{(l - x)^3}{3(-1)} + C_1 = \frac{-w}{6EI}(l - x)^3 + C_1.$$

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Now $\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore 0 = -\frac{w}{6EI}l^3 + C_1,$$

$$\therefore C_1 = \frac{w}{6EI}l^3$$

$$\therefore \frac{dy}{dx} = \frac{wl^3}{6EI} - \frac{w}{6EI}(l - x)^3.$$

$$\text{Thus,} \qquad y = \int \left\{ \frac{wl^3}{6EI} - \frac{w}{6EI}(l - x)^3 \right\} dx + C_2$$

$$= \frac{wl^3x}{6EI} + \frac{w}{24EI}(l - x)^4 + C_2.$$

But $y = 0$ when $x = 0$,

$$\therefore 0 = \frac{w}{24EI}l^4 + C_2, \quad \therefore C_2 = \frac{-wl^4}{24EI},$$

$$\text{i.e. } y = \frac{w}{24EI}[4l^3x + (l - x)^4 - l^4],$$

$$\text{i.e. } y = \frac{w}{24EI}[6x^2l^2 - 4x^3l + x^4]$$

$$= \frac{wx^2}{24EI}[x^2 - 4lx + 6l^2].$$

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From the diagram, it is clear that the greatest deflection occurs when $x = l$ (i.e. at the free end), and when $x = l$,

$$y = \frac{wl^2}{24EI}[l^2 - 4l^2 + 6l^2] = \frac{wl^4}{8EI}.$$

Approximate Integration. (Left-hand figure). Since the area under any curve can always be represented by an integral, if an approximate value can be found for the area under a given curve this will also represent an approximation to a certain integral.

One method of finding the area under a portion of curve is by counting squares but this is very laborious.

A second method is by using the *trapezoidal rule*, where the area under the curve is divided up into a series of intervals by means of n equidistant ordinates $y_1, y_2, \ldots y_n$, each successive pair of which are h units apart.

Considering the first space as a trapezium, its area will be $\frac{1}{2}h(y_1 + y_2)$ approximately.

Similarly, the area of the second interval is approximately $\frac{1}{2}h(y_2 + y_3)$ and so on.

Hence, the approximate area under the curve is

$$\tfrac{1}{2}h[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + \ldots (y_{n-1} + y_n)]$$
$$= \tfrac{1}{2}h[(y_1 + y_n) + 2(y_2 + y_3 + \ldots + y_{n-1})].$$

It can be seen from the diagram that the smaller the value of h, the greater is the degree of accuracy obtained.

This trapezoidal rule is not usually as accurate as Simpson's rule that follows, which is based upon the assumption that each small portion of curve is a straight line, or parabola, or a cubic curve.

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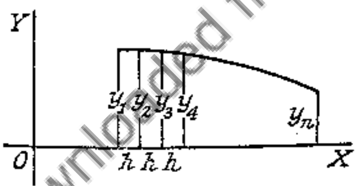

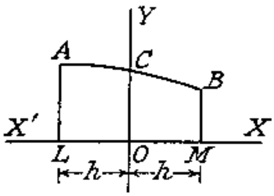

Simpson's Rule. (Right-hand figure.) Let it be required to find the area between a portion of curve AB, the line $X'X$ and the perpendiculars AL, BM from A and B on the line $X'X$, which is chosen as the x-axis.

O, the mid-point of LM, is taken as the origin, and OY, the y-axis, cuts the curve at C.

Let $LO = OM = h$, $AL = y_1$, $CO = y_2$, $BM = y_3$.

Then, $A \equiv (-h, y_1)$, $C \equiv (0, y_2)$, $B \equiv (h, y_3)$.

Let the equation of the curve ACB with respect to the axes OX, OY be $y = ax^3 + bx^2 + cx + d$ (this covers a parabola when $a = 0$, and a straight line when both a and b are zero).

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Since A, C, B lie on the curve,

$$y_1 = -ah^3 + bh^2 - ch + d,$$
$$y_2 = d$$
$$y_3 = ah^3 + bh^2 + ch + d.$$
$$\therefore\ y_1 + 4y_2 + y_3 = 2bh^2 + 6d,$$
$$\therefore\ \tfrac{1}{3}h(y_1 + 4y_2 + y_3) = \tfrac{2}{3}h(bh^2 + 3d) \quad \ldots\ldots\ldots\ldots (1)$$

Now the area under the curve ACB

$$= \int_{-h}^{+h} y\,dx$$
$$= \int_{-h}^{+h} (ax^3 + bx^2 + cx + d)dx$$
$$= \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-h}^{+h}$$
$$= \left(\frac{ah^4}{4} + \frac{bh^3}{3} + \frac{ch^2}{2} + dh\right) - \left(\frac{ah^4}{4} - \frac{bh^3}{3} + \frac{ch^2}{2} - dh\right)$$
$$= \frac{2bh^3}{3} + 2dh = \tfrac{2}{3}h(bh^2 + 3d),$$

therefore from result (1), the area under the curve ACB

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$$= \tfrac{1}{3}h(y_1 + 4y_2 + y_3).$$

By extending this result, a rule can be obtained for the area under a curve of any length, provided an *odd* number of equidistant ordinates be chosen.

Let there be $(2n + 1)$ ordinates $y_1, y_2, \ldots, y_{2n+1}$ equally spaced, each pair being h units apart.

From the above result, the area in the first two spaces

$$= \tfrac{1}{3}h(y_1 + 4y_2 + y_3),$$

the area in the next two spaces $= \tfrac{1}{3}h(y_3 + 4y_4 + y_5)$, and so on. Therefore the total area under the curve

$$= \tfrac{1}{3}h(y_1 + 4y_2 + y_3) + \tfrac{1}{3}h(y_3 + 4y_4 + y_5) + \ldots + \tfrac{1}{3}h(y_{2n-1} + 4y_{2n} + y_{2n+1})$$
$$= \tfrac{1}{3}h[(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + \ldots + y_{2n})]$$

NOTE. The result will be absolutely accurate if the curve ACB is a straight line, parabola, or cubic curve, and in other cases the degree of accuracy obtained will depend upon the number of ordinates chosen, which should not be more than about eleven, as for large numbers of ordinates the work becomes too laborious.

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Theorem. *To find the indefinite integral $I = \int e^{ax+b}\,dx$, where a and b are constants.*

Now $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$, therefore since integration is the converse of differentiation,

$$\int ae^{ax+b}\,dx = e^{ax+b},$$

$$\text{i.e. } a\int e^{ax+b}\,dx = e^{ax+b},$$

$$\text{i.e. } \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}.$$

When $a = 1$ and $b = 0$, we have $\int e^x\,dx = e^x$.

EXAMPLE. Find

$$\text{(i)} \int \frac{du}{2e^{2u}}, \qquad \text{(ii)} \int 5e^{3-5x}\,dx, \qquad \text{(iii)} \int e^{0\cdot 4y}\,dy.$$

$$\text{(i)} \int \frac{du}{2e^{2u}} = \tfrac{1}{2}\int \frac{du}{e^{2u}} = \tfrac{1}{2}\int e^{-2u}\,du = \tfrac{1}{2}\left(\frac{e^{-2u}}{-2}\right) + C$$

$$= -\frac{1}{4e^{2u}} + C.$$

$$\text{(ii)} \int 5e^{3-5x}\,dx = \frac{5e^{3-5x}}{-5} + C = C - e^{3-5x}.$$

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$$\text{(iii)} \int e^{0\cdot 4y}\,dy = \frac{e^{0\cdot 4y}}{0\cdot 4} + C = \tfrac{5}{2}e^{0\cdot 4y} + C.$$

Theorem. *To find the indefinite integral*

$$I = \int \frac{dx}{ax+b},$$

where a and b are constants.

Now $$\frac{d}{dx}[\log_e(ax+b)] = \frac{a}{ax+b}.$$

Using integration as the converse of differentiation,

$$\int \frac{a}{ax+b}\,dx = \log_e(ax+b),$$

$$\text{i.e. } a\int \frac{dx}{ax+b} = \log_e(ax+b)$$

$$\therefore \int \frac{dx}{ax+b} = \frac{1}{a}\log_e(ax+b).$$

When $a = 1$, $b = 0$ the result is $\int \frac{dx}{x} = \log_e x$.

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EXAMPLE. Find

(i) $\int \frac{2\,dx}{4+3x}$, (ii) $\int \frac{du}{2-5u}$, (iii) $\int 4(2-y)^{-1}\,dy$.

(i) $$\int \frac{2}{4+3x}\,dx = 2\int \frac{dx}{4+3x} = 2 \times \tfrac{1}{3}\log_e(4+3x) + C$$
$$= \tfrac{2}{3}\log_e(4+3x) + C.$$

(ii) $$\int \frac{du}{2-5u} = \frac{1}{-5}\log_e(2-5u) + C = C - \tfrac{1}{5}\log_e(2-5u)$$

(iii) $$\int 4(2-y)^{-1}\,dy = 4\int \frac{dy}{2-y} = \frac{4}{(-1)}\log_e(2-y) + C$$
$$= C - 4\log_e(2-y).$$

EXAMPLE. Find an approximate value for

$$\int_1^2 \frac{dx}{x},$$

and compare the result with that obtained by integration.

Let $y = 1/x$, therefore

$$\int_1^2 \frac{1}{x}\,dx$$

represents the area under the curve $y = 1/x$ from $x = 1$ to $x = 2$.

Using reciprocal tables, the following is obtained:

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x	1	1·1	1·2	1·3	1·4	1·5
y	1 y_1	0·90909 y_2	0·83333 y_3	0·76923 y_4	0·71429 y_5	0·66667 y_6

x	1·6	1·7	1·8	1·9	2
y	0·625 y_7	0·58824 y_8	0·55556 y_9	0·52632 y_{10}	0·5 y_{11}

Using Simpson's rule, viz.,

$$\text{approximate area} = \tfrac{1}{3}h\left\{\begin{array}{l}(y_1 + y_{11}) + 2(y_3 + y_5 + y_7 + y_9) \\ \quad + 4(y_2 + y_4 + y_6 + y_8 + y_{10})\end{array}\right\},$$

where $h = 0{\cdot}1$, the approximate value of the integral (i.e.the area under $y = 1/x$ from $x = 1$ to $x = 2$) is

$$\tfrac{1}{3}(0{\cdot}1)\left\{\begin{array}{l}(1 + 0{\cdot}5) + 2(0{\cdot}83333 + 0{\cdot}71429 + 0{\cdot}625 + 0{\cdot}55556) \\ \quad + 4(0{\cdot}90909 + 0{\cdot}76923 + 0{\cdot}66667 + 0{\cdot}58824 + 0{\cdot}52632)\end{array}\right\}$$
$$= \tfrac{1}{30}[1{\cdot}5 + 2 \times 2{\cdot}72818 + 4 \times 3{\cdot}45955]$$
$$= \tfrac{1}{30}[20{\cdot}79456] = 0{\cdot}69315 \text{ to 5 decimal places.}$$

Now $$\int_1^2 \frac{dx}{x} = \Big[\log_e x\Big]_1^2 = \log_e 2 - \log_e 1 = \log_e 2$$
$$= 0{\cdot}69315.$$

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EXAMPLES XI

1. Find (i) $\int_1^2 \frac{x^3 - 1}{x^2}\,dx$, (ii) $\int_0^1 (\sqrt{x} - 1)^2 dx$, (iii) $\int_0^{\frac{1}{2}\pi} \sin 2x \cos x\, dx$.

2. (i) Show that $\int_0^{\pi} \cos^2 x\, dx = \int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\pi$.

(ii) A curve passes through the points (0, 1), (3, 10), and at any point on it

$$\frac{d^2y}{dx^2} = 4x - 6.$$

Find the equation of the curve and show that y increases as x increases for all real values of x.

3. (a) Differentiate $\cos 6x$ with respect to x from first principles.

(b) Evaluate $\int_1^2 (x + 1/x)^4 dx$, and $\int_0^{\pi/6} \sin 3x \cos 3x\, dx$.

(c) The sides of an equilateral triangle are all increasing uniformly at the rate of 1 mm. per second.

Show that, when the sides are each 20 cm. long, the area is increasing at the rate of $100\sqrt{3}$ sq. mm. per second.

4. If y denotes the deflection of a beam of length l at a distance x from one end, it is found that

$$\frac{d^2y}{dx^2} = A(l - x)^2 + B(l - x),$$

where A, B are constants.

Obtain, by integration, general expressions for $\frac{dy}{dx}$ and y.

If $\frac{dy}{dx} = 0$ and $y = 0$ when $x = 0$, find their values when $x = l$.

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5. Draw the curve $y = 2\sin^3 x$ for values of x between 0 and π, taking 2 inches as unit for both axes.

Express $\sin^3 x$ in terms of $\sin x$ and $\sin 3x$, and thence find by integration the area between the curve and the axis of x.

6. Find the equation of a curve $y = f(x)$, given that

$$\frac{d^2y}{dx^2} = 6x - 4,$$

and that the curve passes through the point (1, 3) at which the gradient is -5.

Determine the co-ordinates of the point on the curve at which the tangent is parallel to the tangent at the point (1, 3), and sketch the curve.

7. Evaluate (a) $\int_1^2 \frac{x^5 + 1}{x^2}\,dx$, (b) $\int_0^{\frac{1}{4}\pi} \sin^2 x\, dx$,

(c) $\int_1^2 \frac{(x^2 - 1)^2}{x^2}\,dx$, (d) $\int_0^{\frac{1}{3}\pi} \sin 2x \cos x\, dx$.

8. A plano-convex lens is in the form obtained by rotating the part of the ellipse $x^2 + 10y^2 = 10$, for which y is positive about the axis of y. Find the volume of the lens and prove that its mean thickness is $\frac{2}{3}$ of the greatest thickness.

9. Sketch the curve $y^2 = 1 - \cos 2x$ for values of x between 0 and π.

A spindle-shaped solid is in the form obtained by rotating the above portion of the curve about the axis of x. Calculate the volume of the solid.

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10. A plane cuts a sphere of radius a into two segments whose curved surfaces are in the ratio 3 : 1.

Find the distance of the plane from the centre of the sphere. Prove that the volume of the larger segment is $\frac{9}{8}\pi a^3$.

11. Show that the curve $y = x^3 - 6x^2 + 9x - 4$ touches the x-axis where $x = 1$, and find the point where the curve intersects the x-axis.

Find also the area enclosed by the curve and the x-axis.

12. Prove that

$$\frac{d}{dx}(x^n) = nx^{n-1},$$

when n is a positive integer.

A solid is formed by the rotation about OY of the part of the curve $y = x^{\frac{1}{3}}$ between $y = 1$ and $y = 8$.

Show that its volume is $93\pi/5$ cubic units.

13. Find the points of intersection of the curves $a^7y = x^8$, $a^7x = y^8$, and calculate the area bounded by the arcs of these two curves between their points of intersection. Find also the volume of the solid generated by rotating about the x-axis the area bounded by the arc of $a^7y = x^8$ between (0, 0) and (a, a) the x-axis and the ordinate $x = a$.

14. Find the area enclosed by the curve $y = 3 + 2x - x^2$ and the x-axis. Show that the line $3x + 2y - 6 = 0$ divides this area into two parts whose ratio is 7 : 9.

15. The gradient of a curve at any point (x, y) is $\frac{1}{2}(x - 3)$, and the curve passes through the point (5, 0).

Find the equation of the curve and the area bounded by the curve and the axis of x.

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16. Find the area contained by the curve $y = 4x^2$ and the line $y = 6x - 2$.

Find the point of contact of the tangent parallel to the given line and determine the ratio of the area found above to the area of the parallelogram formed by the chord, the parallel tangent, and lines through the extremities of the chord parallel to the x-axis.

17. Find the equations of the tangents to the curve

$$y = (x - 1)(x - 3)(x + 4)$$

which are parallel to the line $y + x = 0$.

Determine the whole area between the curve, the axis of x, and the ordinates of the points of contact of these tangents.

18. Prove that the area enclosed by the curve $y = ax^2 + bx + c$, the x-axis, and the ordinates at $x = -h$, $x = h$ is $\frac{1}{3}h(y_1 + 4y_2 + y_3)$, where y_1, y_2, y_3 are the ordinates at $x = -h$, $x = 0$ and $x = h$ respectively.

The ordinates of a curve at eleven equidistant points on the x-axis, the distance between every two consecutive points being unity, are given in the table: 0, 4·2, 4·7, 4·9, 4·7, 5·2, 5·5, 5·7, 4·9, 4, 0.

Use the above result to find approximately the area between the curve and the x-axis.

19. Prove that the volume of a segment of height h cut from a sphere of radius R is $\pi h^2(R - \frac{1}{3}h)$.

A sphere of radius R touches a right circular cone along the circumference of a small circle c.

If the semi-angle at the vertex of the cone is 30°, find the volume of the segment of the sphere bounded by the circle c, and show that the volume of the space between the cone and the sphere is $\frac{1}{6}\pi R^3$.

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20. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is rotated about the x-axis through four right angles. Show that the volume obtained is

$$\int_a^b \pi y^2 \, dx.$$

Find the volume of the portion of a sphere generated by the rotation about the x-axis of the area inside the circle $x^2 + y^2 = 100$ and between the lines $x = -6$, $x = 6$.

What fraction of this volume lies within the cylinder obtained by rotating the line $y = 8$ about the x-axis?

21. (i) Evaluate $\int_1^4 \frac{dx}{(5x + 2)^3}$, $\int_0^{\frac{1}{2}\pi} 2 \sin 3x \cos x \, dx$.

(ii) The part of the curve $x^2y = x^4 + 3$ between the ordinates $x = 1$ and $x = 2$ is rotated about the x-axis. Calculate the volume generated.

22. The inner surface of a vessel is formed by the revolution of the curve $x^2 = y - 2$ about the y-axis, which is vertical, the unit on each axis being 1 inch. Show that, if the vessel contains liquid to a depth of h inches, the volume of liquid is $\frac{1}{2}\pi h^2$ cubic inches.

If liquid be poured into the vessel at a uniform rate of 2 cubic inches per second, find the rate at which the surface of the liquid is rising when its depth is 3 inches.

23. A trapezium $OABC$ is such that O is the origin, A is the point (6, 0) and the equations of the sides OC, CB, BA are respectively $y = x$, $y = 2$, $y = -x + 6$. Find the area of the trapezium.

If the trapezium be rotated about OA, find the area of surface generated and the volume it encloses.

24. Sketch the curve $y = 8 + 2x - x^2$, and find the area between it and the x-axis.

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Find also the co-ordinates of the centroid of the area contained between the curve and the x-axis.

25. Draw the graph of $x = 1 + \frac{1}{2} \sin y$ from $y = 0$ to $y = 2\pi$.

A vase is formed by the rotation about the y-axis of the part of the curve $x = 1 + \frac{1}{2} \sin y$ between $y = 0$ and $y = 2\pi$, the base being formed by the rotation of the x-axis. If x and y be measured in inches, find the volume in cubic inches (correct to one decimal place) of the vase.

26. Prove that the area cut off on the surface of a sphere between two parallel planes is equal to the corresponding area on the circumscribing cylinder whose axis is perpendicular to the planes.

A cone of height h and a hemisphere are on the same side of their common circular base of radius $r (h > r)$. Prove that the area of that part of the surface of the hemisphere which is outside the cone is $4\pi r^3 h/(h^2 + r^2)$.

27. Three spheres each of radius a rest on a horizontal plane with their centres at the vertices of an equilateral triangle of side $2a$. A fourth equal sphere rests symmetrically on top of the other three.

Find the height of the highest point of the fourth sphere above the horizontal plane, and prove that the ratio of the volume of the tetrahedron whose vertices are the centres of the spheres to the volume of a sphere is $1 : \pi\sqrt{2}$.

28. A sphere rests in a horizontal circular hole of radius 4 cm., and the lowest point of the sphere is 2 cm. below the plane of the hole. Calculate the area in square cm. of the part of the surface of the sphere below the hole and the volume in cc. of this part of the sphere, each to three significant figures.

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29. The height of a segment of a sphere of radius a is c. Prove that the volume of the segment is $\frac{1}{3}\pi c^2(3a - c)$.

A plane divides a sphere into two parts whose surface areas are in the ratio 5 : 1. Find the ratio of the volumes of the two parts.

30. A vessel containing water has the form of an inverted hollow pyramid without base. The open top is a square of side 12 inches, the faces are four equal isosceles triangles and the vertex is at a depth 8 inches below the top. A sphere of radius $1\frac{1}{2}$ inches is lowered into the water and is found to be just covered when it makes contact with the four faces.

Find the volume of water in the vessel and its initial depth.

31. The vertical cross-section of a horizontal trough, of length h, is an isosceles triangle with vertex downwards, the angle between the sloping sides being 2α. A cylinder of radius a rests inside the trough with its axis of length h horizontal. If the volume of the trough below the cylinder is half that of the cylinder, prove that $\cot \alpha + \alpha = \pi$.

32. A sphere touches the curved surface and the plane base of a hollow right circular cone, the radius of whose base is 9 inches and whose height is 12 inches.

Find the difference between the volumes of the sphere and the cone.

33. Three spheres, each of radius 6 inches, rest on a horizontal table and each of them touches the other two. A right circular cone of semi-vertical angle 45° is placed with its vertex downwards and its axis vertical so as to rest on all three spheres. Find the height of its vertex above the table.

34. Prove that the area of a zone of a sphere of radius R contained between two parallel circles whose radii subtend angles θ_1, θ_2 at the centre of the sphere is $2\pi R^2(\cos\theta_1 \sim \cos\theta_2)$, and show that, if $\theta_1 \sim \theta_2$ is a small angle, the area is approximately $2\pi R^2(\theta_1 \sim \theta_2)\sin\frac{1}{2}(\theta_1 + \theta_2)$.

The radius of the earth being 3,980 miles, calculate the area of the state of Colorado in North America, which is bounded by the meridians 102°, 109° of west longitude and the parallels of 37°, 41° north latitude.

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35. A sphere of radius a is cut by a plane at a distance b from the centre, and radii of the sphere are drawn through all the points of section. Show that the sphere is divided into two sectors whose areas are in the ratio $(a - b) : (a + b)$.

Through a solid sphere of radius 5 inches is bored a hole in the form of a frustum of a cone, whose axis passes through the centre of the sphere.

The circles at the two ends of the boring are of radii 3 inches and 4 inches respectively. Calculate the volume of material removed from the sphere.

36. Prove that the volume of a segment of height h cut from a sphere of radius R is $\pi h^2(R - \frac{1}{3}h)$.

A sphere of radius R touches a right circular cone along the circumference of a small circle c. If the semi-vertical angle of the cone is 30°, find the volume of the segment of the sphere bounded by the circle c, and show that the volume of the space between the cone and the sphere is $\frac{1}{6}\pi R^3$.

37. Prove that the volume of a sphere is two-thirds that of the enclosing cylinder.

The surface of a solid is formed by a segment of a sphere of radius a greater than a hemisphere, and by a cone whose generating lines are tangential to the sphere. The base of the cone is the plane face of the segment. If the vertical angle of the cone be 60°, find the volume of the solid.

38. PA, PB are tangents to a circle centre O and radius R. PO meets the circle in C and AB in D. If $PC = h$, and h is small compared with R, prove that PA is approximately equal to $\sqrt{2Rh}$ and CD is approximately equal to h.

Find the distance of the horizon and the surface area of the earth visible

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from a point half a mile above the earth assuming it to be a sphere of radius 3,960 miles.

39. A hut stands on a rectangular base measuring 24 feet by 10 feet. The sides and ends are rectangular and of height 8 feet, and the roof has four faces each making an angle of 35° with the horizontal. Find the cubic content of the hut and the area of the roof.

40. ABC is an equilateral triangle of side a. A circle is drawn through A, B, and C, and another circle with centre C passing through A and B.

Find the area common to both circles.

41. A column is composed of six solid circular cylinders which are such that the linear dimensions of each one are three-quarters of the corresponding dimensions of the one on which it rests. If the greatest cylinder be of radius 3 feet and height 4 feet, find the total volume of the cylinders and express this volume as a fraction of the volume of a cone of radius 4 feet and height 16 feet.

If the number of cylinders in the column be n, find the limiting value of the above fraction as n increases indefinitely.

42. Equilateral triangles DBC and EBC are drawn on opposite sides of the base BC of a triangle ABC.

Prove that $AD^2 + AE^2 = a^2 + b^2 + c^2$.

Show that the area common to the circles through B and C, having centres at D and E respectively, is $\frac{1}{6}a^2(2\pi - 3\sqrt{3})$.

43. Prove that the volume of a segment of height h cut from a sphere of radius R is $\pi h^2(R - \frac{1}{3}h)$.

A regular tetrahedron is inscribed in a sphere of radius R.

Prove that (i) the height of the tetrahedron is $\frac{4}{3}R$, (ii) the plane of the base of the tetrahedron cuts the sphere into segments having volumes in the ratio 7 : 20.

44. Obtain an expression for the area of the smaller portion of the surface of a sphere of radius r cut off by a plane at a distance d from its centre.

Assuming that the earth is a sphere of radius 3,960 miles, show that the area of the portion visible to an airman at a height of one mile is about 24,900 square miles.

45. Prove that the volume of a right circular cone is equal to one-third of the product of the area of the base and the height.

A right circular cone of height 3 inches is lowered with its vertex downwards into a cylinder of the same radius as the cone containing water. Find the difference of the water levels when the vertex and the base of the cone are just in the surface of the water.

46. An open conical cup whose height is 12 inches and the radius of whose base is 5 inches, is placed with its axis vertical and vertex downwards. A sphere of radius 13 inches rests symmetrically on the cup. Find the area of the surface of the part of the sphere inside the cup, and the volume contained between the sphere and the cup.

47. Three solid spheres of radii $2a$, a, a, are placed inside a right circular cylinder of radius $2a$, whose length is such that each sphere touches the other two and touches one of the plane circular ends of the cylinder. Find the volume of the cylinder.

48. Show that the curved surface area of a segment of height h cut from a sphere of radius r is $2\pi rh$, and hence, or otherwise, that the volume of a sector of a sphere formed by the addition of a cone to the segment is $\frac{2}{3}\pi r^2 h$.

If the semi-vertical angle of the cone is 60°, find the ratio of the volume of he cone to that of the segment.

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49. Obtain an expression for the volume of a right circular cone in terms of its height and the radius of its base.

From one end of a right circular cylinder of height h and radius a ($< h$), a hemispherical portion of radius a is removed; from the other end is removed a right circular cone, whose base is that end of the cylinder which just reaches the hemisphere. Find the volume of the remaining solid.

50. OA is a fixed radius of a sphere, OP is a variable radius which makes a constant angle with OA.

Prove that the locus of P is (a) a plane curve, (b) a circle.

Prove that the difference between the area of the curved surface of a segment of a sphere and the area of its plane base is equal to the area of a circle whose radius is the height of the segment.

51. The diameters of the ends of a frustum of a cone being a and b, and the length of the slant side L, prove that the surface area is $\frac{1}{2}\pi L(a + b)$.

A sphere is placed in contact with the inner surface of a right circular cone, and tangent planes to the sphere are drawn perpendicular to the axis of the cone.

Prove that the area intercepted on the cone between the planes bears to the surface of the sphere the ratio $\sec^2 \alpha : 1$, 2α being the vertical angle of the cone.

52. One of the states of North America is defined by two parallels of latitude l_1, l_2 and meridians of longitude L_1, L_2. Prove that the area is

$$R^2(L_1 - L_2)(\sin l_1 - \sin l_2),$$

where R is the earth's radius.

Calculate the area, taking the parallels to be 41° and 45°, the meridians 104° and 111°, and the earth's radius 3,960 miles.

53. A regular tetrahedron is made with each edge 1 foot in length. Calculate the angle between two adjacent faces, and prove that the radius of the sphere which circumscribes the tetrahedron is about 7·35 inches.

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54. A right circular cone is inscribed in a sphere. If V be the volume of the sphere and A its surface area, and U is the volume of the cone, and B the area of its curved surface, prove that

$$\frac{U}{V} = 2\left(\frac{B}{A}\right)^2.$$

55. A sheet of paper is in the form of an isosceles triangle, whose vertical angle is 120° and whose equal sides are 12 inches long. A sector of a circle is cut from the sheet, the centre of the circle being at the vertex where the equal sides meet, and the sector is formed into a cone. Determine the cubical content of the cone, when the radius of the sector is as large as possible.

It is required to draw a line on the surface of the cone, starting from a point of the base, passing around the cone and returning to the starting-point. What is the length of the shortest line of this sort that can be drawn?

56. A sphere of radius R is cut by a plane whose perpendicular distance from the centre of the sphere is x. Prove that the area of the smaller portion of surface cut off is $2\pi R(R - x)$.

Show that, if a pair of compasses with equal legs is opened to a given angle and is used to draw a circle on the surface of a sphere, the area of the smaller portion of the sphere bounded by this circle is independent of the radius of the sphere.

57. A pyramid is formed by drawing planes through the sides of a horizontal rectangle, meeting at a point vertically over one corner of the rectangle and distant 10 cm. from it. If the sides of the rectangle are 5 cm. and 10 cm. long,

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determine the area of each triangular face, and find the inclinations of the sloping faces to the horizontal.

58. A triangular prism *ABC, PQR* has parallel faces *ABC, PQR* and parallel edges *AP, BQ, CR*. Prove that the volume of the tetrahedron *APQR* is one-third the volume of the prism.

A right circular cone has its vertex at *O*. Two points *A, B* are taken on the circumference of the base of the cone so that the minor arc *AB* is a quadrant of a circle. Prove that the plane *OAB* divides the volume of the cone approximately in the ratio 1 : 10.

59. A sphere of radius a is cut by a plane whose distance from the centre of the sphere is $(a - x)$, where x is positive but less than a. Write down (without proof) the expression in terms of a and x for the area of the smaller portion of the surface cut off.

Assuming the earth to be a sphere of radius 4,000 miles, prove that the area of the portion which is visible to an observer at a height of h feet above the surface is about $\frac{1}{4}(19h)$ square miles.

60. Show that one sphere, and only one, can pass through four given points which do not lie in a plane.

Show that the radius of the sphere circumscribing a regular tetrahedron, each edge of which is $2a$, is $\frac{1}{2}a\sqrt{2}$.

61. The curved surface of a bucket is part of the surface of a right circular cone. The upper rim of the bucket is a circle of diameter 12 inches and its vertical height is 10 inches, whilst its base is a circle of diameter 8 inches, where all three are internal measurements. Prove that the capacity of the bucket is nearly 796 cubic inches.

(If any formula for the volume of a frustum of a cone be used, a proof of the correctness of the formula must be given.)

62. Find the area of the surface of a sphere.

An observer is at a distance of 37 feet from the centre of a sphere of diameter 24 feet. Find the area of that part of the spherical surface which is seen by the observer.

63. A sphere of radius a is cut by a plane at distance $(a - x)$ from the centre. Prove that the volume of the smaller segment formed is $\frac{1}{3}\pi x^2(3a - x)$. (It may be assumed that the area of spherical surface cut off by the plane is $2\pi ax$.)

A hole of radius 3 inches is bored symmetrically through a solid sphere of radius 5 inches. Find the volume of the remaining portion of the sphere.

64. Show that the section of a sphere made by a plane is a circle.

A plane cuts two concentric spheres. Show that the area intercepted on the plane is the same for all positions of the plane.

65. Prove that the volume of a tetrahedron *ABCD* is $\frac{1}{3}\Delta p$, where Δ is the area of the triangle *ABC* and p is the perpendicular distance of *D* from the plane *ABC*.

A regular tetrahedron *ABCD* is cut into two parts by a plane through *D* parallel to *BC* and inclined at an angle $\sin^{-1}(\frac{1}{3})$ to the perpendicular from *D* to the base *ABC* and on the same side of this perpendicular as *DA*. Show that the volumes of the parts are in the ratio of 1 : 8.

66. A hollow cone with its axis vertical and vertex downwards has a semi-vertical angle of 30°.

A sphere of diameter $2\frac{1}{2}$ inches is dropped into the cone and touches it along a horizontal circle.

Find (i) the radius of this circle, (ii) the distance of its centre from the vertex of the cone, and (iii) the part of the volume of the cone below this circle and included between the sphere and the cone.

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67. A segment of a sphere is of height h and the radius of its circular face is r. Prove that its volume is

$$\frac{\pi h}{6}(3r^2 + h^2),$$

and prove also that this volume may be expressed in the form

$$\frac{\pi h^2}{3}(3R - h),$$

where R is the radius of the sphere.

When a balloon is at a height h from the ground, prove that the fraction of the earth's surface which can be seen from it is $h/2(R + h)$, where R is the radius of the earth.

68. (i) OA, OB, OC are three mutually perpendicular straight lines whose lengths are a, b, c respectively.

Prove that the radius of the sphere which passes through the four points O, A, B, C is $\frac{1}{2}\sqrt{(a^2 + b^2 + c^2)}$.

(ii) A point P is taken inside a cube, the length of an edge of which is a. Prove that the sum of the squares of the distances of P from each of the eight corners of the cube is $6a^2 + 8d^2$, where d is the distance of P from the point of intersection of the diagonals of the cube.

69. Prove that the radius of the sphere, which circumscribes a right circular cone of height h and vertical angle 2α, is $\frac{1}{2}h \sec^2 \alpha$.

Prove that the area of the segment of the sphere which contains the cone is $\pi h^2 \sec^2 \alpha$, and that the volume of this segment is

$$\frac{\pi}{6}h^3(1 + 3\tan^2 \alpha).$$

70. A sphere, radius a, is cut into two by a plane distant h from its centre ($h < a$). Find the area of the surface of the smaller part of the sphere.

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Two intersecting spheres, whose radii are 6 inches and 7 inches, have their centres 9 inches apart. Find what proportion of the surface of each sphere is inside the other.

71. Find the radius of the sphere which passes through the corners of a tetrahedron $OABC$, in which the triangle ABC is equilateral with sides each 10 units in length, while the lengths of the three edges passing through O are each 14 units.

72. A right circular cone and a circular cylinder of equal volume are described on the same circular base of radius a, and on the same side of it, the height of the cylinder being $3a$.

Find (i) the semi-vertical angle of the cone,
(ii) the area of the curved surface of the frustum of the cone within the cylinder,
(iii) the volume contained within the cylinder and outside the cone.

73. Show that the volume of the smaller segment cut off a sphere of radius a by a plane distant c from the centre of the sphere is $\frac{1}{3}\pi(a - c)^2(2a + c)$.

Find the ratio in which the volume of a sphere is divided by a plane that divides the surface area in the ratio 5 : 1.

74. The base of a pyramid is a regular hexagon $ABCDEF$ of side a. V is the vertex and each edge passing through V is of length $2a$. Find the volume of the pyramid and the angle between the faces VBA, VBC.

75. A given circle is intersected by part of the arc of an equal circle in the points XY. Prove that the crescent-shaped area cut off from the given circle by this arc is equal to the difference of the areas into which the chord XY divides the circle.

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Find this area when the radius of the circle is a and the centres of the arc and circle are $2x$ apart.

76. Prove that the volume of the frustum of a right circular cone is $\frac{1}{3}\pi h(a^2 + ab + b^2)$, where a, b are the radii of the plane ends, and h is the height of the frustum.

The volume of a frustum of a cone is 125 cubic inches and the radii of the plane ends are 2 and 4 inches; find the total superficial area.

77. A sphere is inscribed in a right circular cone of which the semi-vertical angle is $\sin^{-1} x$.

Prove that the ratio of the volume of the sphere to that of the cone is $4x(1 - x)(1 + x)^{-2}$, and show that the possible values of this ratio range from 0 to $\frac{1}{2}$.

78. Prove that the area of the portion of the surface of a sphere of radius r intercepted between two parallel planes distant a and b from the centre is $2\pi r(a \pm b)$, distinguishing between the two cases.

A mound is in the form of a spherical dome, of height h, and base radius a. Prove that the area of its curved surface is $\pi(a^2 + h^2)$ and that its volume is $\frac{1}{6}\pi h(3a^2 + h^2)$.

Deduce expressions for the surface area and volume of a complete sphere.

79. Show that, if the cross-section of a prism is a triangle, any portion of the prism contained between two parallel planes may be divided into three tetrahedra whose volumes are equal.

Find the volume of a regular tetrahedron whose edges are each of length $2a$.

80. Prove that the area of the curved surface of a spherical cap is $2\pi Rh$, where R is the radius of the sphere and h the height of the cap.

Show that, if a right circular cone of semi-vertical angle α is inscribed in a sphere, the area of the sphere is divided by the base of the cone into two parts whose ratio is $\tan^2 \alpha$.

81. Find the following integrals:

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(i) $\int \frac{du}{3e^{0 \cdot 3u}}$, (ii) $\int \frac{1 + e^{6\theta}}{e^{3\theta}} d\theta$,

(iii) $\int \frac{e^{5x}}{\sqrt{e^{3x}}} dx$, (iv) $\int \frac{2}{1 - 3y} dy$,

(v) $\int (2 + 3y)^{-1} dy$, (vi) $\int_0^1 \frac{dv}{1 + 2v}$.

82. Using tables where necessary, calculate the value of

$$\int_{0 \cdot 1}^{0 \cdot 5} e^{-x}\, dx,$$

(i) by direct integration,
(ii) by Simpson's Rule, using 5 ordinates spaced at intervals of 1/10 unit.
(Give your answers to four places of decimals.)

83. Using Simpson's Rule with four intervals calculate

$$\int_0^4 e^{x^2/50}\, dx$$

working with four places of decimals throughout.

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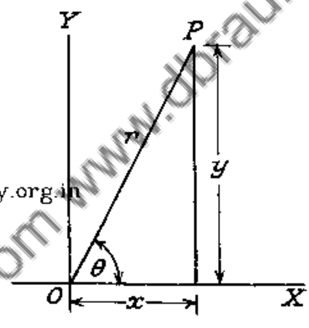
CHAPTER XII

Polar Co-ordinates; Logarithmic, Sine and Cosine Series; Coaxal Circles; Determinants

Polar Co-ordinates. With the usual axes of co-ordinates let P be the point (x, y) and let the line OP be of length r and make an angle θ with OX. Then r and θ are known as the *polar co-ordinates* of the point P, and we write $P \equiv (r, \theta)$, where r is always taken positive.

The diagram shows P taken in the first quadrant, but the above holds good when P is in any quadrant.



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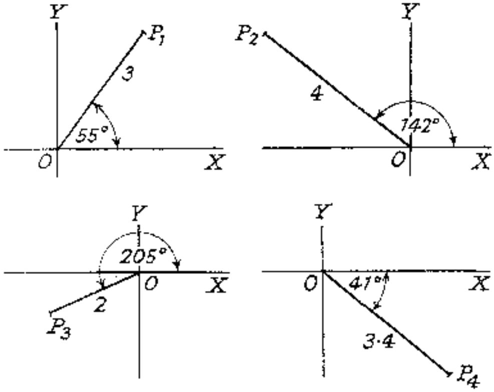
From the diagram it can be seen that $x = r\cos\theta$, $y = r\sin\theta$. Squaring and adding these the result is

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

It can be seen from what has been said that (3, 55°) are the polar co-ordinates of a point P_1 three units distant from O such that OP_1 makes an angle 55° with OX; (4, 142°) are the polar co-ordinates of a point P_2 at a distance four units from O with OP_2 making an angle 142° with OX; (2, 205°) are the polar co-ordinates of a point P_3 two units distant from O such that OP_3 makes an angle of 205° with OX; (3·4, −41°) is a point P_4 at a distance 3·4 units from O with OP_4 making an angle of −41° with OX.

The points P_1, P_2, P_3, P_4 are shown in the following diagrams.

If a curve be given in the form $r = f(\theta)$, or $F(r, \theta) = 0$ the equation is known as the *polar equation* of the curve. In order to draw the



curve a table of values of r corresponding to various values of θ, such as 0°, 30°, 45°, etc., is formed and the points are then plotted on graph paper and joined up by a smooth curve.

EXAMPLE. Draw the following curves:

$$\text{(i) } r = 3(1 + \cos\theta), \quad \text{(ii) } r = 4\sin 2\theta.$$

(i) The following is the table of corresponding values of r and θ, and the curve is shown.

θ	0°	30°	45°	60°	90°	120°	135°	150°
r	6	5·6	5·1	4·5	3	1·5	0·9	0·4

180°	210°	225°	240°	270°	300°	315°	330°	360°
0	0·4	0·9	1·5	3	4·5	5·1	5·6	6

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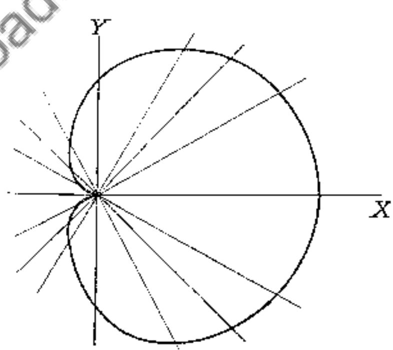

Owing to its heart-shaped appearance this curve is known as a *cardioid*.

(ii) The following is the table of values connecting θ and r, and the curve is drawn from it.

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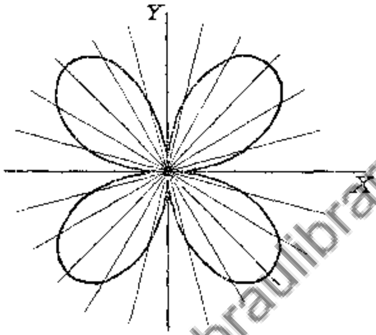
θ	$0°$	$15°$	$30°$	$45°$	$60°$	$75°$	$90°$
r	0	2	3·5	4	3·5	2	0

$105°$	$120°$	$135°$	$150°$	$165°$	$180°$	etc.
-2	$-3·5$	-4	$-3·5$	-2	0	



Conversion of Cartesian Equation to polar form and vice versa. It has been shown earlier that $x = r\cos\theta$ and $y = r\sin\theta$, and hence in order to convert the Cartesian equation (x, y relationship) to the polar form, x is replaced by $r\cos\theta$ and y by $r\sin\theta$.

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Thus the equation $ax + by = c$ of the straight line becomes $ar\cos\theta + br\sin\theta = c$ in polar form which can be written

$$r(a\cos\theta + b\sin\theta) = c$$

$$\text{i.e. } r\sin(\theta + \alpha) = c/\sqrt{(a^2 + b^2)},$$

where $a/\sqrt{(a^2 + b^2)} = \sin\alpha$ and $b/\sqrt{(a^2 + b^2)} = \cos\alpha$, and α is a constant.

Thus a polar equation of the form $r\sin(\theta + \alpha) =$ a constant represents a straight line. Similarly it can be shown that the equation $r\cos(\theta + \alpha) =$ a constant represents a straight line in the polar form.

EXAMPLE. Find the polar equations of

(i) the ellipse $(x^2/a^2) + (y^2/b^2) = 1$,
(ii) the circle $x^2 + y^2 + 2gx + 2fy + c = 0$,
(iii) the parabola $y^2 = 4ax$.

NOTE. In each case x is replaced by $r\cos\theta$ and y by $r\sin\theta$.

(i) The polar equation of the given ellipse is

$$\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1,$$

$$\text{i.e. } r^2(b^2\cos^2\theta + a^2\sin^2\theta) = a^2b^2.$$

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(ii) The polar equation of the given circle is

$$r^2\cos^2\theta + r^2\sin^2\theta + 2gr\cos\theta + 2fr\sin\theta + c = 0,$$
$$\text{i.e. } r^2(\cos^2\theta + \sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0,$$
$$\text{i.e. } r^2 + 2r(g\cos\theta + f\sin\theta) + c = 0.$$

(iii) The polar equation of the parabola $y^2 = 4ax$ is

$$r^2\sin^2\theta = 4ar\cos\theta,$$
$$\text{i.e. } r\sin^2\theta = 4a\cos\theta, \qquad \text{since } r \neq 0.$$

To convert the polar form of the equation of a curve into its Cartesian equation, $r\cos\theta$ is replaced by x and $r\sin\theta$ by y. Also

$$r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2,$$
$$\text{i.e. } r^2 = x^2 + y^2.$$

Hence r^2 can be replaced by $x^2 + y^2$ wherever it occurs.

EXAMPLE. Find the Cartesian equations of the following curves represented in the polar form by

(i) $r = a(2 + \sin\theta)$, (ii) $r = a\cos 2\theta$, (iii) $r^2 = 3\cos\theta + 2\sin\theta$.

(i)
$$r = a(2 + \sin\theta)$$
$$\therefore\ r^2 = a(2r + r\sin\theta)$$
$$= 2ar + ay, \qquad (r\sin\theta = y)$$
$$\text{i.e. } x^2 + y^2 = 2ar + ay, \qquad (r^2 = x^2 + y^2)$$
$$\therefore\ x^2 + y^2 - ay = 2ar$$
$$\therefore\ (x^2 + y^2 - ay)^2 = 4a^2r^2,$$
$$\text{i.e. } (x^2 + y^2 - ay)^2 = 4a^2(x^2 + y^2).$$

(ii)
$$r = a\cos 2\theta = a(\cos^2\theta - \sin^2\theta)$$
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$$\therefore\ r^3 = a(r^2\cos^2\theta - r^2\sin^2\theta)$$
$$= a(x^2 - y^2)$$
$$\therefore\ r^6 = a^2(x^2 - y^2)^2.$$

But $r^2 = x^2 + y^2$ $\therefore$ $r^6 = (x^2 + y^2)^3$, and the Cartesian equation is

$$(x^2 + y^2)^3 = a^2(x^2 - y^2)^2.$$

(iii)
$$r^2 = 3\cos\theta + 2\sin\theta$$
$$\therefore\ r^3 = 3r\cos\theta + 2r\sin\theta = 3x + 2y,$$
squaring $r^6 = (3x + 2y)^2$.

But $r^6 = (x^2 + y^2)^3$ as in (ii) $\therefore$ the Cartesian equation is

$$(x^2 + y^2)^3 = (3x + 2y)^2.$$

EXAMPLE (G.C.E.)

(*a*) Prove that $r = 4\sec(\theta - 30°)$ is the equation of a straight line in polar co-ordinates. State the polar co-ordinates of the foot of the perpendicular on the line from the pole.

(*b*) A point P moves so that its distance from the pole is equal to its perpendicular distance from the straight line $r = 2a\sec\theta$. Show that the polar equation of the locus is $r = a\sec^2\frac{1}{2}\theta$.

(*c*) Eliminate r and θ from the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r(1 + \cos\theta) = 2.$$

(*a*) The equation $r = 4\sec(\theta - 30°)$ is the same as

$$r\cos(\theta - 30°) = 4, \qquad \ldots\ldots\ldots\ldots (1)$$

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$$\text{i.e. } r(\cos\theta\cos 30^\circ + \sin\theta\sin 30^\circ) = 4,$$

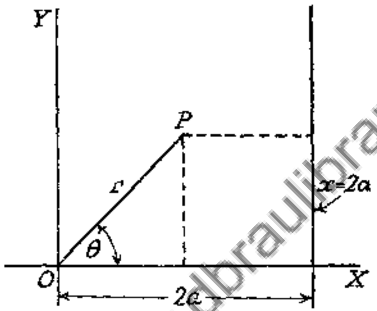
$$\text{i.e. } \frac{\sqrt{3}}{2}r\cos\theta + \frac{1}{2}r\sin\theta = 4.$$

But $r\cos\theta = x$ and $r\sin\theta = y$, $\therefore$ the Cartesian equation is

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4,$$

$$\text{i.e. } x\sqrt{3} + y = 8,$$

which is the equation of a straight line.



From the equation it can be seen that the foot of the perpendicular from the pole on the line has polar co-ordinates $r = 4$, $\theta = 30^\circ$.

(*b*) $P \equiv (r, \theta)$ is one position of P.

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The line $r = 2a\sec\theta$ can be written $r\cos\theta = 2a$. But $r\cos\theta = x$, therefore the Cartesian equation of the line is $x = 2a$.

The projection of OP on OX is $r\cos\theta$, hence the distance of P from the line $x = 2a$ (i.e. $r = 2a\sec\theta$) is

$$2a - r\cos\theta.$$

But this is given equal to r therefore the locus of P is

$$r = 2a - r\cos\theta,$$

$$\text{i.e. } r(1 + \cos\theta) = 2a,$$

$$\text{i.e. } 2r\cos^2\tfrac{1}{2}\theta = 2a,$$

$$\text{i.e. } r = a\sec^2\tfrac{1}{2}\theta.$$

(*c*)

$$r(1 + \cos\theta) = 2$$

$$\therefore r + r\cos\theta = 2,$$

$$\text{i.e. } r + x = 2, \qquad (\text{since } r\cos\theta = x)$$

$$\therefore r = 2 - x.$$

$$\text{Squaring } r^2 = (2 - x)^2$$

$$\text{i.e. } x^2 + y^2 = (2 - x)^2, \qquad (r^2 = x^2 + y^2)$$

$$= 4 - 4x + x^2,$$

$$\text{i.e. } y^2 = 4 - 4x,$$

$$\therefore y^2 = 4(1 - x).$$

Series and Approximations. The following series, for which no proofs are given, should be memorized.

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$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \text{ for } |x| < 1.$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots, \text{ for } |x| < 1.$$

$$\left.\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \end{aligned}\right\} x \text{ in radians.}$$

When x is small compared with unity it is clear that the following approximations hold good.

$\log_e(1+x)$	First approximation	x.
	Second approximation	$x - x^2/2$.
	Third approximation	$x - x^2/2 + x^3/6$, and so on.
$\log_e(1-x)$	First approximation	$-x$.
	Second approximation	$-x - x^2/2$.
	Third approximation	$-x - x^2/2 - x^3/6$, and so on.
$\sin x$	First approximation	x.
	Second approximation	$x - x^3/3!$.
	Third approximation	$x - x^3/3! + x^5/5!$ and so on.
$\cos x$	First approximation	1.
	Second approximation	$1 - x^2/2!$
	Third approximation	$1 - x^2/2! + x^4/4!$ and so on.

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NOTE. The approximation taken in any particular example is dependent upon the degree of accuracy required in the question.

From the logarithmic series we have, for $|x| < 1$,

$$\begin{aligned} \log_e(1+x) - \log_e(1-x) &= (x - x^2/2 + x^3/3 - x^4/4 + \ldots) \\ &\quad -(-x - x^2/2 - x^3/3 - x^4/4 - \ldots) \\ &= 2x + 2x^3/3 + 2x^5/5 + \ldots \end{aligned}$$

$$\therefore \log_e(1+x)/(1-x) = 2(x + x^3/3 + x^5/5 + \ldots).$$

Theorem. *To find a series for* $\log_e(n+1)/n$ *in terms of* $1/(2n+1)$, *where* $n > 0$.

Now $\log_e(1+x)/(1-x) = 2(x + x^3/3 + x^5/5 + \ldots)$.

Using $x = 1/(2n+1)$ in this it follows that

$$\log_e \frac{1 + 1/(2n+1)}{1 - 1/(2n+1)} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\}$$

$$\text{But the L.H.S.} = \log_e \frac{2n+1+1}{2n+1-1} = \log_e \frac{2n+2}{2n}$$

$$= \log_e \frac{n+1}{n},$$

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$$\therefore \log_e \frac{n+1}{n} = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\right\}$$

EXAMPLE. Find $\log_e 2$ to four decimal places.

Now $\log_e \dfrac{n+1}{n} = 2\left\{\dfrac{1}{2n+1} + \dfrac{1}{3(2n+1)^3} + \dfrac{1}{5(2n+1)^5} + \ldots\right\}$

Using $n = 1$ in this

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3^4} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \ldots\right\}$$

Since the result is required to four decimal places it is necessary to take the quantity inside the bracket to six decimal places owing to the factor 2 outside the bracket.

$\therefore \log_e 2 = 2 \times 0{\cdot}346573$
$= 0{\cdot}6931$ to four decimal places.

$$\frac{1}{3} = 0{\cdot}333333$$
$$\frac{1}{81} = 0{\cdot}012346$$
$$\frac{1}{5 \times 3^5} = \frac{1}{15} \times \frac{1}{81} = 0{\cdot}000823$$
$$\frac{1}{7 \cdot 3^7} = \frac{5}{63} \times \frac{1}{5 \times 3^5} = 0{\cdot}000065$$
$$\frac{1}{9 \times 9^9} = \frac{7}{81} \times \frac{1}{7 \times 3^7} = 0{\cdot}000006$$
$$0{\cdot}346573$$

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Exponential Series. It was proved on p. 293 that

$$e^x = 1 + x + x^2/2! + x^3/3! + \ldots$$

and if x be replaced by $-x$ in this we obtain

$$e^{-x} = 1 - x + x^2/2! - x^3/3! + \ldots$$

When x is small the following are the approximations obtained.

e^x	First approximation	1.
	Second approximation	$1 + x$.
	Third approximation	$1 + x + x^2/2!$, etc.
e^{-x}	First approximation	1.
	Second approximation	$1 - x$.
	Third approximation	$1 - x + x^2/2!$, etc.

EXAMPLE. (i) If a and b are positive and $(a - b)/(a + b) = x$, prove that $a/b = 1 + 2(x + x^2 + x^3 + \ldots)$.

Prove further that, if x be so small that powers of x above the third be neglected, then

$$e^{a/b} = e\left(1 + 2x + 4x^2 + \frac{22}{3}x^3\right).$$

(ii) Write down the first six terms of each of the expansions for $\log_e (1 + x)$ and $\log_e (1 + x^3)$ in ascending powers of x. Deduce the

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expansion for $\log_e(1 - x + x^2)$ in ascending powers of x as far as the term in x^6.

$$(a - b)/(a + b) = x$$
$$\therefore (a - b) = x(a + b),$$
$$= ax + bx$$
$$\therefore a - ax = b + bx,$$
$$\text{i.e. } a(1 - x) = b(1 + x).$$

Hence,

$$a/b = (1 + x)/(1 - x)$$
$$= (1 - x + 2x)/(1 - x)$$
$$= 1 + 2x/(1 - x)$$
$$= 1 + 2x(1 - x)^{-1}$$
$$= 1 + 2x(1 + x + x^2 + \ldots \to \infty)$$

(using the Binomial Theorem)

Neglecting the powers of x above the third throughout

$$e^{a/b} = e^{1+2x(1+x+x^2)} = e \cdot e^{2x+2x^2+2x^3}.$$

Now $\qquad e^y = 1 + y + y^2/2! + y^3/3! + \ldots,$

$$\therefore e^{2x+2x^2+2x^3} = 1 + (2x + 2x^2 + 2x^3) + \frac{1}{2!}(2x + 2x^2 + 2x^3)^2 + \frac{1}{3!}(2x + 2x^2 + 2x^3)^3$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^2(1 + x + x^2)^2 + \frac{4}{3}x^3(1 + x + x^2)^3$$

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^2(1 + 2x) + 4x^3/3$$

(neglecting x^4 etc.)

$$= 1 + 2x + 2x^2 + 2x^3 + 2x^2 + 4x^3 + \frac{4}{3}x^3$$

$$= 1 + 2x + 4x^2 + \frac{22}{3}x^3$$

$$\therefore e^{a/b} = e\left(1 + 2x + 4x^2 + \frac{22}{3}x^3\right)$$

(ii) $\log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6$ to six terms.

Replacing x by x^3 in the above

$$\log_e(1 + x^3) = x^3 - x^6/2 + x^9/3 - x^{12}/4 + x^{15}/5 - x^{18}/6 \text{ to six terms.}$$

$$\log_e(1 + x^3) - \log_e(1 + x) = \log_e(1 + x^3)/(1 + x)$$
$$= \log_e(1 - x + x^2)(1 + x)/(1 + x)$$
$$= \log_e(1 - x + x^2).$$

But $\quad \log_e(1 + x^3) - \log_e(1 + x)$

$$= \left(x^3 - \frac{1}{2}x^6 + \ldots\right) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\right)$$

$$= -x + \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3} \text{ as far as the term in } x^6.$$

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Hence, as far as the term in x^6,

$$\log_e (1 - x + x^2) = -x + \frac{x^2}{2} + 2\frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{3}.$$

EXAMPLE. Obtain the expansion of $\cos 2x - 2 \sin x - \log_e (1 - 2x)$ in ascending powers of x as far as the term in x^3.

$$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$$

$$\therefore \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots$$

$$= 1 - 2x^2 \text{ neglecting } x^4 \text{ etc.}$$

$$\log_e (1 - y) = -y - y^2/2 - y^3/3 - \ldots$$

$$\therefore \log_e (1 - 2x) = -(2x) - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \ldots$$

$$= -2x - 2x^2 - \frac{8}{3}x^3,$$

as far as the term in x^3.

$$\sin x = x - \frac{x^3}{3!} + \ldots$$

Thus, neglecting x^4 and higher powers of x,
$\cos 2x - 2 \sin x - \log_e (1 - 2x)$

$$= 1 - 2x^2 - 2\left(x - \frac{x^3}{6} - \right) - \left(-2x - 2x^2 - \frac{8}{3}x^3\right)$$

$$= 1 - 2x^2 - 2x + \frac{x^3}{3} + 2x + 2x^2 + \frac{8}{3}x^3$$

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$$= 1 + 3x^3.$$

Radical Axis of Two Circles and Coaxal Circles.

Theorem. *If two circles be represented by $S_1 = 0$ and $S_2 = 0$,*

where $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1,$
and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2,$

to find what the equation $S_1 - S_2 = 0$ represents.

Using the values of S_1 and S_2, the equation $S_1 - S_2 = 0$ can be written

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1$$
$$- (x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$$
$$\text{i.e. } 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0 \ldots\ldots\ldots(1),$$

which is the equation of a straight line.

Now a point of intersection of the circles $S_1 = 0$ and $S_2 = 0$ will also satisfy the equation $S_1 - S_2 = 0$. Hence the equation $S_1 - S_2 = 0$ will pass through the points of intersection (real, coincident, or imaginary) of the given circle.

Thus the line (1) is a *straight line passing through the points (real, coincident, or imaginary) of the given circles.*

This straight line is known as the *radical axis* of the circles $S_1 = 0$

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and $S_2 = 0$, and is their common chord when they intersect in real points and their common tangent when they touch.

Using the equation $S_1 = S_2$ it can be seen that the tangents from any point on the radical axis to the two circles $S_1 = 0$ and $S_2 = 0$ are equal, since S_1 represents the square of the tangent from the point (x, y) to the circle $S_1 = 0$, and S_2 represents the square of the tangent from (x, y) to the circle $S_2 = 0$.

Hence the radical axis can be defined in either of the following ways:

(i) The straight line passing through the points of intersection (real, coincident, or imaginary) of the two given circles.

(ii) The locus of points from which the tangents to the two given circles are equal.

Theorem. *To prove that the radical axis of the two circles $S_1 = 0$ and $S_2 = 0$ given in the previous theorem is perpendicular to their line of centres.*

The radical axis of $S_1 = 0$ and $S_2 = 0$ has for its equation

$$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0,$$

by the previous theorem, and its slope is $-(g_1 - g_2)/(f_1 - f_2)$.

The centres of the circles $S_1 = 0$ and $S_2 = 0$ are $(-g_1, -f_1)$ $(-g_2, -f_2)$ respectively, and therefore the slope of the line of centres is $(f_1 - f_2)/(g_1 - g_2)$. The product of the two slopes is -1, and hence the *radical axis is perpendicular to the line of centres.*

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Theorem. *To prove that the three radical axes of the circles $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, where*

$$S_1 \equiv (x^2 + y^2 + 2g_1x + 2f_1y + c_1)$$
$$S_2 \equiv (x^2 + y^2 + 2g_2x + 2f_2y + c_2),$$
$$S_3 \equiv (x^2 + y^2 + 2g_3x + 2f_3y + c_3),$$

are concurrent.

The radical axis of $S_1 = 0$ and $S_2 = 0$ is

$$S_1 - S_2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the radical axis of $S_2 = 0$ and $S_3 = 0$ is

$$S_2 - S_3 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Taking (1) + (2), an equation of a line passing through the point of intersection of the lines (1) and (2) is obtained, and this equation is $S_1 - S_3 = 0$ which is the equation of the radical axis of the circles $S_1 = 0$ and $S_3 = 0$.

Thus *the three radical axes are concurrent.* The point in which these three radical axes meet is known as the *radical centre.*

Theorem. *To find the equation of a system of circles every pair of which has the same radical axis.*

Let the y-axis be chosen as the common radical axis of the system of circles, and the x-axis be the common line of centres.

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Two circles of the system will be

$$x^2 + y^2 + 2g_1x + c_1 = 0 \quad \text{.............(1)},$$

and

$$x^2 + y^2 + 2g_2x + c_2 = 0 \quad \text{.............(2)},$$

since the centres lie on the x-axis.

(1) − (2) gives for the radical axis $2(g_1 - g_2)x + c_1 - c_2 = 0$, and since the radical axis is $x = 0$, it must follow that $c_1 = c_2 = c$ (say).

Thus the required equation is

$$x^2 + y^2 + 2gx + c = 0 \quad \text{.............(3)},$$

where g is arbitrary and c is constant.

A system of circles having the same radical axis for any pair is known as a *coaxal system of circles*.

N.B. From the equation (3), the radical axis will cut the circles in real points if c is negative and imaginary points if c is positive ($y^2 = -c$).

Point circles of the system, which are known as *limiting points* of the coaxal system are obtained when $g^2 = c$, i.e. $g = \pm\sqrt{c}$, since (3) can be written $(x + g)^2 + y^2 = g^2 - c$ which has zero radius when $g^2 - c = 0$.

These limiting points will be real if c is positive and imaginary if c is negative.

Definition. Two curves are said to cut *orthogonally* (or be orthogonal) if the tangents at a point of intersection are at right angles.

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Theorem. *To find the condition that the circles*

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0,$$

and

$$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

shall cut orthogonally.

By geometry, the radii at a point of intersection of two orthogonal circles must be perpendicular, and hence, using Pythagoras' theorem, it follows that the sum of the squares of the radii must be equal to the square on the line of centres.

The centre of the first circle is $(-g_1, -f_1)$ and its radius is $\sqrt{(g_1^2 + f_1^2 - c_1)}$. Also the centre of the second circle is $(-g_2, -f_2)$ and its radius is $\sqrt{(g_2^2 + f_2^2 - c_2)}$.

The square on the line of centres is $(g_1 - g_2)^2 + (f_1 - f_2)^2$.

Thus the required condition is

$$g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$$
$$= g_1^2 - 2g_1g_2 + g_2^2 + f_1^2 - 2f_1f_2 + f_2^2$$

$$\text{i.e. } 2g_1g_2 + 2f_1f_2 = c_1 + c_2.$$

EXAMPLE 1. One circle of a coaxal system is $x^2 + y^2 - 4x - 8y + 10 = 0$ and the radical axis is $2x + 4y - 5 = 0$.

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Show that any equation of any circle of the system may be written in the form $x^2 + y^2 + k(2x + 4y - 5) = 0$.

Find the co-ordinates of the limiting points of the system.

Verify that the circle $x^2 + y^2 - x - 2y = 0$ is orthogonal to each circle of the system.

The circle of equation $x^2 + y^2 - 4x - 8y + 10 = 0$ can be written in the form

$$x^2 + y^2 - 2(2x + 4y - 5) = 0,$$

which is a member of the coaxial system (having $2x + 4y - 5 = 0$ as the radical axis) $x^2 + y^2 + k(2x + 4y - 5) = 0$, with k having the value -2.

Hence any circle of the system will have the equation

$$x^2 + y^2 + k(2x + 4y - 5) = 0 \qquad (1).$$

This equation can be written

$$(x + k^2) + (y + 2k)^2 - (k^2 + 4k^2 + 5k) = 0$$

which has centre $(-k, -2k)$.

These circles will be limiting points when

$$k^2 + 4k^2 + 5k = 0,$$
$$\text{i.e. } 5k^2 + 5k = 0,$$
$$\text{i.e. } 5k(k + 1) = 0,$$
$$\text{i.e. } k = 0 \text{ or } -1.$$

When $k = 0$ the limiting point is $(0, 0)$ and when $k = -1$ the limiting point is $(1, 2)$.

The equation (1) can be written

$$x^2 + y^2 + 2kx + 4ky - 5k = 0 \qquad (2).$$

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Now the circle $\qquad x^2 + y^2 - x - 2y = 0 \qquad (3)$

will be orthogonal to this if $2g_1g_2 + 2f_1f_2 = c_1 + c_2$, where $g_1 = k$, $g_2 = -\frac{1}{2}$, $f_1 = 2k$, $f_2 = -1$, $c_1 = -5k$, $c_2 = 0$,

$$\text{i.e. if } 2k(-\tfrac{1}{2}) + 2 \times 2k(-1) = 5k,$$
$$\text{i.e. if } -k - 4k = -5k,$$
$$\text{i.e. if } -5k = -5k, \text{ which is true.}$$

Hence *the circle* (3) *is orthogonal to all circles of system* (2).

EXAMPLE 2. Find the general equation of a circle which cuts orthogonally each of the two circles

$$x^2 + y^2 - 6ax + 5a^2 = 0 \qquad (1),$$
$$x^2 + y^2 - 6ay + 5a^2 = 0 \qquad (2).$$

Show that all such circles belong to a coaxal system with real common points, and find the common points.

Let the required circle have the equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (3).$$

The conditions that circle (3) shall be orthogonal to circles (1) and (2) are

$$2(-3a)g + 2 \times 0 \times f = c + 5a^2,$$
$$\text{i.e. } -6ag = c + 5a^2 \qquad (4),$$

and

$$2 \times 0 \times g + 2(-3a)f = c + 5a^2,$$
$$\text{i.e. } -6af = c + 5a^2 \qquad (5).$$

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From (4) and (5) $\quad -6ag = -6af$

$$\therefore g = f \qquad \text{(since } a \neq 0\text{)}$$

Let $g = f = \lambda$, then (4) becomes $-6a\lambda = c + 5a^2$, and $c = -6a\lambda - 5a^2$. Hence the equation of the circle is

$$x^2 + y^2 + 2\lambda x + 2\lambda y - 6a\lambda - 5a^2 = 0,$$
$$\text{i.e. } x^2 + y^2 + 2\lambda(x + y) - (6a\lambda + 5a^2) = 0.$$

This equation can be written

$$x^2 + y^2 - 5a^2 + 2\lambda(x + y - 3a) = 0 \quad \ldots\ldots\ldots\ldots (6).$$

Hence, the radical axis of the system is

$$x + y - 3a = 0,$$
$$\text{i.e. } y = 3a - x \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Using (7) in (6) we have for the common points

$$x^2 + (3a - x)^2 - 5a^2 = 0,$$
$$\text{i.e. } x^2 + 9a^2 - 6ax + x^2 - 5a^2 = 0,$$
$$\text{i.e. } 2x^2 - 6ax + 4a^2 = 0,$$
$$\text{i.e. } x^2 - 3ax + 2a^2 = 0,$$
$$\text{i.e. } (x - a)(x - 2a) = 0,$$
$$\text{i.e. } x = a \text{ or } 2a.$$

From (7) using $x = a$, $y = 2a$, and using $x = 2a$, $y = a$. Hence the common points are real (assuming a is real) and are $(a, 2a)$ and $(2a, a)$.

Determinants. Consider the equations

$$a_2x + b_2y + c_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$a_3x + b_3y + c_3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

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In order to solve these we take

(1) $\times c_3$ giving $\quad a_2c_3x + b_2c_3y + c_2c_3 = 0 \ldots\ldots\ldots (3),$

(2) $\times c_2$ giving $\quad a_3c_2x + b_3c_2y + c_2c_3 = 0 \ldots\ldots\ldots (4).$

(3) $-$ (4) gives $\quad (a_2c_3 - a_3c_2)x + (b_2c_3 - b_3c_2)y = 0$

$$\therefore (a_2c_3 - a_3c_2)x = -(b_2c_3 - b_3c_2)y,$$

$$\text{i.e. } \frac{x}{b_2c_3 - b_3c_2} = \frac{-y}{a_2c_3 - a_3c_2}$$

Similarly from (1) $\times b_3$ $-$ (2) $\times b_2$ is obtained the result

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{1}{a_2b_3 - a_3b_2}$$

Hence $$\frac{x}{b_2c_3 - b_3c_2} = \frac{-y}{a_2c_3 - a_3c_2} = \frac{1}{a_2b_3 - a_3b_2} \ldots\ldots\ldots\ldots (5).$$

The expressions $(b_2c_3 - b_3c_2)$, $(a_2c_3 - a_3c_2)$, $(a_2b_3 - a_3b_2)$ are usually written in the form

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \text{ respectively}$$

and are known as *second order determinants* having two rows and two columns in each case.

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If, in addition, the variables x, y also satisfy the equation

$$a_1x + b_1y + c_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

the condition required is obtained by substituting the values of x and y from (5) in (6) giving

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0 \ldots (7).$$

This condition is known as the *eliminant* of equations (1), (2) and (6).

Equation (7) can also be written in the determinant form

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0 \ldots\ldots\ldots (8).$$

The left-hand side of either of the equations (7) or (8) can be written briefly in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

which is known as a *third order determinant* and is usually denoted by Δ, containing three rows and three columns.

N.B. A determinant has the same number of columns as rows.

It can be seen by what has been said previously that the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is obtained by multiplying a_1 by the determinant obtained by omitting the row and column containing a_1, subtracting from this the result of multiplying b_1 by the determinant obtained by omitting the row and column containing b_1, and adding to this result the product of c_1 and the determinant obtained by omitting the row and column containing c_1.

The quantities $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are known as the *elements* of the determinant and the determinant obtained by omitting the row and column containing a particular element is known as the *minor* of that element.

Thus in the third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ the determinant } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

is known as the minor of a_1 and is denoted by A_1.

Similarly $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ is the minor of b_2 and is denoted by B_2, $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ is the minor of c_1 and is denoted by C_1, and so on.

N.B. By expanding the left-hand side of equation (7) it can be seen that each term of the expansion of the determinant contains one, and only one, element from each row and column of the determinant.

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EXAMPLE 1. Evaluate (i) $\Delta \equiv \begin{vmatrix} 15 & 7 \\ 23 & 19 \end{vmatrix}$, (ii) $\Delta \equiv \begin{vmatrix} 2x+5 & 4-3x \\ 3-4x & 2+3x \end{vmatrix}$

(i) $\Delta = 15 \times 19 - 7 \times 23 = 285 - 161 = 124.$

(ii) $\Delta = (2x+5)(2+3x) - (4-3x)(3-4x)$
$= 6x^2 + 19x + 10 - (12 - 25x + 12x^2)$
$= 6x^2 + 19x + 10 - 12 + 25x - 12x^2$
$= -6x^2 + 44x - 2 = -2(3x^2 - 22x + 1)$

EXAMPLE 2. Solve the equation

$$\begin{vmatrix} 3x & 5+x \\ 8 & 2x \end{vmatrix} = \begin{vmatrix} x+1 & 4 \\ 11 & x \end{vmatrix}.$$

Expanding the determinants the equation becomes

$$6x^2 - (5+x) \times 8 = (x+1)x - 44,$$
$$\text{i.e. } 6x^2 - 40 - 8x = x^2 + x - 44,$$
$$\text{i.e. } 5x^2 - 9x + 4 = 0,$$
$$\text{i.e. } (5x-4)(x-1) = 0,$$
$$\therefore\ 5x - 4 = 0, \text{ or } x - 1 = 0,$$
$$\therefore\ x = 4/5 \text{ or } 1.$$

EXAMPLE 3. Evaluate (i) $\Delta \equiv \begin{vmatrix} 2 & 5 & 7 \\ 5 & 9 & 11 \\ 4 & 3 & 8 \end{vmatrix}$, (ii) $\Delta \equiv \begin{vmatrix} 15 & 11 & 8 \\ 8 & 15 & 11 \\ 11 & 8 & 15 \end{vmatrix}$.

(i) $\Delta = 2\begin{vmatrix} 9 & 11 \\ 3 & 8 \end{vmatrix} - 5\begin{vmatrix} 5 & 11 \\ 4 & 8 \end{vmatrix} + 7\begin{vmatrix} 5 & 9 \\ 4 & 3 \end{vmatrix}$
$= 2(72 - 33) - 5(40 - 44) + 7(15 - 36)$
$= 2 \times 39 - 5(-4) + 7(-21)$
$= 78 + 20 - 147 = -49.$

(ii) $\Delta = 15\begin{vmatrix} 15 & 11 \\ 8 & 15 \end{vmatrix} - 11\begin{vmatrix} 8 & 11 \\ 11 & 15 \end{vmatrix} + 8\begin{vmatrix} 8 & 15 \\ 11 & 8 \end{vmatrix}$
$= 15(225 - 88) - 11(120 - 121) + 8(64 - 165)$
$= 15 \times 137 - 11(-1) + 8(-101)$
$= 2055 + 11 - 808 = 1258.$

The Rule of Sarrus. This is a rule that can be used for writing down the expansion of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and is stated as follows:

Write down the determinant without the lines and repeat the first two columns. Draw lines diagonally through each of three elements as shown in the diagram.

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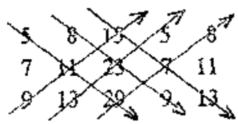
The terms formed by each line will be terms of the expansion, those with arrows downwards are positive and those with arrows upwards are negative.

EXAMPLE. Using the rule of Sarrus evaluate

$$\Delta \equiv \begin{vmatrix} 5 & 8 & 15 \\ 7 & 11 & 23 \\ 9 & 13 & 29 \end{vmatrix}$$

Using the rule of Sarrus



$$\Delta = 5 \times 11 \times 29 + 8 \times 23 \times 9 + 5 \times 7 \times 13 - 9 \times 11 \times 15 - 13 \times 23 \times 5 - 29 \times 7 \times 8$$
$$= 1595 + 1656 + 1365 - 1485 - 1495 - 1624$$
$$= 4616 - 4604 = 12.$$

NOTE. Many determinants can be simplified previous to evaluation and the theorems that follow can be used for the simplification of third order determinants, and the notation is that previously stated.

Theorem. *A determinant is unaltered in value if the rows are changed to columns.*

Let the determinant be $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ www.dbraulibrary.org.in

Then $\Delta = a_1A_1 - b_1B_1 + c_1C_1.$

Expanding
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$
$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \text{ (rearranging terms)}$$
$$= a_1A_1 - b_1B_1 + c_1C_1 = \Delta.$$

Thus the theorem is proved.

From this result $\Delta = a_1A_1 - b_1B_1 + c_1C_1$
$= a_1A_1 - a_2A_2 + a_3A_3,$

and hence *the determinant can be expanded by the first column instead of the first row.*

Theorem. *If two rows (or two columns) of a determinant be interchanged, the determinant changes sign.*

Let
$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1.$$

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Interchanging the first and third rows in Δ we have

$$\Delta_1 \equiv \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}, \text{ and expanding this by}$$

the first row
$$\begin{aligned} \Delta_1 &= a_3(b_2c_1 - b_1c_2) - b_3(a_2c_1 - a_1c_2) + c_3(a_2b_1 - a_1b_2) \\ &= -a_1b_2c_3 + a_1b_3c_2 - a_3b_1c_2 + a_2b_1c_3 \\ &\qquad + a_3b_2c_1 - a_2b_3c_1 \\ &= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) \\ &\qquad + c_1(a_2b_3 - a_3b_2)] \\ &= -[a_1A_1 - b_1B_1 + c_1C_1] \\ &= -\Delta. \end{aligned}$$

A similar result is obtained by interchanging any other two rows or any two colums, hence the theorem is proved.

N.B. When two interchanges are made the determinant will change sign twice and hence its value will be unaltered. For three interchanges the result will be the original determinant with a negative sign.

Theorem. *If two rows (or columns) of a determinant are the same the value of the determinant is zero.*

Let the determinant be $\Delta \equiv \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$

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having the first and third columns the same.

Interchanging the equal columns we have the determinant

$$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = \Delta$$

By the previous theorem the left-hand side of this result is $-\Delta$,

$$\therefore \quad -\Delta = \Delta,$$
$$\text{i.e.} \quad -2\Delta = 0,$$
$$\therefore \quad \Delta = 0.$$

The same result will be obtained when any two other columns, or any two rows are identical.

From this result it can be seen that each of the determinants

$$\begin{vmatrix} 7 & 7 & 19 \\ 11 & 11 & 27 \\ 15 & 15 & 48 \end{vmatrix}, \quad \begin{vmatrix} 3 & 8 & 21 \\ 5 & 7 & 29 \\ 5 & 7 & 29 \end{vmatrix}, \quad \begin{vmatrix} 73 & 13 & 13 \\ 28 & 25 & 25 \\ 32 & 38 & 38 \end{vmatrix}$$

has zero value since in each case either two rows or two columns are equal.

Theorem. *If the elements of any row (or column) of a determinant have a common factor, the value of the determinant is the product of*

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that factor and the determinant obtained by cancelling that factor in the row or column of the original determinant.

Thus if p be a factor in the first row, then

$$\Delta_1 \equiv \begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ with the usual notation for minors.

Expanding by the first row

$$\Delta_1 = pa_1(b_2c_3 - b_3c_2) - pb_1(a_2c_3 - a_3c_2) + pc_1(a_2b_3 - a_3b_2)$$
$$= p(a_1A_1 - b_1B_1 + c_1C_1) = p\Delta.$$

The same result can be proved in the case of other rows having a common factor, and also for columns having common factors, thus proving the theorem.

NOTE. The theorem can be extended in the case of two columns or two rows having common factors to the elements.

Thus
$$\begin{vmatrix} pa_1 & b_1 & qc_1 \\ pa_2 & b_2 & qc_2 \\ pa_3 & b_3 & qc_3 \end{vmatrix} = pq\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Also the extension can be carried further to the case when all three rows, or three columns, have common factors to the elements giving for the case of the rows

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$$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ qa_2 & qb_2 & qc_2 \\ ra_3 & rb_3 & rc_3 \end{vmatrix} = pqr\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

EXAMPLE. Evaluate (i) $\Delta \equiv \begin{vmatrix} 3 & 5 & 19 \\ 6 & 12 & 18 \\ 4 & 8 & 11 \end{vmatrix}$, (ii) $\Delta \equiv \begin{vmatrix} 2 & 4 & 5 \\ 5 & 8 & 10 \\ 9 & 16 & 20 \end{vmatrix}$,

(iii) $\Delta \equiv \begin{vmatrix} 7 & 14 & 21 \\ 5 & 20 & 10 \\ 3 & 9 & 6 \end{vmatrix}$.

(i) 6 is a common factor of the second row,

$$\therefore \Delta = 6\begin{vmatrix} 3 & 5 & 19 \\ 1 & 2 & 3 \\ 4 & 8 & 11 \end{vmatrix} = 6[3(22 - 24) - 5(11 - 12) + 19(8 - 8)]$$
$$= 6[3(-2) - 5(-1)] = 6(-6 + 5)$$
$$= 6(-1) = -6.$$

(ii) 4 is a common factor of the second column and 5 is a common factor of the third column,

$$\therefore \Delta = 4 \times 5\begin{vmatrix} 2 & 1 & 1 \\ 5 & 2 & 2 \\ 9 & 4 & 4 \end{vmatrix} = 0,$$

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for the determinant has the last two columns the same and must therefore be zero.

(iii) 7, 5, 3 are the common factors of the first, second, and third rows respectively of the determinant,

$$\therefore\ \Delta = 7 \times 5 \times 3\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 3 & 2 \end{vmatrix}$$
$$= 105[1(8-6) - 2(2-2) + 3(3-4)]$$
$$= 105[2 - 0 - 3]$$
$$= 105(-1) = -105.$$

Theorem. *If each element of any row (or column) of a determinant consist of the algebraic sum of r terms, the determinant is equal to the algebraic sum of r other determinants in each of which the elements consist of single terms.*

With the usual notation for minors let

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ and } \Delta_1 \equiv \begin{vmatrix} a_1+l_1-m_1 & b_1 & c_1 \\ a_2+l_2-m_2 & b_2 & c_2 \\ a_3+l_3-m_3 & b_3 & c_3 \end{vmatrix}.$$

Expanding by the first column,

$$\Delta_1 = (a_1+l_1-m_1)A_1 - (a_2+l_2-m_2)A_2 + (a_3+l_3-m_3)A_3$$
$$= (a_1A_1 - a_2A_2 + a_3A_3) + (l_1A_1 - l_2A_2 + l_3A_3)$$
$$- (m_1A_1 - m_2A_2 + m_3A_3)$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & b_1 & c_1 \\ l_2 & b_2 & c_2 \\ l_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix},$$

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which proves the theorem for the case $r = 3$, and similarly it can be proved for $r = 4$, $r = 5$, etc.

This theorem can be extended to cover the case when the elements of two different columns (or rows) consist of two or more terms, as follows:

With the previous notation for Δ consider the determinant

$$\Delta_2 \equiv \begin{vmatrix} a_1+l_1 & b_1+m_1 & c_1 \\ a_2+l_2 & b_2+m_2 & c_2 \\ a_3+l_3 & b_3+m_3 & c_3 \end{vmatrix}$$

By the previous result

$$\Delta_2 = \begin{vmatrix} a_1 & b_1+m_1 & c_1 \\ a_2 & b_2+m_2 & c_2 \\ a_3 & b_3+m_3 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & b_1+m_1 & c_1 \\ l_2 & b_2+m_2 & c_2 \\ l_3 & b_3+m_3 & c_3 \end{vmatrix}$$
$$= -\begin{vmatrix} b_1+m_1 & a_1 & c_1 \\ b_2+m_2 & a_2 & c_2 \\ b_3+m_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1+m_1 & l_1 & c_1 \\ b_2+m_2 & l_2 & c_2 \\ b_3+m_3 & l_3 & c_3 \end{vmatrix}$$

(interchanging first two columns in each determinant)

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$$= -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} m_1 & a_1 & c_1 \\ m_2 & a_2 & c_2 \\ m_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} b_1 & l_1 & c_1 \\ b_2 & l_2 & c_2 \\ b_3 & l_3 & c_3 \end{vmatrix} - \begin{vmatrix} m_1 & l_1 & c_1 \\ m_2 & l_2 & c_2 \\ m_3 & l_3 & c_3 \end{vmatrix}$$

(using previous result)

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & b_1 & c_1 \\ l_2 & b_2 & c_2 \\ l_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l_1 & m_1 & c_1 \\ l_2 & m_2 & c_2 \\ l_3 & m_3 & c_3 \end{vmatrix}$$

(interchanging the first two columns in each determinant).

Theorem. *If the elements of any row (or column) be increased or diminished by equimultiples of the corresponding elements of any other row (or column), the value of the determinant is unaltered.*

Considering the first column, this means in algebraical language that, if λ_1 and λ_2 can take any values, then

$$\Delta_1 \equiv \begin{vmatrix} a_1+\lambda_1 b_1+\lambda_2 c_1 & b_1 & c_1 \\ a_2+\lambda_1 b_2+\lambda_2 c_2 & b_2 & c_2 \\ a_3+\lambda_1 b_3+\lambda_2 c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

By the previous theorem

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 b_1 & b_1 & c_1 \\ \lambda_1 b_2 & b_2 & c_2 \\ \lambda_1 b_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_2 c_1 & b_1 & c_1 \\ \lambda_2 c_2 & b_2 & c_2 \\ \lambda_2 c_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \lambda_1 \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + \lambda_2 \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}.$$

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Since the last two determinants each have two columns identical the value of each is zero.

$$\text{Thus } \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

A similar result can be obtained by using rows instead of columns.

N.B. This theorem can be used to simplify a determinant before evaluation, and the simplification can be done quicker in certain cases if equimultiples of the elements of one column (or row) be added to the corresponding elements of the two remaining columns (or rows) simultaneously.

$$\text{Thus } \begin{vmatrix} a_1+\lambda_1 c_1 & b_1+\lambda_2 c_1 & c_1 \\ a_2+\lambda_1 c_2 & b_2+\lambda_2 c_2 & c_2 \\ a_3+\lambda_1 c_3 & b_3+\lambda_2 c_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 c_1 & b_1 & c_1 \\ \lambda_1 c_2 & b_2 & c_2 \\ \lambda_1 c_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 c_1 & \lambda_2 c_1 & c_1 \\ \lambda_1 c_2 & \lambda_2 c_2 & c_2 \\ \lambda_1 c_3 & \lambda_2 c_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \lambda_2 c_1 & c_1 \\ a_2 & \lambda_2 c_2 & c_2 \\ a_3 & \lambda_2 c_3 & c_3 \end{vmatrix}$$

(by the previous theorem)

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$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \lambda_1 \begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} + \lambda_1\lambda_2 \begin{vmatrix} c_1 & c_1 & c_1 \\ c_2 & c_2 & c_2 \\ c_3 & c_3 & c_3 \end{vmatrix} + \lambda_2 \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}$$

(since the last three determinants are zero each having at least two columns identical).

It must be carefully noted that, when using these results, one column must be kept in its original form, or it will be found that an impossible result will be obtained.

In what follows R_1, R_2, R_3, C_1, C_2, C_3 will be used for the first, second, third rows and columns respectively. (i.e. R_2 is the second row, C_3 is the third column, etc.).

EXAMPLE 1. Evaluate (i) $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 21 & 17 & 29 \\ 15 & 23 & 36 \end{vmatrix}$, (ii) $\Delta = \begin{vmatrix} 15 & 23 & 17 \\ 2x & 4x & 6x \\ 19 & 27 & 33 \end{vmatrix}$.

(i) $\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 21 & -4 & 8 \\ 15 & 8 & 21 \end{vmatrix}$ $(C_2 - C_1)$ and $(C_3 - C_1)$

$= 1\{(-4) \times 21 - 8 \times 8\} = (-84 - 64) = -148.$

(ii) Taking out the common factor $2x$ in the second row,

$\Delta = 2x \begin{vmatrix} 15 & 23 & 17 \\ 1 & 2 & 3 \\ 19 & 27 & 33 \end{vmatrix} = 2x \begin{vmatrix} 15 & -7 & -28 \\ 1 & 0 & 0 \\ 19 & -11 & -24 \end{vmatrix}$ $(C_2 - 2C_1)$ and $(C_3 - 3C_1)$

$= -2x \begin{vmatrix} 1 & 0 & 0 \\ 15 & -7 & -28 \\ 19 & -11 & -24 \end{vmatrix}$ (Interchanging the first and second rows)

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$= -2x \begin{vmatrix} -7 & -28 \\ -11 & -24 \end{vmatrix} = (-2x)(-7) \begin{vmatrix} 1 & 4 \\ -11 & -4 \end{vmatrix}$

$= 14x\{1(-4) - 4(-11)\} = 14x(-4 + 44)$

$= 14x \times 40 = 560x.$

EXAMPLE 2. Indicating clearly your method in each case,

(i) evaluate the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{vmatrix}$,

(ii) solve the equation $\begin{vmatrix} x & 1 & 1 \\ 2 & x+1 & 2 \\ 3 & 3 & x+2 \end{vmatrix} = 0$,

(iii) factorise completely the determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

(i) $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 6 & -7 & -14 \\ 7 & -6 & -12 \end{vmatrix}$, $(C_2 - 2C_1)$ and $(C_3 - 3C_1)$.

$= 1\{(-7)(-12) - (-6)(-14)\}$
$= (84 - 84) = 0.$

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(ii) The determinant $= \begin{vmatrix} x+5 & x+5 & x-5 \\ 2 & x+1 & 2 \\ 3 & 3 & x+2 \end{vmatrix}$, $R_1 + (R_2 + R_3)$,

$$= (x+5)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+1 & 2 \\ 3 & 3 & x+2 \end{vmatrix}$$

$$= (x+5)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-1 & 0 \\ 3 & 0 & x-1 \end{vmatrix}, \quad (C_2 - C_1) \text{ and } (C_3 - C_1)$$

$$= (x+5)(x-1)^2.$$

Hence the equation reduces to

$$(x+5)(x-1)^2 = 0,$$

from which $x = -5$ or 1 (twice).

(iii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}$, $(C_2 - C_1)$ and $(C_3 - C_1)$

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} b-a & c-a \\ (b-a)(b^2+ab+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\{(c^2+ca+a^2) - (b^2+ab+a^2)\}$$

$$= (b-a)(c-a)(c^2+ca-b^2-ab)$$

$$= (b-a)(c-a)\{(c^2-b^2)+(ca-ab)\}$$

$$= (b-a)(c-a)\{(c-b)(c+b)+a(c-b)\}$$

$$= (b-a)(c-a)(c-b)(a+b+c)$$

$$= (b-c)(c-a)(a-b)(a+b+c).$$

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EXAMPLE 3. (i) Show that for all values of θ the determinant

$$\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

lies between 2 and 4 inclusive.

State one value of θ for which the determinant has the value 2, and one for which it has the value 4.

(ii) Expand the determinant

$$y = \begin{vmatrix} x & x^2 & x^3 \\ a & b & c \\ p & q & r \end{vmatrix}$$

by the first row, and from the expansion find dy/dx.

Express dy/dx in the form of a determinant, and, hence or otherwise, find the two values of x for which the determinant

$$y = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$$

has stationary values.

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$$\text{(i)}\quad \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}, (R_1 + R_2),$$
$$= 2(\sin^2\theta + 1)$$

The least value of this is when $\sin\theta = 0$, and is therefore 2, and the greatest value when $\sin\theta = \pm 1$ and is therefore 4. Hence the determinant lies between 2 and 4 inclusive.

When the determinant has the value 2 one value of θ is 0°, and when it has the value 4 one value of θ is 90°.

(ii) Expanding by the first row
$$y = x(br - cq) - x^2(ar - cp) + x^3(aq - bp).$$
$$\therefore\ dy/dx = (br - cq) - 2x(ar - cp) + 3x^2(aq - bp).$$

In determinant form $\dfrac{dy}{dx} = \begin{vmatrix} 1 & 2x & 3x^2 \\ a & b & c \\ p & q & r \end{vmatrix}$.

When $a = 1$, $b = 2$, $c = 3$, $p = 1$, $q = -2$, $r = 3$,
$$y = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix} \text{ and } \frac{dy}{dx} = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix}$$
$$\therefore\ \frac{dy}{dx} = 2 \times 3 \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 6 \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 - x & 1 - x^2 \\ 0 & -1 - x & 1 - x^2 \end{vmatrix},$$
$(R_2 - R_1)$ and $(R_3 - R_1)$
$$= 6\{(1 - x)(1 - x^2) + (1 + x)(1 - x^2)\}$$
$$= 6(1 - x^2)\{1 - x + 1 + x\}$$
$$= 12(1 - x^2).$$

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y has stationary values when $dy/dx = 0$,
i.e. when $12(1 - x^2) = 0$,
i.e. when $(1 - x)(1 + x) = 0$,
i.e. when $x = \pm 1$.

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EXAMPLES XII

1. (*a*) Make rough sketches of the loci given by the following Cartesian equations: (i) $y^2 = x$ and $y = -\sqrt{(-x)}$,
(ii) $y = x - 1$ and $y = |x - 1|$.

(*b*) Sketch the locus given by the polar equation $r = \theta/2\pi$ between the values $\theta = 0$, $\theta = 2\pi$.

2. Use the relations $x = r\cos\theta$, $y = r\sin\theta$ to find Cartesian equations for the loci (i) $r(3\cos\theta + 4\sin\theta) = 1$, (ii) $r = 3\cos\theta + 4\sin\theta$.

Show that one of these loci is a circle and the other a straight line. Find the points where the loci cut the Cartesian axes, determine the radius and the co-ordinates of the centre of the circle and sketch both loci on the same diagram.

3. Plot the portion of the curve whose equation in polar co-ordinates is $r = \sin 2\theta$, for values of θ which lie between 0 and $\frac{1}{2}\pi$.

Prove that the point on the curve whose perpendicular distance from the initial line ($\theta = 0$) is greatest, lies at an extremity of the chord through the origin which makes the angle $\cos^{-1}\sqrt{\frac{1}{3}}$ with the initial line.

4. Plot, using values of θ at $30°$ intervals from $0°$ to $360°$, the curve whose equation in polar co-ordinates is $r = 5 + 4\cos\theta$.

Show that all chords PQ drawn through the pole O are of length 10 units.

Calculate the values of θ for which O trisects the chord PQ.

5. Plot the curve whose equation in polar co-ordinates is $r = 1 + \cos\theta$, and on the same diagram draw the straight line $r = \sec\theta$. Find by calculation the values of r and θ for the points where the curve and the line intersect.

6. Construct, correct to one place of decimals, a table of values for r, where $r = 1 + 2\cos\theta$, taking values of θ at intervals of $30°$ from $0°$ to $360°$. Hence give a sketch of the curve whose polar equation is $r = 1 + 2\cos\theta$, showing it to be a closed curve with an interior loop.

A straight line OPQ is drawn through the pole O making a positive acute angle α with the initial line so as to meet the curve again at two points P and Q on the same side of O. Calculate the length of PQ.

7. Obtain the polar equation of the conic $ax^2 + by^2 + 2gx + 2fy + c = 0$, taking the pole as origin, and the axis of x for initial line.

Hence, or otherwise, prove that the locus of the mid-points of the chords of a conic drawn through a fixed point is another conic, with its axes parallel to those of the original conic, and passing through the points of contact of the tangents from the fixed point to the original conic.

8. Sketch the curve $r = a(1 + 2\cos\theta)$ showing how the curve is described as θ increases from $0°$ to $360°$.

A line through the pole O meets the curve again at P and Q, Q lying between O and P. Show that $PQ = 2a$.

Show also that, if OPQ is inclined at $30°$ to the initial line, the tangents at P and Q intersect the initial line at the same point R, where $OR = 4a$.

9. Write down the first three terms of the expansions of $\sin\theta$ and $\cos\theta$ in powers of θ.

Neglecting θ^4 and higher powers of θ, find the positive root of the equation $2\cos 3\theta + 3\sin 2\theta = 2 + 5\theta$, giving your answer to one place of decimals.

10. By expanding the integrand of

$$\int_0^x \frac{1}{1+x}\,dx$$

as a series of powers of x and integrating term by term, find the series for $\log_e(1+x)$, assuming your method to be valid providing that $|x| < 1$.

Write down the series for $\log_e(1-x)$, obtain the series for $\log_e(1+x)/(1-x)$ and deduce a series for $\log_e m/n$ in terms of $(m-n)/(m+n)$.

Hence calculate $\log_e 8$ correct to five places of decimals, given that $\log_e 7 = 1{\cdot}945910$.

11. (a) A chord of length $2x$ divides a circle of radius a into two segments; prove that the heights of the segments are $a \pm (a^2 - x^2)^{1/2}$. Deduce that, if powers of x/a above the sixth can be neglected, the height h of the smaller segment is given by $h = x^2/2a + x^4/8a^3 + x^6/16a^5$.

(b) If x be so small that x^5 and higher powers of x are negligible, find the values of the constants a, b, c, d in the approximation

$$(e^x - 1)/(e^x + 1) = ax + bx^2 + cx^3 + dx^4.$$

12. (a) Determine the range, or ranges, of values of x for which $|3x - 5| > 7$. If x has a value satisfying this condition, prove that the sum of the infinite series

$$1 + 2\left(\frac{7}{3x-5}\right) + 3\left(\frac{7}{3x-5}\right)^2 + 4\left(\frac{7}{3x-5}\right)^3 + \ldots \text{ is } \frac{1}{9}\left(\frac{3x-5}{x-4}\right)^2.$$

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(*b*) Write down the first four terms of the expansion of $\log_e(1 + x)$ and e^x in ascending powers of x, assuming that the expansions are valid.

If x be so small that x^4 may be neglected, prove the approximate formula

$$e^{-x}(1 + x)^{1+\frac{1}{2}x} = 1 - x^2 + \tfrac{7}{12}x^3.$$

13. If $a=b(1+h)$ where h is small and $a>b>0$, expand $[2(a-b)]/(a+b)$ as a series in ascending powers of h.

Show that when a is nearly equal to b, $\log_e a/b$ differs from $[2(a - b)]/(a + b)$ by approximately

$$\frac{1}{12}\left(\frac{a - b}{b}\right)^3.$$

14. (*a*) If $y = 2x - 3x^2$ and $\log_e(z - 2) = y$, express z in ascending powers of x as far as the term in x^4.

(*b*) Write down the roots of the quadratic equation $ax^2 + bx + c = 0$, and use the binomial expansion to prove that, if $4ac/b^2$ is so small compared with unity that its cube and higher powers can be neglected, then the approximate values of the roots are

$$-\frac{b}{a}(k + k^2) \text{ and } -\frac{b}{a}(1 - k - k^2),$$

where $k = ac/b^2$.

Show that one root is then small compared with the other.

15. Two circles, S, S' are respectively concentric with and enclosed by two intersecting circles T, T' and have the same radical axis as T, T'. The circles S, T lie on opposite sides of the radical axis from the circles S', T'. Draw figures showing each of the cases (i) S and S' not intersecting, (ii) S and S' intersecting.

Prove that the length of the tangent from any point on T to S is equal to the length of the tangent from any point on T' to S'.

16. Show that the x-axis is the radical axis of the system of circles given by the equation $x^2 + y^2 - 4x - 2ky + 3 = 0$, where k varies. Find the co-ordinates of the two common points of the system.

Find the equations of: (i) the circle of the system which passes through the point (4, 3); (ii) the two circles of the system which touch the y-axis; (iii) the two circles of the system which touch the line $x + y = 5$.

17. A fixed circle S has centre O and radius r. A fixed line l is drawn not intersecting S. The perpendicular from O to l meets l at N. A variable point P is taken on l, and a circle with centre P cuts S orthogonally and meets ON at A. Prove that $ON^2 - AN^2 = r^2$.

Deduce that the point A is independent of the position of P. Hence show that the circles which cut the circle S orthogonally and whose centres lie on l form a coaxal system.

18. Define the radical axis of two circles, and show that the y-axis is the radical axis of the circles $x^2 + y^2 - 4x - 9 = 0$, $x^2 + y^2 + 6x - 9 = 0$.

Find the equation of the smallest circle through the common points A and B of these two circles. Find also the equations of the circles through A and B which have radius 5.

Show that any circle which cuts orthogonally all circles through A and B has its centre on the y-axis. If such a circle cuts orthogonally the circle $x^2 + y^2 - 2x - 4y - 25 = 0$, find its equation.

19. Define the radical axis of two circles.

A circle S cuts a circle S_1 in two points P_1, Q_1, and cuts a circle S_2 in two points P_2, Q_2; prove that the lines P_1Q_1, P_2Q_2 intersect on the radical axis of S_1, S_2.

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Deduce a geometrical construction for drawing the radical axis of two given non-intersecting circles.

Show how to construct the limiting points of the coaxal system to which the two given circles belong.

If the radii of the two circles are 7 in. and 1 in. and the distance between their centres is 10 in., show that the distance between the limiting points is 4·8 in.

20. Show that any circle through the limiting points of a non-intersecting system of coaxal circles cuts all members of the system orthogonally.

Show that the polars of a given point P for the system pass through another point P'.

Given P and the limiting points L, L', how would you fix the position of P'?

21. L, M are fixed points and a point P moves in a plane through LM so that PL/PM has a constant value k. Show that the locus of P is a circle.

Prove that, as k varies, the circles so obtained form a coaxal system with L, M as limiting points.

Show how to construct the circles of the system which touch a given line.

22. (*a*) Evaluate the following determinants:

(i) $\begin{vmatrix} 17 & 25 \\ 18 & 23 \end{vmatrix}$, (ii) $\begin{vmatrix} 2x+5 & x-1 \\ 3x+2 & 4x-2 \end{vmatrix}$, (iii) $\begin{vmatrix} 2\sin\theta & 3\cos\theta \\ -2\cos\theta+1 & 3\sin\theta+2 \end{vmatrix}$.

(*b*) Solve the equations:

(i) $\begin{vmatrix} 2x-1 & 3 \\ 3x-2 & 2 \end{vmatrix} = 0$, (ii) $\begin{vmatrix} x & 2x \\ 1+x & 3-2x \end{vmatrix} = \begin{vmatrix} 3x & 2-x \\ 4x & 5-2x \end{vmatrix}$.

23. (*a*) By expanding by the first row evaluate:

$$\text{(i)} \begin{vmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 18 \end{vmatrix}, \quad \text{(ii)} \begin{vmatrix} 4 & 5 & -3 \\ -8 & 11 & 2 \\ 17 & -6 & 3 \end{vmatrix}.$$

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(*b*) By expanding by the first column evaluate:

$$\text{(i)} \begin{vmatrix} 4 & 5 & -3 \\ 3 & -2 & 6 \\ 7 & 8 & 11 \end{vmatrix}, \quad \text{(ii)} \begin{vmatrix} 13 & 4 & -9 \\ 11 & 7 & 2 \\ -8 & 3 & 12 \end{vmatrix}.$$

(*c*) Using the rule of Sarrus determine the value of:

$$\begin{vmatrix} 3 & 9 & 11 \\ -7 & 13 & -21 \\ 5 & 23 & 15 \end{vmatrix}$$

24. (*a*) Find the co-ordinates of the point of intersection of the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, given that $a_1b_2 - a_2b_1$ is not zero. Express in a form which involves a determinant the condition that this point should also lie on the line $a_3x + b_3y + c_3 = 0$.

(*b*) The rows of a third order determinant each contain, in different orders, the elements a, b, c. Show that $a + b + c$ is a factor of the determinant.

Indicating clearly the method used for each step in your calculation, evaluate

$$\begin{vmatrix} 7 & 9 & 4 \\ 8 & 14 & 18 \\ 36 & 16 & 28 \end{vmatrix}.$$

25. (*a*) The points A, B, C lie on the curve $y = x(x + 1)$ and their x-co-ordinates are $(k - 1)$, k, $(k + 1)$ respectively. By evaluating a determinant show that the area of the triangle ABC is independent of the value of k.

(*b*) Prove that
$$\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ -\sin x & \cos x & \sin y \\ -\cos x & \sin x & \cos y \end{vmatrix}$$
is equal to $a + b\cos 2y$, where a and b are independent of x and y.

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26. (*a*) Show that there is only one real value of x which satisfies the equation

$$\begin{vmatrix} x & 2 & 3 \\ 3 & x & 2 \\ 2 & 3 & x \end{vmatrix} = 0,$$

and find this value.

(*b*) Factorise completely $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$.

27. (*a*) Solve the equation $\begin{vmatrix} x & 2 & -2 \\ 2 & x & -2 \\ -2 & 2 & x \end{vmatrix} = 0.$

(*b*) Write down the first and second derivatives with respect to x of the product vy, where v and y are functions of x.

By eliminating y and dy/dx, deduce that, if $u = vy$, then

$$v^3 \frac{d^2y}{dx^2} = \begin{vmatrix} u & du/dx & d^2u/dx^2 \\ v & dv/dx & d^2v/dx^2 \\ 0 & v & 2\, dv/dx \end{vmatrix}.$$

28. (*a*) Prove that $\begin{vmatrix} 2\cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix} = \frac{\sin 4\theta}{\sin\theta}$,

when θ is not a multiple of π. Also find the value of the determinant when $\theta = 0$ and when $\theta = \pi$.

(*b*) If θ is the radian measure of a small angle, show that $\cos\theta = 1 - \frac{1}{2}\theta^2$ approximately.

If $3 - 2\cos A = 1{\cdot}04$ and $0 < A < 90°$, show, without using tables, that $A = 11{\cdot}5°$ approximately.

29. Prove that the value of a third-order determinant is unaltered by adding to the elements of any row the same multiples of the corresponding elements of the remaining rows.

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Prove that $a^2b^2c^2 \begin{vmatrix} 1 & \dfrac{x-c^2}{ab} & \dfrac{x-b^2}{ac} \\ \dfrac{x-c^2}{ab} & 1 & \dfrac{x-a^2}{bc} \\ \dfrac{x-b^2}{ca} & \dfrac{x-a^2}{bc} & 1 \end{vmatrix} = 4(x-a^2)(x-b^2)(x-c^2),$

where $2x = a^2 + b^2 + c^2$.

30. (i) Factorise $(b-c)(b^3+c^3) + (c-a)(c^3+a^3) + (a-b)(a^3+b^3)$.

(ii) Find the factors of the determinant

$$\begin{vmatrix} a^2 & a^2-b^2-c^2+2bc & bc \\ b^2 & b^2-c^2-a^2+2ca & ca \\ c^2 & c^2-a^2-b^2+2ab & ab \end{vmatrix}$$

31. Express the determinant $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

as the product of four factors.

If a and b are given unequal numbers, show that there are three values of c for which the equations

$$ax + by + cz = 0, \quad a^2x + b^2y + c^2z = 0, \quad bcx + cay + abz = 0$$

are consistent, and find the ratios of $x:y:z$ for each of these values of c.

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CHAPTER XIII

Curvature, Differentiation of Inverse Trigonometric Functions, Integration by Partial Fractions, and Substitution, Mean Values

Curvature. Let P and Q be two adjacent points on a given curve, with the arc PQ of length δs, such that the tangents at P and Q make angles ψ and $\psi+\delta\psi$ with OX. Then the angle between the tangents at P and Q is $\delta\psi$.

The *curvature* at a certain point of a curve is defined as being the rate of change of the angle between the tangents with respect to the arc s. Thus the curvature at P in the present case is

$$\underset{\delta s\to 0}{\mathrm{Lt}}\ \frac{\delta\psi}{\delta s}=\frac{d\psi}{ds}.$$

Consider now a circle, centre A, radius r, having the same adjacent points P and Q on it and the tangents at P and Q making ψ and $\psi+\delta\psi$ with OX.

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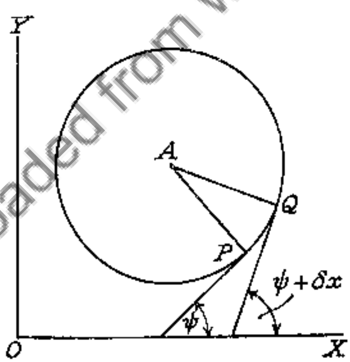

From the properties of a circle the angle PAQ equals the angle between the tangents at P and $Q = \delta\psi$.

Now arc $PQ = r\delta\psi$ and arc $PQ = \delta s$, $\therefore\ \delta s = r\delta\psi$.

Hence $\delta\psi/\delta s = 1/r$, and it follows that the curvature at P

$$=\underset{\delta s\to 0}{\mathrm{Lt}}\ \frac{\delta\psi}{\delta s}=\frac{d\psi}{ds}=\frac{1}{r}.$$

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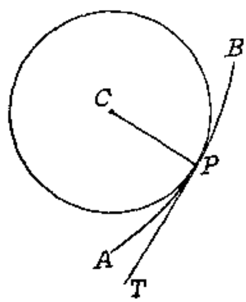
From this it can be seen that the curvature at all points of a circle is the same and is equal to the *inverse of the radius*.



Let $P \equiv (x, y)$ be any point on a portion AB of a given curve with PT the tangent at P. C is the centre of the circle on the same side of the tangent as the curve AB, and having the same curvature as the curve at P. This circle is known as the *circle of curvature* of the curve AB for the point P, its centre C is the *centre of curvature* for the curve at P, and its radius CP (normal to the curve and usually denoted by ρ) is the *radius of curvature* of the curve at P. This circle of curvature can also be defined as the circle cutting arc AB in three coincident points at P.

Using the first definition, and the previous result, it can be seen that the radius of curvature

$$\rho = \frac{1}{\text{curvature}} = \frac{1}{d\psi/ds} = \frac{ds}{d\psi}.$$

Theorem. *To find the formula for the radius of curvature ρ at the point (x, y) of the curve $y = f(x)$.*

Using standard notation $\tan\psi = dy/dx$. Differentiating this with respect to s,

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$$\frac{d}{ds}(\tan\psi) = \frac{d}{ds}\left(\frac{dy}{dx}\right),$$

$$\text{i.e. } \frac{d}{d\psi}(\tan\psi)\cdot\frac{d\psi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{ds},$$

(function of a function theorem)

$$\text{i.e. } \sec^2\psi\cdot\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\cdot\frac{dx}{ds} \quad \dots\dots\dots\dots\dots\dots \quad (1)$$

Now $\sec^2\psi = 1 + \tan^2\psi = 1 + (dy/dx)^2$, and $d\psi/ds = 1/\rho$.
Also from the diagram shown, using Pythagoras' theorem,

$$(\delta s)^2 \to (\delta x)^2 + (\delta y)^2, \text{ as } \delta x \to 0,$$

$$\text{i.e. } (\delta s/\delta x)^2 \to 1 + (\delta y/\delta x)^2, \text{ as } \delta x \to 0,$$

$$\text{i.e. } \underset{\delta x \to 0}{\text{Lt}} \left(\frac{\delta s}{\delta x}\right)^2 = \underset{\delta x \to 0}{\text{Lt}} \left\{1 + \left(\frac{\delta y}{\delta x}\right)^2\right\},$$

$$\text{i.e. } (ds/dx)^2 = 1 + (dy/dx)^2$$

$$\therefore ds/dx = \sqrt{[1 + (dy/dx)^2]},$$

$$\text{and } \frac{dx}{ds} = \frac{1}{ds/dx} = \frac{1}{\sqrt{[1 + (dy/dx)^2]}}.$$

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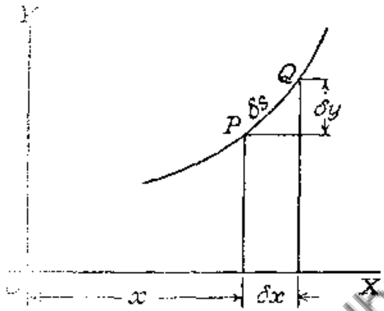
Using these in (1)

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{1}{\rho} = \frac{d^2y/dx^2}{\sqrt{[1 + (dy/dx)^2]}}$$

$$\therefore \frac{1}{\rho} = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}.$$

Hence $$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$



N.B. The usual convention adopted is that the positive root shall be taken in the numerator of the fraction giving the value of ρ, and the radius of curvature will be positive when d^2y/dx^2 is positive, i.e. when the curve is *concave upwards*, and negative when d^2y/dx^2 is negative, i.e. when the curve is *convex upwards*. www.dbraulibrary.org.in

EXAMPLE 1. Find the radius of curvature at the point (1, 4) of the parabola $y = 4x^2$.

$$y = 4x^2 \quad \therefore \; dy/dx = 8x, \text{ and } d^2y/dx^2 = 8.$$

When $x = 1$, $dy/dx = 8$.

Now $$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2},$$

$$\therefore \text{when } x = 1, \rho = \frac{\{1 + 8^2\}^{3/2}}{8} = \frac{65^{3/2}}{8} = \frac{65\sqrt{65}}{8}.$$

EXAMPLE 2. Find the radius of curvature at the point (x, y) of the curve $xy = 4$ in terms of x, and deduce the radius of curvature when $x = 2$.

$$xy = 4 \qquad \therefore \; y = 4/x.$$

$$\frac{dy}{dx} = \frac{-4}{x^2}, \text{ and } \frac{d^2y}{dx^2} = \frac{8}{x^3}.$$

If ρ be the radius of curvature at the point (x, y) then

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2}$$

$$= \frac{\{1 + 16/x^4\}^{3/2}}{8/x^3} = \frac{x^3}{8}\left\{\frac{x^4 + 16}{x^4}\right\}^{3/2}$$

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$$= \frac{x^3}{8x^6}\{x^4 + 16\}^{3/2} = \frac{1}{8x^3}\{x^4 + 16\}^{3/2}.$$

When $x = 2$, $\rho = \frac{1}{8 \times 8}\{16 + 16\}^{3/2} = \frac{1}{64}\{16 \times 2\}^{3/2}$

$$= \frac{1}{64}\{16^{3/2} \times 2^{3/2}\} = \frac{1}{64} \times 4^3 \times 2\sqrt{2}$$

$$= 1/64 \times 64 \times 2\sqrt{2} = 2\sqrt{2}.$$

EXAMPLE 3. Given the curve $y = 2 \log_e x$, find the radius of curvature when $x = 1$.

$$y = 2 \log_e x \quad \therefore \frac{dy}{dx} = \frac{1}{2x}, \text{ and } \frac{d^2y}{dx^2} = \frac{-1}{2x^2}.$$

Radius of curvature ρ at (x, y) is given by

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{\{1 + 1/4x^2\}^{3/2}}{-1/2x^2}$$

$$= -2x^2\left\{\frac{4x^2 + 1}{4x^2}\right\}^{3/2} = -\frac{2x^2}{8x^3}\{4x^2 + 1\}^{3/2}$$

$$= -\frac{\{4x^2 + 1\}^{3/2}}{4x}.$$

When $x = 1$ the value of ρ is $-\frac{1}{4}\{4 + 1\}^{3/2} = -\frac{(5)^{3/2}}{4} = -\frac{5\sqrt{5}}{4}.$

EXAMPLE 4. Find the radius of curvature at the point (x, y) of the curve $y = e^{-2x}$.

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$y = e^{-2x}$ $\therefore dy/dx = -2e^{-2x}$, and $d^2y/dx^2 = 4e^{-2x}$.

Now the radius of curvature ρ at (x, y) is given by

$$\rho = \frac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \frac{\{1 + 4e^{-4x}\}^{3/2}}{4e^{-2x}} = \frac{e^{2x}}{4}\{1 + 4e^{-4x}\}^{3/2}.$$

Theorem. *To find the radius of curvature ρ when the curve is given in the parametric form $x = f_1(t)$, $y = f_2(t)$.*

In this case $\dot{x}$, $\dot{y}$ are used for dx/dt and dy/dt respectively, and $\ddot{x}$, $\ddot{y}$ for d^2x/dt^2 and d^2y/dt^2 respectively.

Now $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (function of a function theorem)

$$= \frac{dy}{dt} \times \frac{1}{dx/dt} = \frac{\dot{y}}{\dot{x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\dot{y}}{\dot{x}}\right) = \left\{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\right\}\frac{dt}{dx}$

(function of a function theorem)

$$= \left\{\frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right)\right\}\frac{1}{dx/dt} = \frac{\dot{x}\frac{d}{dt}(\dot{y}) - \dot{y}\frac{d}{dt}(\dot{x})}{(\dot{x})^2} \times \frac{1}{\dot{x}}$$

(differential of a quotient)

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$$= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x})^3} \quad \dots\dots\dots\dots \quad (2).$$

Now at the point (x, y) the radius of curvature ρ

$$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$

Using (1) and (2) in this $\rho = \dfrac{[1 + (\dot{y}/\dot{x})^2]^{3/2}}{(\dot{x}\ddot{y} - \ddot{x}\dot{y})/(\dot{x})^3}$

$$= \left\{\frac{(\dot{x})^2 + (\dot{y})^2}{(\dot{x})^2}\right\}^{3/2} \times \frac{(\dot{x})^3}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$= \frac{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}{(\dot{x})^3} \times \frac{(\dot{x})^3}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$$

$$= \frac{[(\dot{x})^2 + (\dot{y})^2]^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}.$$

N.B. It is sometimes easier to find dy/dx in terms of t and differentiate the result with respect to x, using the function of a function theorem to obtain d^2y/dx^2, and then use these values in the formula

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$$

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EXAMPLE 1. A curve is given by the equations $x = \sin 2t$, $y = 2 \cos t$. Find the radius of curvature ρ when $t = \pi/3$.

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$x = \sin 2t \;\; \therefore \; \dot{x} = 2 \cos 2t$, and $\ddot{x} = -4 \sin 2t$.
$y = 2 \cos t \;\; \therefore \; \dot{y} = -2 \sin t$, and $\ddot{y} = -2 \cos t$.
When $t = \pi/3$, $\dot{x} = 2(-\tfrac{1}{2}) = -1, \ddot{x} = -4(\sqrt{3}/2) = -2\sqrt{3}$,
$\dot{y} = (-2)(\sqrt{3}/2) = -\sqrt{3}, \ddot{y} = (-2)(\tfrac{1}{2}) = -1.$

Now $\qquad \rho = \dfrac{\{(\dot{x})^2 + (\dot{y})^2\}^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}}$,

$\therefore$ when $t = \dfrac{\pi}{3}$, $\rho = \dfrac{\{(-1)^2 + (-\sqrt{3})^2\}^{3/2}}{(-1)(-1) - (-2\sqrt{3})(-\sqrt{3})}$

$$= \frac{(1 + 3)^{3/2}}{1 - 6} = -\frac{4^{3/2}}{5} = -\frac{8}{5}.$$

EXAMPLE 2. Find the radius of curvature ρ at the point $(2at, at^2)$ of the curve $x^2 = 4ay$, and find its value at the point where $t = 2$.

$$x = 2at \;\; \therefore \; \dot{x} = 2a. \qquad\qquad y = at^2 \;\; \therefore \; \dot{y} = 2at.$$

$$\therefore \; \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{2at}{2a} = t, \text{ and } \frac{d^2y}{dx^2} = \frac{dt}{dx} = \frac{1}{dx/dt} = \frac{1}{\dot{x}} = \frac{1}{2a}.$$

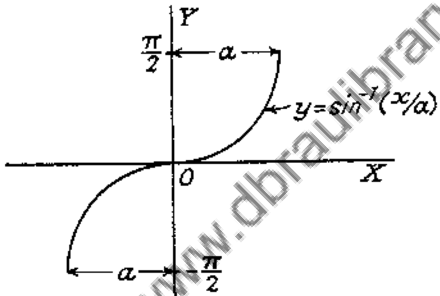
Now $\qquad \rho = \dfrac{\{1 + (dy/dx)^2\}^{3/2}}{d^2y/dx^2} = \dfrac{\{1 + t^2\}^{3/2}}{1/2a} = 2a(1 + t^2)^{3/2}.$

When $t = 2$, $\rho = 2a(1 + 4)^{3/2} = 2a \times 5^{3/2} = 10a\sqrt{5}.$

Differentiation of Inverse Trigonometric Functions. As stated previously, $\sin^{-1} x/a$ is the angle between $-\pi/2$ and $\pi/2$ whose sine has the value x/a; $\cos^{-1} x/a$ is the angle between 0 and π whose cosine has the value x/a; and $\tan^{-1} x/a$ is the angle between $-\pi/2$ and $\pi/2$ whose tangent has the value x/a. (a is a constant.)

N.B. Any ambiguity in sign in the derivative is decided by using the graph.

Theorem. *To find $\frac{d}{dx}$ ($\sin^{-1} x/a$), and to deduce the value of $\frac{d}{dx}$ ($\sin^{-1} x$).*



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Let $$y = \sin^{-1}(x/a)$$
$$\therefore \sin y = x/a,$$
$$\text{and } x = a \sin y \quad \ldots\ldots\ldots\ldots \quad (1)$$

Differentiating (1) with respect to y,

$$\frac{dx}{dy} = a \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{a \cos y}$$

$$= \frac{1}{\pm a\sqrt{(1 - \sin^2 y)}} = \frac{\pm 1}{\sqrt{(a^2 - a^2 \sin^2 y)}} = \frac{\pm 1}{\sqrt{(a^2 - x^2)}}.$$

From the graph it can be seen that the slope of the tangent at all points of the curve is positive, and therefore dy/dx is always positive.

Hence $$\frac{d}{dx} (\sin^{-1} x/a) = \frac{1}{\sqrt{(a^2 - x^2)}}.$$

Using $a = 1$ in the result,

$$\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{(1 - x^2)}}.$$

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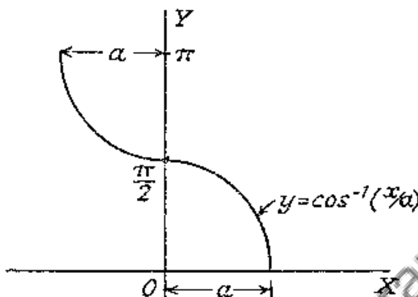
Theorem. *To find* $\frac{d}{dx}$ $(\cos^{-1} x/a)$, *and to deduce the value of* $\frac{d}{dx}(\cos^{-1} x)$.

Let

$$y = \cos^{-1}(x/a)$$

$$\therefore \cos y = x/a,$$

$$\text{and } x = a \cos y \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$



Differentiating (1) with respect to x,

$$1 = \frac{d}{dx}(a \text{ co s} y)$$

$$= \frac{d}{dy}(a \cos y) \cdot \frac{dy}{dx}$$

(function of a function theorem)

$$= -a \sin y\,(dy/dx)$$

$$\therefore \frac{dy}{dx} = \frac{-1}{a \sin y} = \frac{-1}{a\sqrt{(1 - \cos^2 y)}}$$

$$\text{i.e. } \frac{d}{dx}(\cos^{-1} x/a) = \frac{\pm 1}{\sqrt{(a^2 - a^2 \cos^2 y)}} = \frac{\pm 1}{\sqrt{(1 - x^2)}}$$

From the graph, since the slope of the tangent is negative at all points of the curve, it follows that dy/dx is always negative.

Thus $\frac{d}{dx}(\cos^{-1} x/a) = \frac{-1}{\sqrt{(a^2 - x^2)}}$

Using $a = 1$, $(d/dx)(\cos^{-1}x) = 1/\sqrt{(a^2 - x^2)}$.

Theorem. *To find the value of* $\frac{d}{dx}$ $(\tan^{-1} x/a)$, *and to deduce the value of* $\frac{d}{dx}(\tan^{-1} x)$.

In this case no graph is required as there is no ambiguity in sign in the result.

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Let $$y = \tan^{-1} x/a$$
$$\therefore \tan y = x/a,$$
$$\text{and } x = a \tan y \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Differentiating (1) with respect to x,

$$1 = \frac{d}{dx}(a \tan y)$$
$$= \left\{\frac{d}{dy}(a \tan y)\right\}\frac{dy}{dx}$$
(function of a function theorem)
$$= a \sec^2 y \frac{dy}{dx} = a(1 + \tan^2 y)\frac{dy}{dx}$$
$$= \frac{1}{a}(a^2 + a^2 \tan^2 y)\frac{dy}{dx} = \frac{a^2 + x^2}{a}\frac{dy}{dx}$$
$$\therefore \frac{dy}{dx} = \frac{a}{a^2 + x^2}, \text{ i.e. } \frac{d}{dx}(\tan^{-1} x/a) = \frac{a}{a^2 + x^2}$$

Using $a = 1$ in this result,

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

NOTE. The derivatives of $\sec^{-1} x/a$, $\text{cosec}^{-1} x/a$, $\cot^{-1} x/a$ will not be dealt with.

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EXAMPLE 1. Find the derivatives with respect to x of

(i) $\sin^{-1} 2x$, (ii) $\cos^{-1} x/3$, (iii) $\tan^{-1}(2x + 1)$.

(i) $$\frac{d}{dx}(\sin^{-1} 2x) = \frac{d}{dx}\left(\sin^{-1}\frac{x}{\frac{1}{2}}\right) = \frac{1}{\sqrt{[(\frac{1}{2})^2 - x^2]}}$$
(using the first theorem)
$$= \frac{1}{\sqrt{(\frac{1}{4} - x^2)}} = \frac{2}{\sqrt{(1 - 4x^2)}}.$$

(ii) $$\frac{d}{dx}(\cos^{-1} x/3) = \frac{-1}{\sqrt{(3^2 - x^2)}}$$ (using the second theorem)
$$= \frac{-1}{\sqrt{(9 - x^2)}}.$$

(iii) Let $z = 2x + 1 \therefore dy/dx = 2.$

$$\frac{d}{dx}\{\tan^{-1}(2x + 1)\} = \frac{d}{dx}(\tan^{-1} z) = \frac{d}{dz}(\tan^{-1} z).\frac{dz}{dx}$$
(function of a function theorem)
$$= \frac{1}{1 + z^2} \times 2 = \frac{2}{1 + (2x + 1)^2} = \frac{2}{4x^2 + 4x + 2}$$
$$= \frac{1}{2x^2 + 2x + 1}.$$

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EXAMPLE 2. Find the following

(i) $\frac{d}{dx}\{\sin^{-1}\sqrt{(1-x)}\}$, (ii) $\frac{d}{dx}\{\cos^{-1}(1-2x^2)\}$,

(iii) $\frac{d}{dx}\left\{\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$.

(i) Let $y = \sqrt{(1-x)}$, $\therefore \frac{dy}{dx} = \frac{-1}{2\sqrt{(1-x)}}$

$$\frac{d}{dx}[\sin^{-1}\sqrt{(1-x)}] = \frac{d}{dx}(\sin^{-1}y) = \left\{\frac{d}{dy}(\sin^{-1}y)\right\}\frac{dy}{dx}$$

(function of a function theorem)

$$= \frac{1}{\sqrt{(1-y^2)}}\cdot\frac{-1}{2\sqrt{(1-x)}}$$

$$= \frac{1}{\sqrt{[1-(1-x)]}}\cdot\frac{-1}{2\sqrt{(1-x)}}$$

$$= \frac{1}{\sqrt{x}}\cdot\frac{-1}{2\sqrt{(1-x)}} = \frac{-1}{2\sqrt{(x-x^2)}}.$$

(ii) Let $z = 1-2x^2$, $\therefore dz/dx = -4x$.

$$\frac{d}{dx}\{\cos^{-1}(1-2x^2)\} = \frac{d}{dx}(\cos^{-1}z) = \frac{d}{dz}(\cos^{-1}z)\frac{dz}{dx}$$

(function of a function theorem)

$$= \frac{-1}{\sqrt{[1-z^2]}}(-4x) = \frac{4x}{\sqrt{[1-(1-2x^2)^2]}}$$

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$$= \frac{4x}{\sqrt{(4x^2-4x^4)}} = \frac{4x}{2x\sqrt{(1-x^2)}}$$

$$= \frac{2}{\sqrt{(1-x^2)}}.$$

(iii) Let $t = \sqrt{\left\{\frac{1-x}{1+x}\right\}} = \frac{\sqrt{(1-x)}}{\sqrt{(1+x)}}$,

$$\therefore \frac{dt}{dx} = \frac{\sqrt{(1+x)}\frac{d}{dx}[\sqrt{(1-x)}] - \sqrt{(1-x)}\frac{d}{dx}[\sqrt{(1+x)}]}{1+x}$$

(differential of a quotient)

$$= \frac{\sqrt{(1+x)}\times\frac{-1}{2\sqrt{(1-x)}} - \sqrt{(1-x)}\times\frac{1}{2\sqrt{(1+x)}}}{1+x}$$

$$= \frac{-\{(1+x)+(1-x)\}}{2(1+x)\sqrt{(1-x^2)}} = \frac{-2}{2(1+x)\sqrt{(1-x^2)}}$$

$$= \frac{-1}{(1+x)\sqrt{(1-x^2)}}$$

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$$\frac{d}{dx}\left\{\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$$

$$= \frac{d}{dx}(\tan^{-1} t) = \frac{d}{dt}(\tan^{-1} t)\cdot\frac{dt}{dx}$$

(function of a function theorem)

$$= \frac{1}{1+t^2}\cdot\frac{dt}{dx}$$

$$= \frac{1}{1+(1-x)/(1+x)}\cdot\frac{-1}{(1+x)\sqrt{(1-x^2)}}$$

$$= \frac{1}{1+x+1-x}\cdot\frac{-1}{\sqrt{(1-x^2)}} = \frac{-1}{2\sqrt{(1-x^2)}}$$

EXAMPLE 3. Given $y = (2+3t)\sin^{-1} t$, find the values of dy/dt and d^2y/dt^2.

Using the differential of a product,

$$\frac{dy}{dt} = (2+3t)\frac{d}{dt}(\sin^{-1} t) + (\sin^{-1} t)\frac{d}{dt}(2+3t)$$

$$= (2+3t)\cdot\frac{1}{\sqrt{(1-t^2)}} + (\sin^{-1} t)\times 3 = \frac{2+3t}{\sqrt{(1-t^2)}} + 3\sin^{-1} t.$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{2+3t}{\sqrt{(1-t^2)}}\right) + \frac{d}{dt}(3\sin^{-1} t)$$

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$$= \frac{\sqrt{(1-t^2)}\frac{d}{dt}(2+3t) - (2+3t)\frac{d}{dt}[\sqrt{(1-t^2)}]}{1-t^2} + \frac{3}{\sqrt{(1-t^2)}}$$

$$= \frac{3\sqrt{(1-t^2)} - (2+3t)\left(\frac{-t}{\sqrt{(1-t^2)}}\right)}{1-t^2} + \frac{3}{\sqrt{(1-t^2)}}$$

$$= \frac{3(1-t^2)+t(2+3t)}{(1-t^2)^{3/2}} + \frac{3}{(1-t^2)^{1/2}}$$

$$= \frac{3-3t^2+2t+3t^2+3-3t^2}{(1-t^2)^{3/2}} = \frac{6+2t-3t^2}{(1-t^2)^{3/2}}.$$

Logarithmic Differentiation of a Product. Consider

$$y = uvw\ldots \qquad \ldots\ldots\ldots\ldots(1),$$

where u, v, w, etc., are all functions of x.

Taking logarithms to the base e in (1),

$$\log_e y = \log_e u + \log_e v + \log_e w + \ldots\ldots\ldots\ldots(2).$$

Differentiating (2) with respect to x,

$$\frac{d}{dx}(\log_e y) = \frac{d}{dx}(\log_e u) + \frac{d}{dx}(\log_e v) + \frac{d}{dx}(\log_e w) + \ldots$$

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Using the function of a function theorem this becomes

$$\frac{d}{dy}(\log_e y)\cdot\frac{dy}{dx} = \frac{d}{du}(\log_e u)\cdot\frac{du}{dx} + \frac{d}{dv}(\log_e v)\cdot\frac{dv}{dx} + \frac{d}{dx}(\log_e w)\cdot\frac{dw}{dx} + \ldots,$$

$$\text{i.e. } \frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \ldots$$

$$\therefore \frac{dy}{dx} = y\left\{\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \ldots\right\}$$

$$= uvw\ldots\left\{\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \ldots\right\}$$

This is known as *logarithmic differentiation* and is extremely useful for the derivative of a product, especially when one of the functions is of the form $e^{f(x)}$.

EXAMPLE. Using logarithmic differentiation, to find the derivatives with respect to x of (i) $x^3 \tan(2 + 3x).\sin 3x$, (ii) $(3x - 2)^2 e^{2x+3} \cos(2 - x)$.

(i) Let $y = x^3 \tan(2 + 3x) \sin 3x$,

$$\therefore \log_e y = 3\log_e x + \log_e\{\tan(2 + 3x)\} + \log_e(\sin 3x) \ldots\ldots (1).$$

Differentiating (1) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{1}{\tan(2 + 3x)}\frac{d}{dx}\{\tan(2 + 3x)\} + \frac{1}{\sin 3x}\frac{d}{dx}(\sin 3x)$$

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$$= \frac{3}{x} + \frac{1}{\tan(2 + 3x)} \times \frac{3}{\cos^2(2 + 3x)} + \frac{1}{\sin 3x}\cdot 3\cos 3x$$

$$= \frac{3}{x} + \frac{3}{\sin(2 + 3x)\cos(2 + 3x)} + 3\cot 3x$$

$$= \frac{3}{x} + \frac{6}{\sin(4 + 6x)} + 3\cot 3x$$

$$\therefore \frac{dy}{dx} = y\left\{\frac{3}{x} + \frac{6}{\sin(4 + 6x)} + 3\cot 3x\right\}$$

$$= x^3\tan(2 + 3x)\sin 3x\left\{\frac{3}{x} + \frac{6}{\sin(4 + 6x)} + 3\cot 3x\right\}.$$

(ii) Let $y = (3x - 2)^2 e^{2x+3}\cos(2 - x)$,

$$\therefore \log_e y = 2\log_e(3x - 2) + 2x + 3 + \log_e\{\cos(2 - x)\} \ldots (1).$$

Differentiating (1) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 2 \times \frac{3}{3x - 2} + 2 + \frac{1}{\cos(2 - x)}\frac{d}{dx}\{\cos(2 - x)\}$$

$$= \frac{6}{3x - 2} + 2 + \frac{1}{\cos(2 - x)} \times \sin(2 - x)$$

$$= \frac{6}{3x - 2} + 2 + \tan(2 - x)$$

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$$\therefore \; \frac{dy}{dx} = y\left\{\frac{6}{3x - 2} + 2 + \tan(2 - x)\right\}$$

$$= (3x - 2)^2 e^{2x+3} \cos(2 - x)\left\{\frac{6}{3x - 2} + 2 + \tan(2 - x)\right\}$$

N.B. Logarithmic differentiation can also be used for differentiating a single function of the variable where the derivative cannot be obtained by previous methods, as shown in the following example.

EXAMPLE. Find the values of dy/dx, where (i) $y = (ax + b)^{cx+d}$, (ii) $y = \{\log_e(ax + b)\}^{cx+d}$, a, b, c, and d being constants.

(i) $y = (ax + b)^{cx+d}$ $\therefore$ $\log_e y = (cx + d)\log_e(ax + b)$ .... (1)

Differentiating (1) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = (cx + d)\frac{d}{dx}\{\log_e(ax + b)\} + \log_e(ax + b)\,.\,\frac{d}{dx}(cx + d)$$

(differential of a product)

$$= (cx + d) \times \frac{a}{ax + b} + c\log_e(ax + b)$$

$$\therefore \; \frac{dy}{dx} = y\left\{\frac{a(cx + d)}{ax + b} + c\log_e(ax + b)\right\}$$

$$= (ax + b)^{cx+d}\left\{\frac{a(cx + d)}{ax + b} + c\log_e(ax + b)\right\}$$

(ii) Let $q = \log_e z$, where $z = \log_e(ax + b)$,

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$$\therefore \; \frac{dq}{dx} = \frac{d}{dz}(\log_e z)\,.\,\frac{dz}{dx} \qquad \text{(function of a function theorem)}$$

$$= \frac{1}{z}\,.\,\frac{a}{ax + b} = \frac{a}{(ax + b)\log_e(ax + b)}$$

$$y = \{\log_e(ax + b)\}^{cx+d} = z^{cx+d}.$$

$\therefore$ $\log_e y = (cx + d)\log_e z = (cx + d)q$ .... (1).

Differentiating (1) with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = (cx + d)\frac{dq}{dx} + q\frac{d}{dx}(cx + d)$$

$$= (cx + d)\,.\,\frac{a}{(ax + b)\log_e(ax + b)} + cq$$

$$= \frac{a(cx + d)}{(ax + b)\log_e(ax + b)} + c\log_e(ax + b),$$

$$\therefore \; \frac{dy}{dx} = y\left\{\frac{a(cx + d)}{(ax + b)\log_e(ax + b)} + c\log_e(ax + b)\right\}$$

$$= \{\log_e(ax + b)\}^{ax+b}\left\{\frac{a(cx + d)}{(ax + b)\log_e(ax + b)} + c\log_e(ax + b)\right\}$$

Integration by Partial Fractions. If the integrand is in the form of an algebraical fraction and the integral cannot be evaluated by previous

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methods, the fraction should be expressed in partial fractions before integration takes place.

Theorem. *To find* $\int \frac{dx}{a^2 - x^2}$, *where* $x^2 < a^2$, *and* a *is a constant.*

Let $\frac{1}{a^2 - x^2} \equiv \frac{A}{a + x} + \frac{B}{a - x}$.

Then $1 \equiv A(a - x) + B(a + x)$.

Using $x = -a$ in this identity,

$$1 = 2aA \quad \therefore A = \tfrac{1}{2}a.$$

Also when $x = a$, $1 = 2aB \quad \therefore B = \tfrac{1}{2}a.$

$$\therefore \int \frac{dx}{a^2 - x^2} = \int \left\{\frac{1}{2a(a + x)} + \frac{1}{2a(a - x)}\right\} dx$$

$$= \frac{1}{2a}\int \left(\frac{1}{a + x} + \frac{1}{a - x}\right) dx$$

$$= \frac{1}{2a}[\log_e (a + x) - \log_e (a - x)] + C$$

$$= \frac{1}{2a} \log_e \frac{a + x}{a - x} + C$$

In a similar manner it can be shown that

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e \frac{x - a}{x + a} + C, \quad \text{where } x^2 > a^2$$

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EXAMPLE 1. Evaluate (i) $\int_0^2 \frac{dx}{9 - x^2}$, (ii) $\int_3^4 \frac{dx}{x^2 - 4}$.

(i) Using the theorem, $\int_0^2 \frac{dx}{9 - x^2} = \left[\frac{1}{2 \times 3} \log_e \frac{3 + x}{3 - x}\right]_0^2$

$$= \left[\frac{1}{6} \log_e \frac{3 + x}{3 - x}\right]_0^2$$

$$= \tfrac{1}{6}\{\log_e 5 - \log_e 1\} = \tfrac{1}{6} \log_e 5$$

$$= \tfrac{1}{6} \times 1{\cdot}60944 = 0{\cdot}2682 \text{ to 4 s. f.}$$

(ii) Using the previous theorem,

$$\int_3^4 \frac{dx}{x^2 - 4} = \left[\frac{1}{2 \times 2} \log_e \frac{x - 2}{x + 2}\right]_3^4 = \frac{1}{4}\left[\log_e \frac{2}{6} - \log_e \frac{1}{5}\right]$$

$$= \frac{1}{4}\left[\log_e \frac{1}{3} - \log_e \frac{1}{5}\right] = \frac{1}{4}[-\log_e 3 + \log_e 5]$$

$$= \tfrac{1}{4}[-1{\cdot}09861 + 1{\cdot}60944] = \tfrac{1}{4}(0{\cdot}51083)$$

$$= 0{\cdot}1277 \text{ to 4 significant figures.}$$

EXAMPLE 2. Find $I = \int \frac{3dx}{9 - 4x^2}$.

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Let $$\frac{3}{9-4x^2} \equiv \frac{A}{3-2x} + \frac{B}{3+2x}$$

$$\therefore\ 3 \equiv A(3+2x) + B(3-2x).$$

In this identity using,

$x = 3/2,$ $3 = 6A\ \therefore\ A = \frac{1}{2},$

$x = 3/2,$ $3 = 6B\ \therefore\ B = \frac{1}{2}.$

Thus $$\frac{3}{9-4x^2} = \int\frac{1}{2}\left\{\frac{1}{3-2x} + \frac{1}{3+2x}\right\}.$$

Hence $$I = \frac{1}{2}\left\{\frac{1}{3-2x} + \frac{1}{3+2x}\right\}\,dx$$

$$= \frac{1}{2}\int\left\{\frac{1}{3+2x} + \frac{1}{3-2x}\right\}\,dx$$

$$= \tfrac{1}{2}\{\tfrac{1}{2}\log_e(3+2x) - \tfrac{1}{2}\log_e(3-2x)\} + C$$

$$= \tfrac{1}{4}\{\log_e(3+2x) - \log_e(3-2x)\} + C$$

$$= \frac{1}{4}\log_e\frac{3+2x}{3-2x} + C.$$

NOTE. In all other cases the integrand will be expressed in partial fractions (as in example 2) before integration takes place.

EXAMPLE 1. Evaluate the following

(i) $\displaystyle\int_0^1 \frac{x}{x^2+5x+6}\,dx,$ (ii) $\displaystyle\int\frac{dx}{x(2x-1)^2},$ (iii) $\displaystyle\int\frac{dx}{x(x-1)(2x+1)}.$

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(i) Let $\dfrac{x}{x^2+5x+6} \equiv \dfrac{A}{x+2} + \dfrac{B}{x+3},\ \therefore\ x \equiv A(x+3) + B(x+2).$

In this identity, using,

$x = -2,$ $-2 = A\ \therefore\ A = -2,$

$x = -3,$ $-3 = -B\ \therefore\ B = 3,$

$$\therefore\ \frac{x}{x^2+5x+6} = \frac{3}{x+3} - \frac{2}{x+2}.$$

Hence $$\int_0^1\frac{x}{x^2+5x+6}\,dx = \int_0^1\left(\frac{3}{x+3} - \frac{2}{x+2}\right)dx$$

$$= \Big[3\log_e(x+3) - 2\log_e(x+2)\Big]_0^1$$

$$= [(3\log_e 4 - 3\log_e 3) - (2\log_e 3 - 2\log_e 2)]$$

$$= 3\log_e 4 - 5\log_e 3 + 2\log_e 2$$

$$= 3\times 1{\cdot}38629 - 5\times 1{\cdot}09861 + 2\times 0{\cdot}69315$$

$$= 4{\cdot}15887 - 5{\cdot}49305 + 1{\cdot}38630$$

$$= 5{\cdot}54517 - 5{\cdot}49305$$

$= 0{\cdot}0521$ to 4 decimal places.

(ii) Let $\dfrac{1}{x(2x-1)^2} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{(2x-1)^2},$

$$\therefore\ 1 \equiv A(2x-1)^2 + Bx(2x-1) + Cx.$$

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In this identity, using,

$x = 0,\qquad 1 = A \quad \therefore\ A = 1,$
$x = \frac{1}{2},\qquad 1 = \frac{1}{2}C \quad \therefore\ C = 2,$
$x = 1,\qquad 1 = A + B + C = 1 + B + 2,$
$\therefore\ B = -2.$

Hence $\dfrac{1}{x(2x-1)^2} = \dfrac{1}{x} - \dfrac{2}{2x-1} + \dfrac{2}{(2x-1)^2}$

Thus,
$$\int \frac{dx}{x(2x-1)^2} = \int \left\{\frac{1}{x} - \frac{2}{2x-1} + \frac{2}{(2x-1)^2}\right\} dx$$
$$= \log_e x - \log_e(2x-1) - \frac{1}{2x-1} + C$$
$$= \log_e \frac{x}{2x-1} - \frac{1}{2x-1} + C.$$

(iii) Let $\dfrac{1}{x(x-1)(2x+1)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{2x+1}$,
$$\therefore\ 1 \equiv A(x-1)(2x+1) + Bx(2x+1) + Cx(x-1).$$

In this identity, using,

$x = 0,\qquad 1 = -A \quad \therefore\ A = -1,$
$x = 1,\qquad 1 = 3B \quad \therefore\ B = 1/3,$
$x = -\frac{1}{2},\qquad 1 = \frac{3}{4}C \quad \therefore\ C = 4/3,$

$$\therefore\ \frac{1}{x(x-1)(2x+1)} = \frac{-1}{x} + \frac{1}{3(x-1)} + \frac{4}{3(2x+1)}$$

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Hence
$$\int \frac{dx}{x(x-1)(2x+1)} = \int \left\{-\frac{1}{x} + \frac{1}{3(x-1)} + \frac{4}{3(2x+1)}\right\} dx$$
$$= -\log_e x + \tfrac{1}{3}\log_e(x-1) + \tfrac{2}{3}\log_e(2x+1) + C$$
$$= \tfrac{1}{3}\{-3\log_e x + \log_e(x-1) + 2\log_e(2x+1)\} + C$$
$$= \tfrac{1}{3}\{-\log_e x^3 + \log_e(x-1) + \log_e(2x+1)^2\} + C$$
$$= \frac{1}{3}\log_e \frac{(x-1)(2x+1)^2}{x^3} + C.$$

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EXAMPLE 2. Express $\dfrac{2 - x + x^2}{(1+x)(1-x)^2}$ in partial fractions, and evaluate
$$\int_0^{1/2} \frac{2 - x + x^2}{(1+x)(1-x)^2}\, dx.$$

Let $\dfrac{2 - x + x^2}{(1+x)(1-x)^2} \equiv \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{C}{(1-x)^2}$,
$$\therefore\ 2 - x + x^2 = A(1-x)^2 + B(1+x)(1-x) + C(1+x).$$

In this identity, using,

$x = -1,\qquad 4 = 4A \quad \therefore\; A = 1,$
$x = 1,\qquad 2 = 2C \quad \therefore\; C = 1,$
$x = 0,\qquad 2 = A + B + C$
$\qquad = 1 + B + 1 = 2 + B$
$\therefore\; B = 0.$

Hence $\dfrac{2 - x + x^2}{(1 + x)(1 - x)^2} \equiv \dfrac{1}{1 + x} + \dfrac{1}{(1 - x)^2}.$

Using the partial fractions obtained,

$$\int_0^{1/2} \frac{2 - x + x^2}{(1 + x)(1 - x)^2}\,dx = \int_0^{1/2} \left\{\frac{1}{1 + x} + \frac{1}{(1 - x)^2}\right\} dx$$
$$= \left[\log_e (1 + x) + \frac{1}{1 - x}\right]_0^{1/2}$$
$$= (\log_e 1{\cdot}5 + 2) - (\log_e 1 + 1)$$
$$= \log_e 1{\cdot}5 + (2 - 1) = \log_e 1{\cdot}5 + 1$$
$$= 0{\cdot}40547 + 1 = 1{\cdot}4055 \text{ to 4 decimal places.}$$

Theorem. *To find* (i) $I = \displaystyle\int \frac{dx}{\sqrt{(a^2 - x^2)}}$, (ii) $I = \displaystyle\int \frac{dx}{a^2 + x^2}$.

(i) Now it has been shown that

$$\frac{d}{dx}(\sin^{-1} x/a) = \frac{1}{\sqrt{(a^2 - x^2)}}, \text{ and } \frac{d}{dx}(\cos^{-1} x/a)$$
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$$= \frac{-1}{\sqrt{(a^2 - x^2)}} \quad (x^2 < a^2).$$

Hence $\displaystyle\int \frac{dx}{\sqrt{(a^2 - x^2)}} = \sin^{-1}\frac{x}{a} + C = -\cos^{-1}\frac{x}{a} + C.$

Using $a = 1$ in this, $\displaystyle\int \frac{dx}{\sqrt{(1 - x^2)}} = \sin^{-1} x + C = -\cos^{-1} x + C.$

(ii) It was shown earlier that

$$\frac{d}{dx}(\tan^{-1} x/a) = \frac{a}{a^2 + x^2}$$

$$\therefore \int \frac{a}{a^2 + x^2}\,dx = \tan^{-1} x/a, \quad \text{(indefinite integral)}$$

$$\text{i.e. } a\int \frac{dx}{a^2 + x^2} = \tan^{-1} x/a, \quad \text{(indefinite integral)}$$

$$\text{i.e. } \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$$

It follows that $\displaystyle\int \frac{dx}{1 + x^2} = \tan^{-1}x + C.$

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EXAMPLE. Find

(i) $\int_0^1 \frac{dx}{\sqrt{(4 - x^2)}}$, (ii) $\int_1^2 \frac{dx}{4 + x^2}$, (iii) $\int \frac{dx}{\sqrt{(4 - 9x^2)}}$, (iv) $\int \frac{dx}{9 + 25x^2}$.

(i) $$\int_0^1 \frac{dx}{\sqrt{(4 - x^2)}} = \left[\sin^{-1}\frac{x}{2}\right]_0^1$$ (using result (i) of the theorem)

$$= [\sin^{-1}\tfrac{1}{2} - 0] = \pi/6.$$

(ii) $$\int_1^2 \frac{dx}{4 + x^2} = \left[\tfrac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^2$$ (using result (ii) of the theorem)

$$= \tfrac{1}{2}\tan^{-1}1 - \tfrac{1}{2}\tan^{-1}0{\cdot}5 = \tfrac{1}{2}(\tan^{-1}1 - \tan^{-1}0{\cdot}5)$$
$$= \tfrac{1}{2}(0{\cdot}78540 - 0{\cdot}46368) = \tfrac{1}{2} \times 0{\cdot}32172$$
$$= 0{\cdot}1609 \text{ to 4 decimal places.}$$

(iii) $$\int \frac{dx}{\sqrt{(4 - 9x^2)}} = \int \frac{dx}{3\sqrt{(4/9 - x^2)}} = \frac{1}{3}\int \frac{dx}{\sqrt{\{(\frac{2}{3})^2 - x^2\}}}$$
$$= \frac{1}{3}\sin^{-1}\frac{x}{2/3} + C$$
$$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + C.$$

(iv) $$\int \frac{dx}{9 + 25x^2} = \int \frac{dx}{25(9/25 + x^2)} = \frac{1}{25}\int \frac{dx}{(3/5)^2 + x^2}$$
$$= \frac{1}{25} \times \frac{1}{3/5}\tan^{-1}\frac{x}{3/5} + C$$
$$= \frac{1}{15}\tan^{-1}\frac{5x}{3} + C.$$

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NOTE. The result $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

can sometimes be used when dealing with integrands involving the theory of partial fractions, as shown in the following examples.

N.B. When a fraction contains a denominator of the form $px^2 + qx + r$, which cannot be factorised further, the corresponding partial fraction is

$$\frac{Ax + B}{px^2 + qx + r}.$$

EXAMPLE. Evaluate the following integrals,

(i) $\int \frac{x^2 + x + 4}{(x+1)(x^2+3)}\,dx$, (ii) $\int \frac{4x^2 + 4x + 25}{x(4x^2 + 25)}\,dx$, (iii) $\int \frac{2x^2 + 6x + 15}{(2x - 1)(x^2 + 9)}\,dx$.

(i) Let $$\frac{x^2 + x + 4}{(x + 1)(x^2 + 3)} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 3},$$

$$\therefore x^2 + x + 4 \equiv A(x^2 + 3) + (x + 1)(Bx + C).$$

In this identity, using,

$x = -1$, $\quad 4 = 4A \quad \therefore A = 1,$

$x = 0$, $\quad 4 = 3A + C = 3 + C \quad \therefore C = 1,$

$x = 1$, $\quad 6 = 4A + 2B + 2C = 4 + 2B + 2$
$\quad = 6 + 2B \quad \therefore B = 0.$

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Thus $\dfrac{x^2+x+4}{(x+1)(x^2+3)} = \dfrac{1}{x+1} + \dfrac{1}{x^2+3}$,

and $\displaystyle\int \frac{x^2+x+4}{(x+1)(x^2+3)}\,dx = \int\left(\frac{1}{x+1} + \frac{1}{x^2+3}\right) dx$

$$= \log_e (x+1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C.$$

(ii) Let $\dfrac{4x^2+4x+25}{x(4x^2+25)} \equiv \dfrac{A}{x} + \dfrac{Bx+C}{4x^2+25}$,

$\therefore\ 4x^2+4x+25 \equiv A(4x^2+25) + x(Bx+C)$.

In this identity, using,

$x = 0$, $\quad 25 = 25A \quad \therefore\ A = 1$,

$x = 1$, $\quad 33 = 29A + B + C = 29 + B + C$

$\therefore\ B + C = 4$ .... (1),

$x = -1$, $\quad 25 = 29A + B - C = 29 + B - C$

$\therefore\ B - C = -4$ .... (2),

(1) + (2) gives $\quad 2B = 0 \quad \therefore\ B = 0$;

and using this in (1) $\quad C = 4$.

Thus $\dfrac{4x^2+4x+25}{x(4x^2+25)} = \dfrac{1}{x} + \dfrac{4}{4x^2+25}$

$$= \frac{1}{x} + \frac{1}{x^2+25/4}.$$

Hence $\displaystyle\int \frac{4x^2+4x+25}{x(4x^2+25)}\,dx = \int\left(\frac{1}{x} + \frac{1}{x^2+25/4}\right) dx$

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$$= \log_e x + \frac{1}{5/2} \tan^{-1} \frac{x}{5/2} + C$$

$$= \log_e x + \frac{2}{5} \tan^{-1} \frac{2x}{5} + C.$$

(iii) Let $\dfrac{2x^2+6x+15}{(2x-1)(x^2+9)} \equiv \dfrac{A}{2x-1} + \dfrac{Bx+C}{x^2+9}$,

$\therefore\ 2x^2+6x+15 \equiv A(x^2+9) + (2x-1)(Bx+C)$.

In this identity, using,

$x = \frac{1}{2}$, $\quad \frac{1}{2} + 3 + 15 = \dfrac{37}{4}A$,

i.e. $\dfrac{37}{2} = \dfrac{37}{4}A \quad \therefore\ A = 2$,

$x = 0$, $\quad 15 = 9A - C = 18 - C \quad \therefore\ C = 3$,

$x = 1$, $\quad 23 = 10A + B + C = 20 + B + 3 = 23 + B$

$\therefore\ B = 0$.

Thus $\displaystyle\int \frac{2x^2+6x+15}{(2x-1)(x^2+9)}\,dx = \int \frac{2}{2x-1}\,dx + \int \frac{3}{x^2+9}\,dx$

$$= 2 \times \tfrac{1}{2} \log_e (2x-1) + 3 \times \tfrac{1}{3} \tan^{-1} x/3 + C$$

$$= \log_e (2x-1) + \tan^{-1} x/3 + C.$$

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Integration by Substitution. Certain integrals cannot be dealt with by the standard methods already employed, and these require the method of substitution, i.e. the introduction of a new variable.

Theorem. *If $y = f(x)$, and $x = \varphi(z)$ to prove that*

$$\int f(x)\,dx = \int F(z)\,.\,\varphi'(z)\,dz,$$

where $F(z)$ is the result of replacing x by $\varphi(z)$ in $f(x)$, and $\varphi'(z)$ is $\frac{d}{dz}[\varphi(z)]$.

Let $I = \int f(x)\ dx, \quad \therefore \frac{dI}{dx} = f(x) = F(z).$

Using the function of a function theorem this becomes

$$\frac{dI}{dz}\cdot\frac{dz}{dx} = F(z)$$

$$\therefore \frac{dI}{dz} = \frac{F(z)}{dz/dx} = F(z)\frac{dx}{dz} = F(z)\varphi'(z).$$

Hence $$I = \int F(z)\varphi'(z)\,dz,$$

$$\text{i.e.} \int f(x)\,dx = \int F(z)\varphi'(z)\,dz.$$

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NOTE. When making the substitution $x = \varphi(z)$ it is customary to write $dx = \varphi'(z)\,dz$, which really means that, in the integral, *dx is replaced by $\varphi'(z)\,dz$.*

EXAMPLE 1. Using the substitution $\sin\theta = z$, find the values of

(i) $I = \int \sin^2\theta\cos\theta\,d\theta$, (ii) $I = \int \cos^5\theta\,d\theta$, (iii) $I = \int \sin^2\theta\cos^3\theta\,d\theta$.

(i) $\sin\theta = z \quad \therefore \cos\theta\,d\theta = dz$

Using these in the integral,

$$I = \int z^2\,dz = \tfrac{1}{3}z^3 + C = \tfrac{1}{3}\sin^3\theta + C.$$

(ii) $\sin\theta = z \quad \therefore \cos\theta\,d\theta = dz.$

$$I = \int \cos^4\theta\,.\cos\theta\,d\theta = \int (1 - \sin^2\theta)^2\,dz$$

$$= \int (1 - z^2)^2\,dz = \int (1 - 2z^2 + z^4)\,dz$$

$$= z - \frac{2}{3}z^3 + \frac{1}{5}z^5 + C$$

$$= \sin\theta - 2/3\sin^3\theta + 1/5\sin^5\theta + C.$$

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(iii) $\sin\theta = z \quad \therefore \quad \cos\theta\, d\theta = dz.$

$$I = \int \sin^2\theta\cos^2\theta . \cos\theta\, d\theta = \int \sin^2\theta\,(1 - \sin^2\theta)\, dz$$

$$= \int z^2(1 - z^2)\, dz = \int (z^2 - z^4)\, dz$$

$$= \frac{1}{3}z^3 - \frac{1}{5}z^5 + C = \frac{1}{3}\sin^3\theta - \frac{1}{5}\sin^5\theta + C.$$

EXAMPLE 2. Using the substitution $\sqrt{(1+x^2)} = z$, find the value of

$$I = \int \frac{dx}{x\sqrt{(1 + x^2)}}.$$

$\sqrt{(1 + x^2)} = z$

$\therefore \quad \dfrac{x}{\sqrt{(1 + x^2)}}\, dx = dz,$ (differential of L.H.S. $\times\, dx$ = differential of R.H.S. $\times\, dz$)

i.e. $\dfrac{dx}{\sqrt{(1 + x^2)}} = \dfrac{dz}{x}.$

Hence $$I = \int \frac{1}{x}\cdot\frac{dx}{\sqrt{(1 + x^2)}} = \int \frac{1}{x}\cdot\frac{dz}{x} = \int \frac{dz}{x^2}$$

$$= \int \frac{dz}{z^2 - 1} \quad (1 + x^2 = z^2 \quad \therefore \quad x^2 = z^2 - 1)$$

$$= \tfrac{1}{2}\log_e \frac{z - 1}{z + 1} + C = \tfrac{1}{2}\log_e \frac{\sqrt{(1 + x^2)} - 1}{\sqrt{(1 + x^2)} + 1} + C.$$

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NOTE. When dealing with definite integrals it is advisable to change the limits at the same time as making the substitution, the new limits being found by using the original limits in the substitution that is being used.

EXAMPLE 1. By using the substitution $\sin^{-1} x/2 = \theta$ find the value of $I = \int_0^1 \sqrt{(4 - x^2)}\, dx.$

$\theta = \sin^{-1} x/2 \quad \therefore$ when $x = 0$, $\theta = \sin^{-1} 0 = 0$,
and when $x = 1$, $\theta = \sin^{-1}\tfrac{1}{2} = \pi/6.$

Hence 0 will be the lowest limit and $\pi/6$ the upper limit in the new integral in terms of θ.

Also $x = 2\sin\theta \quad \therefore \quad dx = 2\cos\theta\, d\theta.$

Hence $$I = \int_0^1 \sqrt{(4 - x^2)}\, dx = \int_0^{\pi/6} \sqrt{(4 - 4\sin^2\theta)} . 2\cos\theta\, d\theta$$

$$= \int_0^{\pi/6} 2\sqrt{(1 - \sin^2\theta)} . 2\cos\theta\, d\theta = 4\int_0^{\pi/6} \cos\theta . \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/6} 2\cos^2\theta\, d\theta = 2\int_0^{\pi/6} (1 + \cos 2\theta)\, d\theta$$

$$= 2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/6} = 2[\pi/6 + \tfrac{1}{2}\sin \pi/3]$$

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$$= \frac{\pi}{3} + \sin\frac{\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$

EXAMPLE 2. Using the substitution $\sqrt[3]{x} = z$, evaluate

$$I = \int_0^1 (2 - x)\sqrt[3]{x}\,dx$$

to four decimal places.

$\sqrt[3]{4x} = z$ $\therefore$ when $x = 0$, $z = 0$, and when $x = 1$, $z = 1$.

Also $x = z^3$ $\therefore$ $dx = 3z^2\,dz.$

$$I = \int_0^1 (2 - x)\sqrt[3]{x}\,dx = \int_0^1 (2 - z^3)z\,.\,3z^2\,dz$$

$$= 3\int_0^1 z^3(2 - z^3)\,dz = 3\int_0^1 (2z^3 - z^6)\,dz$$

$$= 3\left[\frac{z^4}{2} - \frac{z^7}{7}\right]_0^1 = 3\left[\frac{1}{2} - \frac{1}{7}\right] = 3 \times \frac{5}{14}$$

$= 15/14 = 1{\cdot}0714$ to 4 decimal places.

EXAMPLE 3. Using the substitution $(1 + x^2)^{1/2} = z$ evaluate

$$I = \int_0^1 \frac{x\,dx}{(1 + x^2)^{3/2}}.$$

$$(1 + x^2)^{1/2} = z \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

$\therefore$ when $x = 0$, $z = 1$, and when $x = 1$, $z = 2$.

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From (1) $\dfrac{x}{(1 + x^2)^{1/2}}\,dx = dz.$

Now $$I = \int_0^1 \frac{1}{1 + x^2}\cdot\frac{x}{(1 + x^2)^{1/2}}\,dx = \int_1^2 \frac{1}{z^2}\,dz$$

$$= \left[-\frac{1}{z}\right]_1^2 = -\tfrac{1}{2} + 1 = \tfrac{1}{2}.$$

Theorem. *Using the substitution $u = a - x$, prove that*

$$\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx.$$

$u = a - x$ $(x = a - u)$ $\therefore$ $du = -dx$, or $dx = -du$.

Also when $x = 0$, $u = a$, and when $x = a$, $u = 0$.

Thus, with this substitution,

$$\int_0^a f(x)\,dx = \int_a^0 f(a - u)\,.\,(-du)$$

$$= -\int_a^0 f(a - u)\,du$$

$$= \int_0^a f(a - u)\,du \qquad \left(\text{since} \int_0^a F(x)\,dx = -\int_a^0 F(x)\,dx\right)$$

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$$= \int_0^a f(a - x)\,dx. \qquad \text{(replacing } u \text{ by } x\text{)}$$

EXAMPLE. By means of this theorem find the values of

$$\text{(i)} \int_0^3 x(3 - x)^3\,dx, \quad \text{(ii)} \int_0^\pi x \sin^2 x\,dx.$$

(i) Using the theorem,

$$\int_0^3 x(3 - x)^3\,dx = \int_0^3 (3 - x)\{3 - (3 - x)\}^3\,dx$$
$$= \int_0^3 (3 - x)x^3\,dx = \int_0^3 (3x^3 - x^4)\,dx$$
$$= \left[\frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^3 = \frac{3}{4} \times 81 - \frac{243}{5}$$
$$= 243(1/4 - 1/5) = 243/20 = 12{\cdot}15.$$

(ii) Using the theorem,

$$\int_0^\pi x \sin^2 x\,dx = \int_0^\pi (\pi - x)\sin^2(\pi - x)\,dx$$
$$= \int_0^\pi (\pi - x)\sin^2 x\,dx$$
$$= \int_0^\pi \pi \sin^2 x\,dx - \int_0^\pi x \sin^2 x\,dx,$$

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$$\therefore\ 2\int_0^\pi x \sin^2 x\,dx = \pi\int_0^\pi \sin^2 x\,dx = \frac{\pi}{2}\int_0^\pi (1 - \cos 2x)\,dx$$
$$= \frac{\pi}{2}\left[x - \tfrac{1}{2}\sin 2x\right]_0^\pi$$
$$= \frac{\pi}{2}[\pi - 0] = \frac{\pi^2}{2}.$$

Hence $\displaystyle\int_0^\pi x \sin^2 x\,dx = \frac{\pi^2}{4}.$

NOTE. Generally, when the integrand contains $\sqrt{(a+x)}$ the substitution to be made (if not given) is $\sqrt{(a+x)} = z$, and when $a = 0$ this reduces to $\sqrt{x} = z$.

EXAMPLE. Evaluate (i) $I = \displaystyle\int \frac{dx}{(x + 4)\sqrt{x}}$, (ii) $I = \displaystyle\int \frac{\sqrt{(x + 3)}}{x + 5}\,dx$.

(i) Let $\sqrt{x} = z$ $\therefore$ $x = z^2$, and $dx = 2z\,dz$. Also $x + 4 = z^2 + 4$.

Hence
$$I = \int \frac{2z\,dz}{(z^2 + 4)z} = 2\int \frac{dz}{z^2 + 4}$$
$$= 2 \times \tfrac{1}{2}\tan^{-1} z/2 + C = \tan^{-1}\sqrt{x}/2 + C.$$

(ii) Let $\sqrt{(x + 3)} = z$ $\therefore$ $x + 3 = z^2$, from which $dx = 2z\,dz$, and $x + 5 = z^2 + 2$.

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Thus
$$I = \int \frac{z}{z^2+2} \cdot 2z\,dz = 2\int \frac{z^2}{z^2+2}\,dz$$
$$= 2\int \frac{(z^2+2)-2}{z^2+2}\,dz = 2\int \left(1 - \frac{2}{z^2+2}\right) dz$$
$$= 2\left[z - 2 \times \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}}\right] + C$$
$$= 2\left[\sqrt{(x+3)} - \sqrt{2} \tan^{-1} \sqrt{\frac{x+3}{2}}\right] + C.$$

Theorem. *To find* $I = \int \frac{dx}{\sqrt{(c+bx-ax^2)}}$, *where a is positive and (b^2+4ac) is positive.*

$$I = \int \frac{dx}{\sqrt{\left[a\left(\frac{c}{a} + \frac{bx}{a} - x^2\right)\right]}} = \int \frac{dx}{\sqrt{a}\sqrt{\left(\frac{c}{a} + \frac{b}{a}x - x^2\right)}}$$
$$= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left[\left(\frac{c}{a} + \frac{b^2}{4a^2}\right) - \left(x - \frac{b}{2a}\right)^2\right]}}$$
(completing the square of the x-portion)
$$= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left[\frac{4ac+b^2}{4a^2} - \left(x - \frac{b}{2a}\right)^2\right]}}$$

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$$= \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{[k^2 - (x - b/2a)^2]}}, \quad \text{where } k = \frac{\sqrt{(4ac+b^2)}}{2a}.$$

Let $\sin^{-1}\left(\frac{x-(b/2a)}{k}\right) = \theta$, $\therefore \frac{x-(b/2a)}{k} = \sin\theta$

$\therefore x - (b/2a) = k\sin\theta,$

$\therefore dx = k\cos\theta\, d\theta$

Hence
$$I = \frac{1}{\sqrt{a}} \int \frac{k\cos\theta\, d\theta}{\sqrt{(k^2 - k^2\sin^2\theta)}} = \frac{1}{\sqrt{a}} \int \frac{k\cos\theta}{k\sqrt{(1-\sin^2\theta)}}\, d\theta$$
$$= \frac{1}{\sqrt{a}} \int \frac{\cos\theta}{\cos\theta}\, d\theta = \frac{1}{\sqrt{a}} \int d\theta$$
$$= \frac{1}{\sqrt{a}}\theta + C = \frac{1}{\sqrt{a}} \sin^{-1}\left(\frac{x-(b/2a)}{k}\right) + C$$
$$= \frac{1}{\sqrt{a}} \sin^{-1}\left(\frac{x-(b/2a)}{\sqrt{(4ac+b^2)}/2a}\right) + C$$
$$= \frac{1}{\sqrt{a}} \sin^{-1}\left(\frac{2ax-b}{\sqrt{(4ac+b^2)}}\right) + C$$

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Example. Evaluate the following:

$$\text{(i) } I = \int \frac{dt}{\sqrt{(9 + 8t - 4t^2)}}, \quad \text{(ii) } I = \int_0^1 \frac{dx}{\sqrt{(2x - x^2)}}.$$

(i)
$$I = \int \frac{dt}{\sqrt{[4(9/4 + 2t - t^2)]}} = \int \frac{dt}{2\sqrt{(9/4 + 2t - t^2)}}$$

$$= \frac{1}{2}\int \frac{dt}{\sqrt{[(9/4 + 1) - (t - 1)^2]}}$$

(completing the square of the t-portion)

$$= \frac{1}{2}\int \frac{dt}{\sqrt{[13/4 - (t - 1)^2]}}.$$

Let $\sin^{-1}\left\{\frac{t-1}{\sqrt{(13/4)}}\right\} = \theta, \quad \therefore \frac{t-1}{\frac{1}{2}\sqrt{13}} = \sin\theta,$

$$\therefore t - 1 = \frac{\sqrt{13}}{2}\sin\theta, \quad \therefore dt = \frac{\sqrt{13}}{2}\cos\theta \, . d\theta.$$

Hence,
$$I = \frac{1}{2}\int \frac{\sqrt{13}/2 \cos\theta \, d\theta}{\sqrt{(13/4 - 13/4 \sin^2\theta)}}$$

$$= \frac{1}{2}\int \frac{\sqrt{13}/2 \cos\theta \, d\theta}{\sqrt{13}/2 \sqrt{(1 - \sin^2\theta)}}$$

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$$= \frac{1}{2}\int \frac{\cos\theta}{\cos\theta} d\theta = \frac{1}{2}\int d\theta = \tfrac{1}{2}\theta + C$$

$$= \tfrac{1}{2}\sin^{-1}\left\{\frac{t-1}{\sqrt{13}/2}\right\} + C = \tfrac{1}{2}\sin^{-1}\left\{\frac{2(t-1)}{\sqrt{13}}\right\} + C.$$

(ii) $I = \int_0^1 \frac{dx}{\sqrt{[1 - (x-1)^2]}}$. (Completing the square of the x-portion.)

Let $\sin^{-1}(x - 1) = \theta$.

When $x = 1$, $\theta = \sin^{-1} 0 = 0$, and when $x = 0$, $\theta = \sin^{-1}(-1) = -\pi/2$.

Also $x - 1 = \sin\theta$, i.e. $x = 1 + \sin\theta$,

$\therefore dx = \cos\theta \, d\theta$.

Hence
$$I = \int_{-\pi/2}^{0} \frac{\cos\theta}{\sqrt{(1 - \sin^2\theta)}} d\theta = \int_{-\pi/2}^{0} \frac{\cos\theta}{\cos\theta} . d\theta$$

$$= \int_{-\pi/2}^{0} d\theta = \Big[\theta\Big]_{-\pi/2}^{0} = 0 - (-\pi/2) = \pi/2.$$

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Theorem. *To find* $I = \int \frac{dx}{ax^2 + bx + c}$, *where* $4ac - b^2$ *is positive and* $ax^2 + bx + c$ *has no real factors.*

$$I=\int\frac{dx}{a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)}$$

(completing the square of x-portion)

$$=\frac{1}{a}\int\frac{dx}{(x+b/2a)^2+k^2},\quad \text{where } k=\sqrt{\left[\frac{4ac-b^2}{4a^2}\right]}$$

$$=\frac{\sqrt{(4ac-b^2)}}{2a}.$$

Let $\tan^{-1}\left(\frac{x+b/2a}{k}\right)=\theta$, $\therefore \frac{x+b/2a}{k}=\tan\theta$,

and $\quad x+b/2a=k\tan\theta$, $\therefore dx=k\sec^2\theta\,d\theta$

$$\text{Thus } I=\frac{1}{a}\int\frac{k\sec^2\theta\,d\theta}{k^2\tan^2\theta+k^2}=\frac{1}{a}\int\frac{k\sec^2\theta}{k^2(1+\tan^2\theta)}\,d\theta$$

$$=\frac{1}{ak}\int\frac{\sec^2\theta}{\sec^2\theta}d\theta=\frac{1}{ak}\int d\theta=\frac{1}{ak}\theta+C$$

$$=\frac{1}{ak}\tan^{-1}\left(\frac{x+b/2a}{k}\right)+C.$$

Example. Evaluate the following:

$$\text{(i) } I=\int\frac{dy}{4y^2+8y+5},\quad \text{(ii) } I=\int_0^{-1}\frac{dt}{t^2+2t+2}$$

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(i)
$$I=\int\frac{dy}{4(y^2+2y+5/4)}=\frac{1}{4}\int\frac{dy}{(y+1)^2+(5/4-1)}$$

$$=\frac{1}{4}\int\frac{dy}{(y+1)^2+\frac{1}{4}}.$$

Let $\tan^{-1}\left(\frac{y+1}{\frac{1}{2}}\right)=\theta$, $\therefore \frac{y+1}{\frac{1}{2}}=\tan\theta$,

i.e. $y+1=\frac{1}{2}\tan\theta$, and $\therefore dy=\frac{1}{2}\sec^2\theta\,d\theta$.

Using these,
$$I=\frac{1}{4}\int\frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{4}\tan^2\theta+\frac{1}{4}}=\frac{1}{4}\int 2\frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$

$$=\frac{2}{4}\int\frac{\sec^2\theta}{\sec^2\theta}\,d\theta=\frac{1}{2}\int d\theta=\tfrac{1}{2}\theta+C$$

$$=\tfrac{1}{2}\tan^{-1}\left(\frac{y+1}{\frac{1}{2}}\right)+C=\tfrac{1}{2}\tan^{-1}\{2(y+1)\}+C.$$

(ii) $I=\int_0^{-1}\frac{dt}{(t+1)^2+1}$ (completing the square of the t portion).

Let $\tan^{-1}(t+1)=\theta$, $\therefore t+1=\tan\theta$, and $dt=\sec^2\theta\,d\theta$.

When $t=0$, $\tan^{-1}1=\theta$, i.e. $\theta=\pi/4$,
and when $t=-1$, $\tan^{-1}0=\theta$, i.e. $\theta=0$.

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$$\text{Hence } I = \int_{\pi/4}^{0} \frac{\sec^2\theta}{\tan^2\theta + 1}\,d\theta = \int_{\pi/4}^{0} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

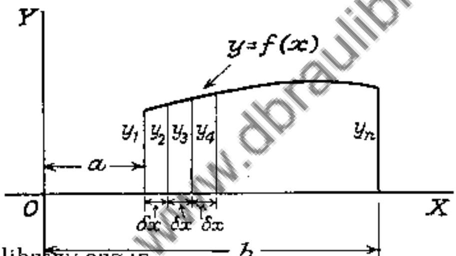
$$= \int_{\pi/4}^{0} d\theta = \Big[\theta\Big]_{\pi/4}^{0} = 0 - \frac{\pi}{4} = -\frac{\pi}{4}.$$

Mean or Average Value. Consider a set of n quantities $y_1, y_2, y_3, \ldots, y_n$. The *average* or *mean value* of these n quantities is

$$(y_1 + y_2 + y_3 + \ldots + y_n)/n.$$

Theorem. *To find the mean value of y between $x = a$ and $x = b$, where $y = f(x)$.*

The area under the curve $y = f(x)$ is divided up into small areas by means of the ordinates $y_1, y_2, y_3, \ldots, y_n$ each consecutive pair being at a small distance δx apart, where y_1 is the ordinate at $x = a$, etc.



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The mean value is given by

$$\underset{n\to\infty}{\text{Lt}} \frac{y_1 + y_2 + y_3 + \ldots + y_n}{n} = \underset{n\to\infty}{\text{Lt}} \frac{\delta x(y_1 + y_2 + y_3 + \ldots + y_n)}{n\delta x}$$

$$= \underset{\delta x\to 0}{\text{Lt}} \left\{ \frac{\sum_{x=a}^{x=b} y\delta x}{(b - a) + \delta x} \right\} \quad [\text{since } (b - a) = (n - 1)\delta x]$$

$$= \frac{\underset{\delta x\to 0}{\text{Lt}} \sum_{\delta x=a}^{x=b} y\delta x}{\underset{\delta x\to 0}{\text{Lt}} (b - a + \delta x)} \quad (\text{Lt of quotient} = \text{quotient of limits})$$

$$= \frac{\int_a^b y\,dx}{b - a} = \frac{\text{area under the curve}}{\text{base}}.$$

Example. Find the mean values of the following functions for the limits stated:

(i) x^2 from $x = 2$ to $x = 4$ (ii) $\sin 2x$ from $x = 0$ to $x = \pi/2$
(iii) $\log_e x$ from $x = 1$ to $x = 4$ (iv) xe^{x^2} from $x = 0$ to $x = 3$

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The previous theorem is used in each case.

(i) $$\text{Mean value} = \frac{\int_2^4 x^2\,dx}{4-2} = \frac{1}{2}\left[\frac{x^3}{3}\right]_2^4$$
$$= \tfrac{1}{6}(4^3 - 2^3) = \tfrac{1}{6} \times 56 = 28/3.$$

(ii) $$\text{Mean value} = \frac{\int_0^{\pi/2} \sin 2x\,dx}{\pi/2 - 0} = \frac{2}{\pi}\left[-\tfrac{1}{2}(\cos 2x)\right]_0^{\pi/2}$$
$$= \frac{2}{\pi}[-\tfrac{1}{2}\cos\pi + \tfrac{1}{2}\cos 0]$$
$$= \frac{2}{\pi}[+\tfrac{1}{2} + \tfrac{1}{2}] = \frac{2}{\pi}.$$

(iii) $$\text{Mean value} = \frac{\int_1^4 \log_e x\,dx}{4-1} = \frac{1}{3}\int_1^4 \log_e x \,.\, 1\,dx$$
$$= \frac{1}{3}\left[\log_e x\int 1\,dx - \int\left\{\frac{d}{dx}(\log_e x)\int 1\,dx\right\}dx\right]_1^4$$
(See Chapter XIV: Integration by parts)
$$= \frac{1}{3}\left[x\log_e x - \int\left(\frac{1}{x}\times x\right)dx\right]_1^4$$
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$$= \frac{1}{3}\left[x\log_e x - \int dx\right]_1^4$$
$$= \frac{1}{3}\left[x\log_e x - x\right]_1^4$$
$$= \tfrac{1}{3}\{(4\log_e 4 - 4) - (0 - 1)\}$$
$$= \tfrac{1}{3}(4\log_e 4 - 3).$$

(iv) Let $I = \int_0^3 x\,e^{x^2}\,dx$, and $z = x^2$.

$\therefore$ $dz = 2x\,dx$, i.e. $x\,dx = \tfrac{1}{2}\,dz$, and $z = 0$ when $x = 0$, $z = 9$ when $x = 3$.

$$\text{Hence } I = \int_0^9 e^z\,\tfrac{1}{2}dz = \frac{1}{2}\left[e^z\right]_0^9$$
$$= \tfrac{1}{2}[e^9 - 1]$$

$$\text{Now the mean value required} = \frac{I}{3-0} = \tfrac{1}{3}\times\tfrac{1}{2}(e^9 - 1)$$
$$= \tfrac{1}{6}(e^9 - 1).$$

NOTE. The *mean centre* of the area under a certain portion of curve is the point given by the average values of x and y for that portion of curve.

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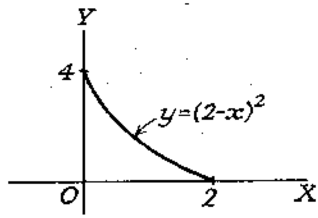
Example. Find the mean centre of the area under the curve $y = (2 - x)^2$ from $x = 0$ to $x = 2$.



When $x = 0$, $y = 4$ and when $x = 2, y = 0$, also $dy = -2(2 - x) \, dx$.

$$\text{Average value of } y = \frac{\int_0^2 y\,dx}{2 - 0} = \frac{1}{2}\int_0^2 (2 - x)^2\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{3}(2 - x)^3\right]_0^2 = \tfrac{1}{2}[0 + \tfrac{1}{3} \times 8] = 4/3.$$

$$\text{Average value of } x = \tfrac{1}{2}\int_{x=2}^{x=0} x\,dy = \tfrac{1}{2}\int_2^0 x\{-2(2 - x)\}dx$$

$$= \int_0^2 (2x^2 - 4x)\,dx = \tfrac{1}{2}\left[\tfrac{2}{3}x^3 - 2x^2\right]_2^0$$

$$= \tfrac{1}{2}[-16/3 + 8] = \tfrac{1}{2} \times 8/3 = 4/3.$$

Hence the mean centre is the point (4/3, 4/3).

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EXAMPLES XIII

1. Show that the radius of curvature ρ at a point $P(a \cos \phi, b \sin \phi)$ on an ellipse of eccentricity e is given by

$$\rho = \frac{a(1 - e^2)}{(1 - e^2 \cos^2 \psi)^{3/2}},$$

where ψ is the inclination of the tangent at P to the x-axis.

A meridian section of the earth, whose polar axis is 7,900 miles and whose equatorial diameter d is 7,925 miles may be taken as an ellipse. Prove that, if powers of e^4 be neglected, the length of the meridian from the equator to latitude θ is given by

$$S = \frac{d}{2{,}536}(1{,}266\theta - 3 \sin 2\theta) \text{ miles.}$$

2. Assuming the formula for the radius of curvature ρ of a plane curve in Cartesian co-ordinates, show that, if the co-ordinates of a point on a curve are given as function of a parameter t, then

$$\rho = \frac{\pm(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \ddot{x}\dot{y}},$$

where the dots denote differentiations with respect to t.

If the normal at the point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$ meets the directrix in Q, and PQ is produced to R so that $PR = 2PQ$, show that PR is equal to the length of the radius of curvature at P.

Show also that the locus of R is the curve $ay^2 = 4x^2(-2a - x)$.

3. (a) If $y = \sqrt{(1 - x^2)}.\sin^{-1} x$, find dy/dx. Express $(1 - dy/dx)(1/x - x)$ in terms of y.

(b) Find the radius of curvature at the point $P(at^2, 2at)$ on the curve $y^2 = 4ax$. If A, B are the points $(0, 2a)$, $(0, -2a)$ respectively, express the radius of curvature at A in terms of the length AB. Hence show that the centre of curvature for the point A has the same y-co-ordinate as B.

4. (a) Differentiate the functions (i) $2x \tan^{-1} x - \log_e (1 + x^2)$, (ii) $\sin^{-1} \sqrt{(x - 1)}$. State the restrictions on the values of x for the second of these functions.

(b) Show that there is just one turning point on the curve $y = x \log_e x$, and determine the nature of this point. Sketch the curve.

Find the curvature at a general point on this curve, and show that at the turning point the radius of curvature has the same magnitude as the y-co-ordinate.

5. (a) Differentiate the functions

$$2 \tan^{-1} \sqrt{x} \text{ and } \cos^{-1}\left(\frac{1 - x}{1 + x}\right)$$

(b) Find the curvature at the point t on the ellipse $x = a \cos t$, $y = b \sin t$. Determine the ratio of the curvatures at the points $A\ (a, 0)$ and $B\ (0, b)$.

6. Show, by graphical considerations or by using the substitution $u = a - x$ that

$$\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$$

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Hence, or otherwise, evaluate $\int_0^2 x(2 - x)^4\, dx$.

(b) Evaluate $\int_0^3 \frac{1}{x^2 + 9}\, dx$.

7. (a) Find the derivatives with respect to x of the functions

(i) $6(x^2 + 1) \log_e (x + 1) - 2x^3 + 3x^2 - 6x$,

(ii) $(x^2 + 2)\sqrt{(1 - x^2)} + 3x^3 \sin^{-1} x$.

(b) The abscissa of a point P moving along the x-axis is given in terms of the time t by the equation $x = e^{-t/2}\, t(t - 3)$ where $t > 0$. Show that the ratio of the extreme distances of P from the origin is $e^{2.5}:9$, and that the accelerations towards the origin in these extreme positions are in the ratio $e^{2.5}:1$.

8. (i) Differentiate with respect to x

$$(a)\ \log_e (1 + x^2) + 2x^3 \tan^{-1} x - x^2; \quad (b)\ \frac{x + 2}{\sqrt{(x^2 + 4x)}}$$

expressing the results in a simplified form.

(ii) Show that when x is positive there is one point P on the curve $y = x^4 e^{-x}$ where dy/dx is zero and two points Q, R, where d^2y/dx^2 is zero. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the co-ordinates of P, Q, R respectively, show that

$$x_2 + x_3 = 2x_1; \quad 256y_2y_3 = 81{y_1}^2.$$

9. (a) Differentiate with respect to x

$$\text{(i) } \tan^{-1}\left(\frac{x - 6}{3x + 2}\right), \quad \text{(ii) } \frac{x + 2}{x - 2} e^{x}.$$

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(*b*) If $xy = h - 9c^2x + x^3 + k \log_e x$ where h, k, c are constants, determine the values of x for which

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0.$$

10. (*a*) Find the derivatives with respect to x of the functions

$$\text{(i)}\ (2x^2 + 2x + 3)\, e^{-x}, \quad \text{(ii)}\ \tan^{-1}\left(\frac{4x}{8 - x^2}\right).$$

(*b*) A curve is given by the parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

The normal at a variable point P meets the x-axis in Q, and the line through Q parallel to the y-axis meets the tangent at P in R. Show that QR is of constant length.

11. (i) Differentiate $x^x + (\log_e x)^x$ with respect to x.

(ii) If $y = \sin (m \sin^{-1} x)$ prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

(iii) If $x = a(t - \sin t)$ and $y = a(1 - \cos t)$, prove that

$$y^2 \frac{d^2y}{dx^2} + a = 0.$$

12. (*a*) Evaluate $\displaystyle\int_1^5 \frac{dx}{\sqrt{(2x - 1)}}$.

(*b*) By means of the substitution $x = a \sin \theta$, or otherwise, evaluate

$$\text{(i)} \int_0^1 \frac{x + 1}{\sqrt{(1 - x^2)}}\, dx, \quad \text{(ii)} \int_{\pi/6}^{\pi/2} \cot \theta\, d\theta.$$

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13. (*a*) Evaluate the integrals (i) $\displaystyle\int_2^3 \frac{dx}{x(x - 1)}$, (ii) $\displaystyle\int_0^{\pi/4} \sin 3x \sin x\, dx$.

(*b*) Find by integration the volume traced out by the complete revolution about the x-axis of the finite area enclosed between the parabola $y^2 = x$ and the line $y = x$.

14. (*a*) Use partial fractions to integrate

$$\frac{x}{(x + 1)(x + 2)}.$$

(*b*) Use the substitution $\cos \theta = x$ to find the value of $\displaystyle\int_0^{\pi/2} \cos^7 \theta \sin^3 \theta\, d\theta$.

15. (*a*) Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{3 + 5 \cos x}\, dx$.

(*b*) If $\tan x/2 = t$ prove that $\cos x = (1 - t^2)/(1 + t^2)$ and $dx/dt = 2/(1 + t^2)$. Hence evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{3 + 5 \cos x}$.

16. By means of the substitution $t = 1 + x^2$, or otherwise, evaluate

$$\int_0^2 \frac{x}{1 + x^2}\, dx.$$

Express in partial fractions $\dfrac{1 + 5x}{(5 - x)(1 + x^2)}$ and evaluate

$$\int_0^2 \frac{1 + 5x}{(5 - x)(1 + x^2)}\, dx.$$

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17. Evaluate (i) $\int_1^4 \frac{x^2+4x-14}{(x+2)(x+5)(x+8)}\,dx$, (ii) $\int_1^4 \frac{x^2+4}{x(x+2)}\,dx$,

(iii) $\int_0^{\pi/4} \sin^2 x \sin 2x\,dx$.

18. Find $\int \frac{dx}{x(x^2-3)}$, and show that $\int_0^2 \frac{\sqrt{(x+2)}}{x+6}\,dx = 0{\cdot}858$ (approx.).

19. Find $\int \frac{dx}{(x+1)\sqrt{x}}$ and $\int \sin^3 x\,dx$, and show that

$$\int_1^3 \frac{x^2+x-3}{x(x^2+3)}\,dx = \frac{\pi\sqrt{3}}{18}.$$

20. (a) Express in partial fractions $1/(x^2-1)$ and hence evaluate

$$\int_2^4 \frac{1}{x^2-1}\,dx.$$

(b) Evaluate $\int_0^{\pi/12} \sin^2 x\,dx$.

21. (i) By means of the substitution $x^3t+1=0$, or otherwise, show that

$$\int_2^3 \frac{dx}{x(x^3-1)} = \frac{1}{3}\log_e \frac{208}{189}.$$

(ii) Evaluate (a) $\int_{-1}^{\infty} \frac{dx}{x^2+2x+5}$, (b) $\int_1^2 \frac{dx}{\sqrt{(3+2x-x^2)}}$

22. (i) If $y = \tan\{m(\tan^{-1} x)\}$, prove that

$$(1+x^2)\frac{d^2y}{dx^2} = 2(my-x)\frac{dy}{dx}.$$

(ii) Given that $x = 4b\cos\theta - b\cos 4\theta$, $y = 4b\sin\theta - b\sin 4\theta$, find dy/dx in terms of θ, and prove that

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$$\frac{d^2y}{dx^2} = -\frac{5}{16b}\sec^3\frac{5\theta}{2}\operatorname{cosec}\frac{3\theta}{2}.$$

23. (i) Differentiate with respect to x (a) $\sin^{-1}(\frac{1}{2}x)$, (b) $x\sin^{-1}x$.

(ii) Evaluate (a) $\int_0^1 \frac{dx}{1+x^2}$, (b) $\int_0^{\pi/2} \frac{dx}{1+2\cos x}$, (c) $\int \cos^3 x\,dx$.

24. (i) By using the substitution $x = a\cos^2\theta + b\sin^2\theta$, or otherwise, show that, when $a < b$,

$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi,$$

and find the indefinite integral

$$\int \frac{dx}{\sqrt{(x-a)(b-x)}}.$$

(ii) By the substitution $x = 3\sin^2\theta + \cos^2\theta$, or otherwise, prove that

$$\int_1^3 \sqrt{\left(\frac{3-x}{x-1}\right)}\,dx = \pi.$$

25. (a) Find $\int_0^2 \frac{dx}{4-x^2}$.

(b) Prove that $\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx = (1-\pi/4)a^2$.

(c) Find a_n if n is a positive integer and

$$a_n\int_0^{\pi/2} \cos^2 nx\,dx = \int_0^{\pi/2} \cos x \cos 2x\,dx$$

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26. A particle describes simple harmonic motion in which the displacement x is given in terms of the time t by the equation $x = a \sin t$.

Find for the interval $t = 0$ to $t = \pi/2$,

(i) the mean value of the velocity with respect to time,
(ii) the mean value of the velocity with respect to distance.

27. By means of Simpson's rule and taking unit intervals of x from $x = 8$ to $x = 12$, find approximately the area enclosed by the curve $y = \log_{10} x$, the lines $x = 8$ and $x = 12$, and the x-axis. Deduce the average value of $\log_{10} x$ between $x = 8$ and $x = 12$.

28. (*a*) Show that $\int_0^t \sin \omega x \cos \omega(t - x)\, dx = \frac{1}{2}t \sin \omega t$.

(*b*) Find the co-ordinates of the mean centre of the area above the x-axis that is bounded by the part of the curve $a^2y = x^2(a - x)$ for which $0 \leqslant x \leqslant a$.

29. Find the mean values of the following functions for the limits stated. (i) $x(2 - x)^2$ for $x = 0$ to $x = 2$; (ii) $\tan x$ for $x = 0$ to $x = \pi/4$; (iii) $x \log_e x$ from $x = 1$ to $x = 3$ (use integration by parts).

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CHAPTER XIV

Integration by Parts and Differential Equations

Integration by Parts. This is a method of integrating the product of two functions, and also of integrating a single function whose integral cannot be determined by previous methods.

Consider u and w to be two functions of x.

Then $$\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx},$$

$$\therefore u\frac{dw}{dx} = \frac{d}{dx}(uw) - w\frac{du}{dx}.$$

Integrating this with respect to x,

$$\int u\frac{dw}{dx}\,.\,dx = uw - \int\frac{du}{dx}\,.\,w\,dx \ldots\ldots\ldots \quad (1).$$

Let $\frac{dw}{dx} = v \quad \therefore \quad w = \int v\,dx$, where the *indefinite* integral is used.

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The result (1) now becomes

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx,$$

and this is the formula for integration by parts.

Stated in words the formula becomes:

The integral of the product of two functions is equal to the product of the first and the integral of the second diminished by the integral of the product of the differential coefficient of the first and the integral of the second.

NOTE 1. The second function (v) must be one whose integral can be obtained by previous methods, and in cases when one of the two functions cannot be integrated by normal means, this function is chosen as the u function.

2. When limits are involved it is safest to insert the limits when the complete integration has been made in the separate parts.

3. Integration by parts can be used in finding integrals of single functions that cannot be determined by other means. In these cases the integrand is chosen as the first function (u), and unity as the second function (v).

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Example 1. Evaluate

$$\text{(i) } I = \int x \log_e x \, dx, \quad \text{(ii) } I = \int x \sin x \, dx, \quad \text{(iii) } I = \int_0^1 x \tan^{-1} x \, dx.$$

(i) In this case

$$\int \log_e x \, dx$$

cannot be obtained by normal methods and hence $\log_e x$ is chosen as the u function.

Integrating by parts,

$$I = \int (\log_e x) . x \, dx = (\log_e x) \int x \, dx - \int \left\{ \frac{d}{dx} (\log_e x) \int x \, dx \right\} dx$$

$$= \frac{x^2}{2} \log_e x - \int \left(\frac{1}{x} \times \frac{x^2}{2} \right) dx$$

$$= \frac{x^2}{2} \log_e x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \log_e x - \frac{x^2}{4} + C.$$

(ii) Choosing x as the u function and integrating by parts,

$$I = x \int \sin x \, dx - \int \left\{ \frac{d}{dx} (x) . \int \sin x \, dx \right\} dx$$

$$= -x \cos x - \int 1(-\cos x) \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C.$$

(iii) $\int \tan^{-1} x \, dx$ cannot be obtained by ordinary means, and hence $\tan^{-1} x$ will be chosen as the u function.

Integrating by parts,

$$I = \int_0^1 (\tan^{-1} x) \, x \, dx$$

$$= \left[(\tan^{-1} x) \int x \, dx - \int \left\{ \frac{d}{dx} (\tan^{-1} x) \int x \, dx \right\} dx \right]_0^1$$

$$= \left[\frac{x^2}{2} \tan^{-1} x - \int \left(\frac{1}{1 + x^2} . \frac{x^2}{2} \right) dx \right]_0^1$$

$$= \left[\frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \, dx \right]_0^1$$

$$= \left[\frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{(1 + x^2) - 1}{1 + x^2} \, dx \right]_0^1$$

$$= \left[\frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1 + x^2} \right) dx \right]_0^1$$

$$= \left[\frac{x^2}{2} \tan^{-1} x - \tfrac{1}{2}(x - \tan^{-1} x) \right]_0^1$$

$$= [\tfrac{1}{2} \tan^{-1} 1 - \tfrac{1}{2}(1 - \tan^{-1} 1)] - (0)$$

$$= \tfrac{1}{2} \times \pi/4 - \tfrac{1}{2}(1 - \pi/4) = \pi/8 - \tfrac{1}{2} + \pi/8 = \pi/4 - \tfrac{1}{2}.$$

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EXAMPLE 2. Evaluate

(i) $I = \int \tan^{-1} x\, dx$, (ii) $I = \int_1^2 \log_e x\, dx$, (iii) $I = \int \sin^{-1} x\, dx$.

(i) $I = \int (\tan^{-1} x) . 1\, dx$ (1 is chosen as the v function)

$$= \tan^{-1} x \int 1\, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) . \int 1\, dx \right\} dx$$

(integration by parts)

$$= x \tan^{-1} x - \int \left(\frac{1}{1+x^2} . x \right) dx = x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx$$

Let $I_1 = \int \frac{x}{1+x^2}\, dx$, and $1 + x^2 = y$,

$\therefore 2x\, dx = dy$ and $x\, dx = \frac{1}{2} dy$.

Hence $I_1 = \int \frac{\frac{1}{2} dy}{y} = \frac{1}{2} \int \frac{dy}{y} = \frac{1}{2} \log_e y - C$

$= \frac{1}{2} \log_e (1 + x^2) - C$,

$\therefore I = x \tan^{-1} x - \frac{1}{2} \log_e (1 + x^2) + C$.

(ii) $I = \int_1^2 (\log_e x) . 1\, dx$ (1 chosen as the v function)

$$= \left[(\log_e x) \int 1\, dx - \left\{ \int \frac{d}{dx}(\log_e x) \int 1\, dx \right\} dx \right]_1^2$$

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(integration by parts)

$$= \left[x \log_e x - \int \left(\frac{1}{x} . x \right) dx \right]_1^2 = \left[x \log_e x - x \right]_1^2$$

$= (2 \log_e 2 - 2) - (0 - 1) = 2 \log_e 2 - 1$.

(iii) $I = \int (\sin^{-1} x) . 1\, dx$ (1 is chosen as the v function)

$$= (\sin^{-1} x) \int 1\, dx - \int \left\{ \frac{d}{dx}(\sin^{-1} x) \int 1\, dx \right\} dx$$

$$= x \sin^{-1} x - \int \left(\frac{1}{\sqrt{(1 - x^2)}} . x \right) dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{(1 - x^2)}}\, dx$$

Let $I_1 = \int \frac{x}{\sqrt{(1 - x^2)}}\, dx$.

Using $1 - x^2 = z$, $-2x\, dx = dz$ and $\therefore x\, dx = -\frac{1}{2} dz$.

Hence $I_1 = \int \frac{-dz}{2\sqrt{z}} = -\sqrt{z} - C = -\sqrt{(1 - x^2)} - C$.

Thus $I = x \sin^{-1} x - I_1 = x \sin^{-1} x + \sqrt{(1 - x^2)} + C$.

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Note 4. In cases when both the u and v functions can be integrated by ordinary methods and one of the functions is $(ax+b)^n$, where n is a positive integer, then this function is chosen as the u function.

Example. Find (i) $I = \int x \cos x\, dx$, (ii) $I = \int (2x+3)^2 \sin x\, dx$.

(i) $I = \int x \cos x\, dx$, where x is the u function and $\cos x$ the v function

$$= x\int \cos x\, dx - \int \left\{\frac{d}{dx}(x)\int \cos x\, dx\right\} dx \qquad \text{(integration by parts)}$$

$$= x \sin x - \int (1 \,.\, \sin x)\, dx = x\sin x + \cos x + C.$$

(ii) Taking $(2x+3)^2$ as the u function, $\sin x$ as the v function and integrating by parts

$$I = (2x+3)^2 \int \sin x\, dx - \int\left\{\frac{d}{dx}(2x+3)^2 \int \sin x\, dx\right\} dx$$

$$= (2x+3)^2(-\cos x) - \int 4(2x+3)(-\cos x)\, dx$$

$$= -(2x+3)^2 \cos x + 4\int (2x+3)\cos x\, dx$$

$$= -(2x+3)^2\cos x + 4\left[(2x+3)\int \cos x\, dx\right.$$

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$$\left. - \int\left\{\frac{d}{dx}(2x+3)\int \cos x\, dx\right\} dx\right]$$

(integration by parts with $(2x+3)$ as u function and $\cos x$ as v function)

$$= -(2x+3)^2 \cos x + 4\left[(2x+3)\sin x - \int 2 \sin x\, dx\right]$$

$$= -(2x+3)^2 \cos x + 4[(2x+3)\sin x + 2\cos x] + C.$$

N.B. In the case of part (ii) it is to be noted that integration by parts has to be applied twice.

Note 5. When the integrand has a factor which involves powers of $\cos x$ or $\sin x$ it is necessary to first convert these into linear expressions in terms of the sines or cosines of the multiple angles as the case may be.

Example. Find the indefinite integrals (i) $I = \int x \sin^2 x\, dx$,

(ii) $I = \int x \cos^3 x\, dx$.

(i) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$,

$$\therefore\ I = \tfrac{1}{2}\int x(1-\cos 2x)\, dx = \tfrac{1}{2}\int x\, dx - \tfrac{1}{2}\int x\cos 2x\, dx$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{2}\int x \cos 2x\, dx$$

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$$= \tfrac{1}{4}x^2 - \tfrac{1}{2}\left[x\int \cos 2x\,dx - \int\left\{\frac{d}{dx}(x)\int \cos 2x\,dx\right\}dx\right]$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{2}\left[x\,.\,\tfrac{1}{2}\sin 2x - \int(1 \times \tfrac{1}{2}\sin 2x)\,dx\right]$$

$$= \tfrac{1}{4}x^2 - \tfrac{1}{2}[(x/2)\sin 2x + \tfrac{1}{4}\cos 2x]$$

$$= \tfrac{1}{8}[2x^2 - 2x\sin 2x - \cos 2x].$$

(ii) Now $\cos 3x = 4\cos^3 x - 3\cos x$,

$\therefore 4\cos^3 x = \cos 3x + 3\cos x$, i.e. $\cos^3 x = \tfrac{1}{4}(\cos 3x + 3\cos x)$.

Hence $$I = \tfrac{1}{4}\int x(\cos 3x + 3\cos x)\,dx$$

$$= \tfrac{1}{4}\int x\cos 3x\,dx + \tfrac{3}{4}\int x\cos x\,dx$$

$$= \tfrac{1}{4}\left[x\int \cos 3x\,dx - \int\left\{\frac{d}{dx}(x)\int \cos 3x\,dx\right\}dx\right] + \tfrac{3}{4}\left[x\int \cos x\,dx - \int\left\{\frac{d}{dx}(x)\int \cos x\,dx\right\}dx\right]$$

$$= \tfrac{1}{4}\left[\tfrac{1}{3}x\sin 3x - \int(1 \times \tfrac{1}{3}\sin 3x)\,dx\right] + \tfrac{3}{4}\left[x\sin x - \int(1 \times \sin x)\,dx\right]$$

$$= \tfrac{1}{4}[\tfrac{1}{3}x\sin 3x + \tfrac{1}{9}\cos 3x] + \tfrac{3}{4}[x\sin x + \cos x]$$

$$= \tfrac{1}{36}[3x\sin 3x + \cos 3x + 27x\sin x + 27\cos x].$$

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NOTE 6. When using the integration by parts theorem, the original integral sometimes occurs in the result, and can thus be found by algebraic methods as shown in the following theorem.

Theorem. *Find the indefinite integrals* $I_1 = \int e^{at}\sin bt\,dt$ *and* $I_2 = \int e^{at}\cos bt\,dt$, *where a and b are constants.*

Integrating by parts with $\sin bt$ as the u function,

$$I_1 = \sin bt\int e^{at}\,dt - \int\left\{\frac{d}{dt}(\sin bt)\int e^{at}\,dt\right\}dt$$

$$= \sin bt \times \frac{1}{a}e^{at} - \int\left(b\cos bt \times \frac{1}{a}e^{at}\right)dt$$

$$= \frac{1}{a}e^{at}\sin bt - \frac{b}{a}\int \cos bt\,.\,e^{at}\,dt \qquad \text{(1)}$$

$$= \frac{1}{a}e^{at}\sin bt - \frac{b}{a}\left[(\cos bt)\int e^{at}\,dt - \int\left\{\frac{d}{dt}(\cos bt)\int e^{at}\,dt\right\}dt\right]$$

$$= \frac{1}{a}e^{at}\sin bt - \frac{b}{a}\left[\cos bt \times \frac{1}{a}e^{at} - \int(-b\sin bt) \times \frac{1}{a}e^{at}\,dt\right]$$

$$= \frac{1}{a} e^{at} \sin bt - \frac{b}{a}\left[\frac{1}{a} e^{at} \cos bt + \frac{b}{a}\int e^{at} \sin bt \, dt\right]$$

$$= \frac{1}{a} e^{at} \sin bt - \frac{b}{a^2} e^{at} \cos bt - \frac{b^2}{a^2} I_1,$$

$$\therefore I_1\left[1 + \frac{b^2}{a^2}\right] = \frac{1}{a^2} e^{at}[a \sin bt - b \cos bt],$$

$$\text{i.e. } I_1(a^2 + b^2) = e^{at}[a \sin bt - b \cos bt]$$

$$\therefore I_1 = \frac{e^{at}[a \sin bt - b \cos bt]}{a^2 + b^2} \quad \ldots\ldots\ldots\ldots (2).$$

Equation (1) can be written,

$$I_1 = \frac{1}{a} e^{at} \sin bt - \frac{b}{a} I_2,$$

$$\therefore \text{ using (2), } \frac{b}{a} I_2 = \frac{1}{a} e^{at} \sin bt - \frac{e^{at}(a \sin bt - b \cos bt)}{a^2 + b^2}$$

$$\therefore I_2 = \frac{a}{b}\left\{\frac{1}{a} e^{at} \sin bt - \frac{e^{at}(a \sin bt - b \cos bt)}{a^2 + b^2}\right\}$$

$$= \frac{a \; e^{at}}{b}\left\{\frac{(a^2 + b^2) \sin bt - a^2 \sin bt + ab \cos bt}{a(a^2 + b^2)}\right\}$$

$$= \frac{e^{at}}{b(a^2 + b^2)}(b^2 \sin bt + ab \cos bt)$$

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$$= \frac{e^{at}(b \sin bt + a \cos bt)}{a^2 + b^2}.$$

EXAMPLE. Find the indefinite integrals (i) $I = \int e^{-t} \cos 2t \, dt$,

(ii) $I = \int e^{2t} \sin 3t \, dt$.

(i) Choosing $\cos 2t$ as the u function,

$$I = \int (\cos 2t) e^{-t} dt = \cos 2t \int e^{-t} dt - \int \left\{\frac{d}{dt} (\cos 2t) \int e^{-t} dt\right\} dt$$

$$= (\cos 2t)(-e^{-t}) - \int (-2 \sin 2t)(-e^{-t}) \, dt$$

$$= -e^{-t} \cos 2t - 2\int (\sin 2t) e^{-t} dt$$

$$= -e^{-t} \cos 2t - 2\left\{(\sin 2t)\int e^{-t} dt\right\} - \int \left\{\frac{d}{dt} (\sin 2t)\int e^{-t} dt\right\} dt$$

$$= -e^{-t} \cos 2t - 2\left\{\sin 2t \, (-e^{-t}) - \int (2 \cos 2t)(-e^{-t}) \, dt\right\}$$

$$= -e^{-t} \cos 2t + 2e^{-t} \sin 2t - 4\int e^{-t} \cos 2t \, dt$$

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$$= e^{-t}(2\sin 2t - \cos 2t) - 4I$$

$$\therefore\ 5I = e^{-t}(2\sin 2t - \cos 2t) \text{ and } I = \tfrac{1}{5}e^{-t}(2\sin 2t - \cos 2t).$$

(ii) Choosing $\sin 3t$ as the u function in integration by parts,

$$I = \int (\sin 3t)e^{2t}\,dt = (\sin 3t)\int e^{2t}\,dt - \int\left\{\frac{d}{dt}(\sin 3t)\int e^{2t}\,dt\right\}dr$$

$$= (\sin 3t)\times \tfrac{1}{2}e^{2t} - \int (3\cos 3t)(\tfrac{1}{2}e^{2t})\,dt$$

$$= \tfrac{1}{2}e^{2t}\sin 3t - \tfrac{3}{2}\int(\cos 3t)e^{2t}\,dt$$

$$= \tfrac{1}{2}e^{2t}\sin 3t - \tfrac{3}{2}\left[(\cos 3t)\int e^{2t}\,dt - \int\left\{\frac{d}{dt}(\cos 3t)\int e^{2t}\,dt\right\}dt\right]$$

$$= \tfrac{1}{2}e^{2t}\sin 3t - \tfrac{3}{2}\left[(\cos 3t)\tfrac{1}{2}e^{2t} - \int(-3\sin 3t)(\tfrac{1}{2}e^{2t})\,dt\right]$$

$$= \tfrac{1}{2}e^{2t}\sin 3t - \tfrac{3}{4}e^{2t}\cos 3t - \tfrac{9}{4}\int e^{2t}\sin 3t\,dt$$

$$= \tfrac{1}{4}e^{2t}(2\sin 3t - 3\cos 3t) - \tfrac{9}{4}I,$$

$$\therefore\ \tfrac{13}{4}I = \tfrac{1}{4}e^{2t}(2\sin 3t - 3\cos 3t)$$

$$\therefore\ I = \frac{e^{2t}}{13}(2\sin 3t - 3\cos 3t).$$

NOTE 7. When finding

$$I = \int x\tan^2 x\,dx$$

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it will be necessary to use the fact $\tan^2 x = \sec^2 x - 1$ before using integration by parts.

Thus $\quad I = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$= \int x\sec^2 x\,dx - x^2/2$$

$$= \left[x\int\sec^2 x\,dx - \int\left\{\frac{d}{dx}(x)\int\sec^2 x\,dx\right\}dx\right] - x^2/2$$

$$= x\tan x - \int(1.\tan x)\,dx - x^2/2$$

$$= x\tan x + \int -\frac{\sin x}{\cos x}\,dx - x^2/2.$$

Let $\quad I_1 = \int -\dfrac{\sin x}{\cos x}\,dx$, and $u = \cos x$.

Then $du = -\sin x\,dx$, and $I_1 = \int \dfrac{du}{u} = \log_e u + C$

$$= \log_e(\cos x) + C.$$

Hence $I = x\tan x + \log_e(\cos x) - x^2/2 + C.$

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Differential Equations. The types of differential equations previously considered were of the form $dy/dx = f(x)$ and $d^2y/dx^2 = \varphi(x)$, the solutions of which could be obtained by straightforward integration.

Further types of differential equations will now be considered, and these will be of the *first order and first degree*, i.e. they will only contain first differential coefficients of y with respect to x and functions of x and y, and will be of the form

$$P\frac{dy}{dx} + Q = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where P and Q are functions of x and y.

The variables in the equation (1) are said to be *separable* if the fraction P/Q can be expressed in the form $F(y)/\varphi(x)$ where $F(y)$ does not contain x and $\varphi(x)$ does not contain y.

Only two types of these equations of the first order and first degree will be dealt with.

1. Variables Separable. (*a*) *y absent* in P and Q.

P and Q must be functions of x only and the original equation

$$P\frac{dy}{dx} + Q = 0 \text{ becomes } \frac{dy}{dx} = \frac{-Q}{P} = f(x).$$

$$\text{Hence } y = \int f(x)\,dx + C.$$

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(*b*) *x also absent* in P and Q.

Here Q/P is a constant and the differential equation becomes

$$\frac{dy}{dx} = \frac{-Q}{P} = \text{constant} = k \text{ (say)},$$

and the solution is $y = kx + C$.

(*c*) *x absent* in P and Q.

The differential equation (1) in this case can be written

$$\frac{P}{Q} \cdot \frac{dy}{dx} = -1,$$

$$\text{i.e. } f(y)\frac{dy}{dx} = -1, \quad \text{where } \frac{P}{Q} = f(y),$$

$$\text{i.e. } f(y) = \frac{-1}{dy/dx} = -\frac{dx}{dy},$$

$$\therefore\ dx/dy = -f(y),$$

and integrating with respect to y

$$x = -\int f(y)\,dy + C.$$

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(c) Both *x and y present* in *P* and *Q*, but the variables separable.

Let $\frac{P}{Q} = \frac{F(y)}{\varphi(x)}$, and the equation (1) becomes

$$\frac{P}{Q}\,\frac{dy}{dx} + 1 = 0, \text{ i.e. } \frac{F(y)}{\varphi(x)}\,\frac{dy}{dx} + 1 = 0,$$

$$\therefore\ F(y)\,dy/dx + \varphi(x) = 0.$$

Integrating this with respect to *x*,

$$\int F(y)\frac{dy}{dx}\,dx + \int \varphi(x)\,dx = C.$$

From integration by substitution it can be seen that the first integral in this equation is equal to

$$\int F(y)\,dy,$$

and hence the equation can be written

$$\int F(y)\,dy + \int \varphi(x)\,dx = C.$$

EXAMPLE 1. (*y* absent in *P* and *Q*).

Solve the equation $(1 - x^2)\,dy/dx = 1 + 2x$.

The equation can be written

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$$\frac{dy}{dx} = \frac{1 + 2x}{1 - x^2} \quad \ldots\ldots\ldots\ldots (1).$$

Let $\frac{1 + 2x}{1 - x^2} = \frac{A}{1 + x} + \frac{B}{1 - x}$,

$\therefore\ 1 + 2x = A(1 - x) + B(1 + x)$.

In this identity using,

$x = 1, \qquad 3 = 2B \quad \therefore \quad B = \frac{3}{2}$,

$x = -1, \qquad -1 = 2A \quad \therefore \quad A = -\frac{1}{2}$.

Thus equation (1) can be written

$$\frac{dy}{dx} = -\frac{1}{2(1 + x)} + \frac{3}{2(1 - x)},$$

$$\therefore\ y = \int\left\{-\frac{1}{2(1 + x)} + \frac{3}{2(1 - x)}\right\}dx$$

$$= C - \tfrac{1}{2}\log_e(1 + x) - \tfrac{3}{2}\log_e(1 - x)$$

$$= C - \tfrac{1}{2}\{\log_e(1 + x) + 3\log_e(1 - x)\}.$$

EXAMPLE 2. (*x* absent in *P* and *Q*).

Solve the equation $(2 + y)\,dy/dx = 4 + y^2$.

The equation can be written

$$\frac{2 + y}{4 + y^2}\,\frac{dy}{dx} = 1,$$

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$$\text{i.e. } \frac{2+y}{4+y^2} = \frac{1}{dy/dx} = \frac{dx}{dy}.$$

integrating with respect to y,

$$x = \int \frac{2+y}{4+y^2}\,dy = \int \left\{\frac{2}{4+y^2} + \frac{y}{4+y^2}\right\} dy$$

$$= 2 \times \tfrac{1}{2}\tan^{-1}\frac{y}{2} + \tfrac{1}{2}\int \frac{2y}{4+y^2}\,dy = \tan^{-1}\frac{y}{2} + \tfrac{1}{2}I,$$

where $$I = \tfrac{1}{2}\int \frac{2y}{4+y^2}\,dy.$$

Let $4 + y^2 = z$, $\therefore 2y\,dy = dz$.

Hence $$I = \tfrac{1}{2}\int \frac{dz}{z} = \tfrac{1}{2}\log_e z + C = \tfrac{1}{2}\log_e(4+y^2) + C.$$

Thus the solution is $x = \tan^{-1} y/2 + \frac{1}{2}\log_e(4 + y^2) + C$.

EXAMPLE 3. (Both x and y present in P and Q.)

Solve the equations:

(i) $x^2(1 - y)\,dy/dx = (1 + x)y$,

(ii) $2\tan x\,dy/dx + y^2 - 1 = 0$, given $y = 2$ when $x = \pi/2$,

(iii) $2x\,dy/dx = 1 - y^2$.

(i) Dividing through the equation by x^2y it becomes

$$\frac{1-y}{y}\frac{dy}{dx} = \frac{1+x}{x^2},$$

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$$\text{i.e. } (1/y - 1)\frac{dy}{dx} = \frac{1}{x^2} + \frac{1}{x}.$$

Integrating this with respect to x,

$$\int \left(\frac{1}{y} - 1\right) dy = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$$

i.e. $\log_e y - y = \log_e x - 1/x + C$,

i.e. $\log_e y - \log_e x - y + 1/x = C$,

i.e. $\log_e y/x - y + 1/x = C$.

(ii) Dividing through the equation by $(y^2 - 1)\tan x$ it becomes

$$\frac{2}{y^2-1}\frac{dy}{dx} + \frac{\cos x}{\sin x} = 0,$$

$$\text{i.e. } \left\{\frac{1}{y-1} - \frac{1}{y+1}\right\}\frac{dy}{dx} + \frac{\cos x}{\sin x} = 0,$$

Integrating with respect to x, using $\log_e C$ for the constant,

$$\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy + \int \frac{\cos x}{\sin x}\,dx = \log_e C,$$

i.e. $\log_e(y - 1) - \log_e(y + 1) + \log_e(\sin x) = \log_e C$,

$$\text{i.e. } \log_e\left(\frac{y-1}{y+1}\sin x\right) = \log_e C,$$

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$$\therefore \frac{y-1}{y+1}\sin x = C.$$

Now $y = 2$ when $x = \pi/2$, $\therefore \frac{1}{3} = C$.

Hence the solution is $$\frac{y-1}{y+1}\sin x = \tfrac{1}{3}.$$

(iii) The equation can be written

$$\frac{2}{1-y^2}\,\frac{dy}{dx} = \frac{1}{x},$$

$$\text{i.e. } \left(\frac{1}{1+y} + \frac{1}{1-y}\right)\frac{dy}{dx} = \frac{1}{x}.$$

Integrating with respect to x and using $\log_e C$ for the constant of integration,

$$\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = \int\frac{1}{x} + \log_e C,$$

$$\text{i.e. } \{\log_e(1+y) - \log_e(1-y)\} = \log_e x + \log_e C,$$

$$\text{i.e. } \log_e\frac{1+y}{1-y} = \log_e Cx,$$

$$\therefore \frac{1+y}{1-y} = Cx.$$

2. Linear Equations. If the dependent variable y and dy/dx both occur in a differential equation in the first degree, the equation is known as a *linear differential equation of the first order*, and a typical equation of this type is

$$dy/dx + Py = Q \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where P and Q are functions of x only or constants.

When P and Q are both constants the equation degenerates to Type 1 and can be readily solved.

If P and Q are not both constants the method for solving is the following:

Multiply the equation (1) by v, where v is an arbitrary function of x whose value will be determined later, known as the *integrating factor* of equation (1) and *which contains no arbitrary constant.*

The equation now becomes

$$v\frac{dy}{dx} + Pvy = vQ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

But $$\frac{d}{dx}(vy) = v\frac{dy}{dx} + y\frac{dv}{dx},$$

$$\text{i.e. } v\frac{dy}{dx} = \frac{d}{dx}(vy) - y\frac{dv}{dx}.$$

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When this is used in equation (2) it becomes

$$\frac{d}{dx}(vy) - y\frac{dv}{dx} + Pvy = vQ,$$

$$\text{i.e. } \frac{d}{dx}(vy) + y\left(Pv - \frac{dv}{dx}\right) = vQ \ \ldots\ldots\ldots \ (3).$$

Now choose v so that the coefficient of y in (3) is zero. Then

$$Pv - \frac{dv}{dx} = 0, \text{ i.e. } \frac{dv}{dx} = Pv,$$

$$\text{i.e. } \frac{1}{v}\frac{dv}{dx} = P.$$

Integrating this with respect to x, with no constant of integration,

$$\int \frac{1}{v}\,dv = \int P\,dx,$$

$$\therefore \log_e v = \int P\,dx,$$

$$\text{i.e. } v = e^{\int P dx}.$$

When this value of v is used in equation (3) it becomes

$$\frac{d}{dx}(y\,e^{\int P dx}) = Q\,e^{\int P dx},$$

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and integrating with respect to x,

$$y\,e^{\int P dx} = \int (Q\,e^{\int P dx})\,dx + C,$$

$$\text{i.e. } y = (e^{-\int P dx})\int (Q\,e^{\int P dx})\,dx + C.$$

EXAMPLE I. Solve the equations:

(i) $x\,dy/dx + 2y = 3x + 2$,
(ii) $dy/dx + 2y = x$,
(iii) $dy/dx + y\cot x = 2\cos 2x$, where $y = 0$ when $x = \pi/2$.

(i) The equation can be written

$$\frac{dy}{dx} + \frac{2}{x}y = 3 + \frac{2}{x} \ \ldots\ldots\ldots\ldots\ldots \ (1).$$

The integrating factor is $e^{\int 2/x\,dx} = e^{2\log_e x} = e^{\log_e x^2} = x^2$.

Multiplying through equation (1) by x^2

$$x^2\frac{dy}{dx} + 2xy = 3x^2 + 2x,$$

$$\text{i.e. } \frac{d}{dx}(x^2y) = 3x^2 + 2x.$$

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Integrating this with respect to x^2

$$x^2 y = \int (3x^2 + 2x)\, dx = x^3 + x^2 + C.$$

Hence the solution is $y = x + 1 + Cx^{-2}$.

(ii) The integrating factor is $e^{\int 2dx} = e^{2x}$

Multiplying through the equation by e^{2x}, we have

$$e^{2x}\, dy/dx + 2e^{2x} y = xe^{2x},$$

$$\text{i.e. } \frac{d}{dx}(y\, e^{2x}) = x\, e^{2x},$$

$$\therefore\ y\, e^{2x} = \int x\, e^{2x}\, dx$$

$$= x \int e^{2x}\, dx + \int \left\{ \frac{d}{dx}(x) \int e^{2x}\, dx \right\} dx$$

(integration by parts)

$$= x \times \tfrac{1}{2} e^{2x} + \int (1 \times \tfrac{1}{2} e^{2x})\, dx$$

$$= \frac{x}{2} e^{2x} + \frac{e^{2x}}{4} + C$$

$$\therefore\ y = x/2 + \tfrac{1}{4} + C e^{-2x}.$$

(iii) The integrating factor is $e^{\int \cot x\, dx}$

Now $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$ www.dbraulibrary.org.in

Let $\sin x = z$, $\therefore \cos x\, dx = dz$.

Hence $\int \cot x\, dx = \int \frac{dz}{z} = \log_e z = \log_e (\sin x)$.

Thus the integrating factor is $e^{(\log_e \sin x)} = \sin x$.

Multiplying through the equation by $\sin x$ it becomes

$$\sin x \frac{dy}{dx} + y \cos x = 2 \cos 2x \sin x,$$

$$\text{i.e. } \frac{d}{dx}(y \sin x) = \sin 3x - \sin x.$$

Integrating this with respect to x

$$y \sin x = \int (\sin 3x - \sin x)\, dx$$

$$= -\tfrac{1}{3} \cos 3x + \cos x + C$$

But $y = 0$ when $x = \pi/2$, $\therefore 0 = C - 0$, i.e. $C = 0$.

Hence required solution is $y \sin x = \cos x - \frac{1}{3} \cos 3x$.

EXAMPLE 2. Solve the equations

(i) $dy/dx + 2xy = 2x$,

(ii) $dy/dx + y = 3\, e^{-x}$,

(iii) $x\, dy/dx - y = 2x^2 \sin 2x$.

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(i) The integrating factor is $e^{\int 2x\,dx} = e^{x^2}$.

Multiplying through the equation by e^{x^2} it becomes

$$e^{x^2} dy/dx + 2x\, e^{x^2} y = 2x\, e^{x^2},$$

$$\text{i.e. } \frac{d}{dx}(y\, e^{x^2}) = 2x\, e^{x^2}$$

Integrating with respect to x,

$$y\, e^{x^2} = \int 2x\, e^{x^2}\, dx = I. \quad \text{(say)}$$

Let $e^{x^2} = z$, $\therefore 2x\, e^{x^2}\, dx = dz$, and $I = \int dz = z + C = e^{x^2} + C$.

The solution therefore is $y\, e^{x^2} = e^{x^2} + C$,

$$\text{i.e. } y = 1 + C\, e^{-x^2}.$$

(ii) The integrating factor is $e^{\int dx} = e^x$.

Multiplying through the equation by e^x it becomes

$$e^x\, dy/dx + y\, e^x = 3,$$

$$\text{i.e. } \frac{d}{dx}(y\, e^x) = 3.$$

From this
$$y\, e^x = \int 3\, dx$$
$$= 3x + C,$$
$$\text{i.e. } y = e^{-x}(3x + C).$$

(iii) The equation can be written

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$$\frac{dy}{dx} - \frac{1}{x} y = 2x \sin 2x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

The integrating factor of equation (1) is

$$e^{\int -\frac{1}{x} dx} = e^{-\log_e x} = e^{\log_e \frac{1}{x}} = 1/x.$$

Multiplying through equation (1) by $1/x$, we have

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 2 \sin 2x,$$

$$\text{i.e. } \frac{d}{dx}\left(\frac{y}{x}\right) = 2 \sin 2x.$$

Integrating with respect to x

$$y/x = \int 2 \sin 2x\, dx = C - \cos 2x,$$

$$\text{i.e. } y = x(C - \cos 2x).$$

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EXAMPLES XIV

1. Evaluate (i) $\int_0^1 x\, e^{-x}\, dx$, (ii) $\int_1^2 x \log_e x\, dx$, (iii) $\int_0^1 \sin^{-1} x\, dx$.

3. Evaluate (i) $\int_0^{\pi} x \sin\, x\, dx$, (ii) $\int \sin x . \log_e (\sin x)\, dx$, (iii) $\int x^2 \sin x\, dx$.

3. Find (i) $\int x \tan^{-1} x\, dx$, (ii) $\int x \sin x \sec^3 x\, dx$.

4. Find (*a*) $\int e^{-x} \sin x\, dx$, (*b*) $\int x \sec^2 x\, dx$, (*c*) $\int_0^{\pi/4} x \tan^2 x\, dx$.

5. If $P = \int e^{ax} \sin bx\, dx$ and $Q = \int e^{ax} \cos\, bx\ \, dx$, show by integrating by parts, that $P = -\dfrac{e^{ax} \cos\, bx}{b} + \dfrac{a}{b} Q$; and $Q = \dfrac{e^{ax} \sin\, bx}{b} - \dfrac{a}{b} P$.

Hence obtain the integrals denoted by P and Q.

6. A particle moves in a straight line and its velocity v at time t is given by $dv/dt = g - kv$, where g and k are constants. Find the velocity at any time t and show that it tends to the value g/k whatever the initial velocity.

If the particle starts from rest at the origin, find how far it moves in time T.

7. (i) If $e^{-x}\, dy/dx = (1 - y)^2$, and $y = 0$ when $x = 0$, express y in terms of x.

(ii) When a mixture of two liquids is being reduced by boiling it is found that the ratio of the rates at which the liquids are separately decreasing at any instant is proportional to the ratio of the amounts x and y remaining at that instant.

Express this fact in the form of a differential equation, and solve the equation to obtain the relation between x and y.

8. (i) If $(x^2 - 1)\, dy/dx + 2y = 0$, find the value of y in terms of x given that $y = 3$ when $x = 2$.

(ii) Solve the differential equation $dy/dx - x \operatorname{cosec}^2 y = 3 \operatorname{cosec}^2 y$.

9. (i) If $e^{-x^2}\, dy/dx = x(y + 2)^2$, find the value of y in terms of x, given that $y = 0$ when $x = 0$.

(ii) Solve the differential equation $x\, dy/dx + 3y = \sqrt{(1 + 3x^3)}$, given that $y = 1$ when $x = 1$.

10. Solve the equation $dy/dx + y = x\, e^{-x}$.

If $y = -3/2$ when $x = 0$, show that y has its minimum value, equal to $-e$, when $x = -1$.

11. (i) Solve the differential equation $x\, dy/dx + 2y = (2 \sin x)/(x \cos^3 x)$.

(ii) Find the general solution of the differential equation

$$\tan x . dy/dx + y = \sin x.$$

12. (i) Solve the differential equation $dy/dx + 2y = e^{-2x}$.

(ii) By means of the substitution $y = zx$, where z is a function of x, reduce the differential equation $x\, dy/dx - y = \tfrac{1}{4}x^2 - y^2$ to a differential equation involving z and x only. Hence solve it, given $y = 0$ when $x = \log_e 2$.

13. Find the function $y = f(x)$ which satisfies the differential equation $x\, dy/dx = y + kx^2 \cos x$ and the condition $y = 2\pi$ when $x = \pi$.

For what values of k do x and y always have the same sign?

Find a first order differential equation which is satisfied by this function but does not involve k.

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EXAMPLES ON FORMAL GEOMETRY

1. Prove that, if two triangles are equiangular, the sides opposite equal angles are proportional.

$ABCD$ is a rectangle; two perpendicular lines are drawn, one cutting AB, CD in E and F respectively, and the other cutting AD, BC in G and H respectively.

Prove that $\dfrac{EF}{GH} = \dfrac{BC}{AB}$.

2. The altitudes AD, BE of a triangle meet at the point H. If AD be produced to meet the circumscribing circle of the triangle at K, prove that $HK = 2HD$.

Prove also that the three altitudes pass through H.

3. $ABCD$ is a plane quadrilateral. Points E, F, G, H are taken on AB, BC, CD, DA respectively, such that

$$\frac{AE}{EB} = \frac{BF}{FC} = \frac{CG}{GD} = \frac{DH}{HA} = \lambda.$$

Prove that (i) the sum of the areas of the triangles EBF and GDH is equal to that of triangles FCG and HAE. (ii) the ratio of the areas of the quadrilaterals $EFGH$ and $ABCD$ is $(\lambda^2 + 1) : (\lambda + 1)^2$.

4. OAA_1, OBB_1, OCC_1, ODD_1 are four concurrent lines. A_1B_1, B_1C_1, C_1D_1 are parallel to AB, BC, CD respectively. Prove that $ABCD$ and $A_1B_1C_1D_1$ are similar quadrilaterals.

PQR is an acute-angled triangle. Show how to construct a square with two vertices on QR, one vertex on PQ and one vertex on PR.

5. Through any point P inside a parallelogram $ABCD$, EPF is drawn parallel to AD meeting AB at E and DC at F, and GPH parallel to AB meeting AD at G and BC at H. HE and CA produced meet at O.

By producing AD to meet OH at X (or otherwise), prove that

$$\frac{OA}{OC} = \frac{AE}{EB} \cdot \frac{AG}{GD}.$$

Hence show that GF produced also meets CA produced at O.

6. AX and BY are parallel lines and AY and BX intersect in P. A line through P parallel to XA meets AB in Q. Prove that

$$AX \cdot BQ = BY \cdot AQ, \text{ and } \frac{1}{AX} + \frac{1}{BY} = \frac{1}{PQ}.$$

7. Prove that the areas of triangles of the same height are proportional to their bases.

Through a point D are drawn two straight lines BDC, EDF meeting two straight lines ABE, AFC drawn through a point A. Prove that

$$\frac{\triangle BAF}{\triangle EAC} = \frac{\triangle BDF}{\triangle EDC}.$$

8. On a given segment BC of a line as base, a triangle ABC is constructed, the bisector of whose angle A passes through a given point D on BC such that $BD : DC = p : q$, where $p > q$.

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Show that the greatest possible value of the angle B of the triangle is $\sin^{-1} q/p$, and that there are always two triangles with an angle B less than this value, the corresponding values of the angle C being unequal.

9. E and F are points on the sides AC and AB respectively of a triangle ABC, such that $AE = \lambda . EC$ and $AF = \lambda . FB$, where λ is a constant. D is any point on BC and DE and DF meet the line through A parallel to BC in L and M respectively. Prove that P, the point of intersection of MB and LC, lies on AD, and that $MB = (\lambda - 1)BP$.

10. If two triangles are equiangular, prove that their corresponding sides are proportional.

ABC is a triangle having a right angle at B. D is a point on AC, and E and F are the feet of the perpendiculars from D on AB and BC respectively.

Prove that the rectangle $AD . DC$ is equal to the sum of the rectangles $AE . EB$ and $BF . FC$.

11. Without assuming any formula for the area of a triangle or parallelogram, prove that the area of a triangle is one-half of that of any parallelogram on the same base and having the same altitude as the triangle.

$AMLN$ is a rectangle; B is a point in LM, C a point in LN; the lines drawn through B parallel to AM and through C parallel to AN meet in P.

Prove that the area of the triangle ABC is half the difference between the areas of the rectangles AL and AP.

12. The internal bisector of the angle A of a triangle ABC meets the base BC in X, and the external bisector of the same angle meets BC produced in Y. Prove that $BA : AC = BX : XC = BY : CY$.

If M be the middle point of XY, prove that $XY = 2MA$.

13. If the triangles ABC, DEF have areas in the ratio $AB . AC : DE . DF$, prove geometrically that the angles at A and D are either equal or supplementary.

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If the diagonals AC, BD of a quadrilateral $ABCD$ meet in X, prove that the areas of the triangles ABC, ADC are in the ratio $BX : XD$.

14. Construct the point C in the segment AB of a straight line, such that $AC^2 = AB . CB$ and give a geometrical proof of the construction.

If the segment AC is again divided at D so that $AD^2 = AC . DC$, prove (by any method) that $AD = CB$.

15. A line drawn through a fixed point O meets two given parallel lines in X and Y. Prove that $OX : OY$ is a constant ratio.

In the same figure, two fixed points A and B lie on one of the lines through O. The lines AX, BY meet in P. Prove that the locus of P is a third straight line parallel to those traced by X and Y.

16. Define similar polygons, and prove that, if two polygons be similar, they may be placed so that corresponding sides are parallel and the lines joining corresponding vertices are concurrent.

Show how to inscribe an equilateral triangle in a given triangle, one side of the inscribed triangle being parallel to one side of the given triangle.

(No proof of the validity of your construction need be given.)

17. If the internal and external bisectors of the angle A of the triangle ABC meet BC in the points X and Y respectively, prove that $BX = ca/(b + c)$, $CY = ab/(b - c)$, if $b > c$. Find the magnitudes of AX and AY in terms of a, b, and c.

18. An angle AOC is bisected internally by a straight line OB. From any point P in the plane of AOC perpendiculars PL, PM, PN are drawn to OA, OB, OC respectively. Prove that $LM = MN$.

If OB bisects $\angle AOC$ externally, prove that the above equality still holds.

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19. Prove that the areas of similar triangles are to each other as the squares on corresponding sides.

$ABCD$ is a parallelogram and X is the point of trisection of AB which is nearer to B. CX and DA produced meet in the point Y. Prove that the area of the triangle AXY is two-thirds of the area of the parallelogram.

20. Prove that, if two triangles are equiangular, their corresponding sides are proportional.

If H be the orthocentre of the triangle ABC and AH meets BC in D, prove that $BD \cdot DC = AD \cdot DH$.

Hence, or otherwise, show that, if AH meets the circumcircle of the triangle in X, $HD = DX$.

21. If two triangles ABC and DEF have their angles ABC and DEF equal and $AB : BC = DE : EF$, prove that the triangles are similar.

If K and M are the feet of the perpendiculars from the vertices A and C of the parallelogram $ABCD$ to the diagonal BD, and L and N are the feet of the perpendiculars from B and D to the diagonal AC, prove that $KLMN$ is a parallelogram which is similar to $ABCD$.

22. If X be the middle point of the side BC of the triangle ABC, prove that $AB^2 + AC^2 = 2BX^2 + 2AX^2$.

If the length of the medians AX, BY, and CZ of the triangle ABC are 2 inches, 3 inches, and 4 inches respectively, find the lengths of the sides and the greatest angle of the triangle.

23. In the side BC of the triangle ABC, points L and M are taken such that the three angles BAL, LAM, MAC are equal. Prove that, if $BL = MC$, the triangle ABC is isosceles.

24. Show how to construct a rectangle $DEFG$ inside an acute-angled triangle ABC, so that D is on the side AB, E is on the side AC, and F and G are on the side BC, DG being twice as long as DE.

If $BC = 3$ inches, and the perpendicular from A on $BC = 4$ inches, find the ratio of the area of the rectangle to that of the triangle.

25. Define similar polygons, and prove that, if two similar polygons are placed with one pair of corresponding sides parallel, then the lines joining pairs of corresponding vertices are concurrent.

ABC is a triangle, right-angled at A; show how to inscribe in it a rectangle $PQRS$ so that PQ lies along BC, R lies on CA, and S lies on AB, while the sides PQ, QR are in the ratio 2 : 1. Express the lengths of PQ, QR in terms of the sides of the triangle ABC.

26. $ABCD$ is a parallelogram. Points X, Y, Z, W are taken on the sides AB, BC, CD, DA respectively so that AX, BY, CZ, DW are each $1/n$th part of the corresponding sides. AY, BZ, CW, DX are joined forming a quadrilateral $PQRS$. Prove that the ratio of the areas $PQRS$ and $ABCD$ is $(n - 1)^2/(n^2 + 1)$.

27. Prove that the straight lines which bisect the vertical angle of a triangle internally and externally divide the base in the ratio of the sides.

Find the locus of the vertex of a triangle on a given base, when the ratio of the sides is given.

28. The side BC of a triangle ABC is divided at X, so that $p \cdot BX = q \cdot XC$. Prove that $(p + q)^2AX^2 = (p + q)(p \cdot AB^2 + q \cdot AC^2) - pq \cdot BC^2$.

If G be the centroid of a triangle ABC, prove that

$$BC^2 + CA^2 + AB^2 = 3(AG^2 + BG^2 + CG^2).$$

29 Prove that two triangles are similar if they have one angle of the one equal to one angle of the other, and the sides about these angles proportional.

The points L, M divide a segment of the line AB internally and externally in the same numerical ratio, and O bisects LM. Show that $OA \cdot OB = OL^2$.

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If P be any point on the circle with diameter LM, show that PL, PM are bisectors of the angle APB.

30. ABC is a triangle with sides $BC = 36$ cm., $CA = 25$ cm., $AB = 29$ cm. A point O lies inside the triangle, and is distant 5 cm. from BC and 10 cm. from CA. Find its distance from AB.

If AO is produced to cut BC in X, find the lengths of the segments into which X divides BC.

31. The perpendiculars of a triangle ABC meet at H, and OA' is the perpendicular from the circumcentre O to the side BC. Prove that $AH = 2OA'$.

If AA' and OH meet at M, prove that $AM = 2MA'$.

32. Prove that the line bisecting an internal angle of a triangle divides the opposite side in the ratio of the sides containing the angle.

A vertical flagstaff 51 feet high, standing on level ground, is divided into two parts by a mark 25 feet from the ground. How far away from its foot must an observer stand in order that the two parts may subtend equal angles at his eye which is 5 feet from the ground?

33. The internal bisector of the angle A of a triangle ABC meets the side BC at D. Prove that $BD : DC = AB : AC$.

If the bisector of the angle ADB meets AB at E and that of angle ADC meets AC at F, prove that

$$\frac{1}{AE} - \frac{1}{AF} = \frac{1}{AB} - \frac{1}{AC}.$$

34. Prove that, if D and E are points on the sides AB and AC respectively of a triangle ABC such that $AD : DB = AE : EC$, then DE is parallel to BC.

$ABCD$ is a trapezium in which AD is parallel to and less than BC. E and F are points on AB and DC respectively, such that AB is n times AE and DC is n times DF. DA is produced to G so that DG is n times DA and H is taken on AD produced so that $AH = BC$. Prove that GH is n times EF.

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35. Three points A, B, C are taken on a straight line such that B lies between A and C, and AB is greater than BC. A line BX is drawn through B making an angle of 45° with AC. Show how to find a point P on BX such that BP bisects the angle APC, and prove that the length of BP is $\frac{AB \cdot BC}{AB - BC}\sqrt{2}$.

36. ABC is a triangle and D is the mid-point of AB. P and Q are the centres of squares described on AC and BC respectively, externally to the triangle. Prove that $DP = DQ$.

37. The mid-points of the sides BC and AB of a triangle ABC are D and F respectively, and E is a point on AC such that $AE = 3EC$. A point G is taken on EF such that $FG = 2GE$. Prove that GD is parallel to AB.

38. In a triangle OAB, which is right-angled at A, the side OA is 2 inches and the side AB is 1 inch in length. Another triangle OBC is right-angled at B and OB is three times the length of BC. If A and C are on opposite sides of OB, prove that the angle COA is 45°.

39. $ABCD$ is a parallelogram, and a line is drawn through A meeting BD in P, CD in Q and BC produced in R. Prove that $PQ : PR = PD^2 : PB^2$ and that PA is the mean proportional between PQ and PR.

40. ABC is a triangle in which $AB = AC$. A line through the mid-point of BC meets AB in P and AC produced (through C) in Q. A line through C parallel to AB meets PQ in R. Prove that:

(i) $BP = CR$,
(ii) $BP : PA = QC : QA$,
(iii) $AB(AP + AQ) = 2AP \cdot AQ$.

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41. A and B are two fixed points on the same side of a fixed straight line XY. If P be a variable point on XY, prove that $AP + PB$ is a minimum when AP and BP are equally inclined to XY.

ABC is an acute-angled triangle and P, Q, R are the feet of the perpendiculars from A, B, C respectively to the opposite sides. Prove that PQR is the triangle of minimum perimeter with vertices on BC, CA, AB respectively.

42. The four points A, B, C, D lie on a circle in that order. From B perpendiculars BX, BY are drawn to AD and CD respectively. Prove that $\angle BYX = \angle BDA$.

Prove also that, if XY meet the circle at P and Q, the triangles BPA, BYQ are similar.

43. Prove that the locus of points from which equal tangents can be drawn to two given circles is a line perpendicular to the line joining the centres of the two circles.

Given a point P and two non-intersecting circles C_1, C_2, give a construction to obtain a point Q such that the tangents from Q to C_1 and C_2 are each equal to the distance PQ.

44. A triangle ABC is inscribed in a circle. Lines drawn through A parallel to the tangents at B and C meet BC in D and E respectively. Prove that: (i) $AD = AE$, (ii) $BD : CE = AB^2 : AC^2$.

45. With a point A on the circumference of a circle S_1 as centre, a circle S_2 is described, cutting S_1 at B and C. A straight line through A meets S_1 at P, S_2 at Q and the chord BC at R. Prove that $AQ^2 = AP \cdot AR$.

46. Tangents are drawn to a circle at points A and B, and P is any other point on the circle. Prove that the product of the perpendiculars from P to the tangents is equal to the square of the perpendicular from P to AB.

47. Prove that, if two triangles be equiangular, the sides opposite the equal angles are proportional.

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Two circles ABP, PDC intersect at P and APD, BPC are straight lines. Prove that, if the radius of the circle APB is twice the radius of the circle PDC, then chord AB is twice chord CD.

48. In a right-angled triangle ABC, prove that the perpendicular AD from the right angle A to the hypotenuse BC is a mean proportional between the segments BD, DC of the hypotenuse.

A variable tangent to a given circle meets two fixed parallel tangents at P, Q, and touches the circle at R. Prove that the rectangle $PR \cdot RQ$ is constant.

49. OPQ and ORS are straight lines and $OP \cdot OQ = OR \cdot OS$. Prove that the points P, Q, R, S are concyclic.

O is a fixed point and P is a variable point on a fixed line l. Q is a point on the line OP such that $OP \cdot OQ$ is constant. Find the locus of Q.

50. Four points A, B, C, D are taken in order on the circumference of a circle, and a point X is taken on the line AC between A and C such that the angles ADX and BDC are equal. Prove that (i) the triangles ADX and BDC are similar, (ii) $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

51. Two circles have four real common tangents and the line joining their centres meets the circles at points A, B, C, D in this order. If t_1 be the distance between the points of contact of a direct common tangent to the two circles, prove that $t_1{}^2 = AC \cdot BD$.

If the distance between the points of contact of a transverse common tangent is t_2, prove that $t_1{}^2 - t_2{}^2 = AB \cdot CD$.

52. AB is a diameter of a circle and LM is a line at right angles to the diameter, intersecting it at C. From A any two lines APQ, ARS are drawn

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cutting the circle at P and R and the line at Q and S. Prove that the triangles APB, AQC are similar, also the triangles APR, AQS.

53. In two triangles ABC and PQR the angles B, Q are equal and $AB : BC = PQ : QR$. Prove that the triangles are similar.

A diameter AB of a circle is produced to O and P is a variable point on the circle. Through O is drawn a line OP' so that POP' is a constant angle and the product $OP \cdot OP'$ is constant. If A', B' are the points corresponding in this way to A and B, prove that P' lies on the circle with $A'B'$ as diameter.

54. Two circles intersect in A and B. The tangent at A to the first circle meets the second circle in R; the tangent at B to the circle ARB meets the first circle in P; PA produced meets the circle ARB in Q; RB produced meets the circle APB in S. Prove that $PQRS$ is a parallelogram.

55. Two circles intersect at A and B, and the chord through A meets the circles again at C and D. Prove that, for all positions of the chord, the angles between the tangents to the circles at C and D are constant. Distinguish between the cases when A is between C and D, and when A is in CD produced.

56. The tangents from a point P to a circle centre O touch the circle at A and B, and the chord AB meets OP at C. Any line through P meets the circle at X and Y. Prove that the points O, C, X, Y are concyclic and that PC is the external bisector of the angle XCY.

57. $ABCDE$ is a regular pentagon. CD is produced beyond D to F, and DC is produced beyond C to G so that CF and DG are each equal to AC. Prove that the triangle AFG and the pentagon are equal in area. Prove also that EF touches the circle circumscribing the pentagon.

58. A line DE parallel to the base BC of the triangle ABC cuts AB, AC in D and E respectively. The circle which passes through D and touches AC at E, meets AB at F. Prove that F, E, B, C are concyclic.

59. Two circles, centres A and B, touch externally at a point C. A common tangent touches the circles at P and Q respectively and meets AB produced at S. If T be the point in PQ such that $PT : TQ = PS : QS$, prove that (i) the triangles PAT, QBT are similar, (ii) the internal bisector of the angle ATB passes through C.

60. Give a geometrical construction for drawing a circle with its centre on one of two intersecting straight lines, touching the other and passing through a given point.

Prove that your construction is correct and that there are in general two solutions; also state the limiting position of the point for the construction to be possible.

61. From a point P, straight lines PA, PB, PC, ... are drawn ending on the straight line ABC ... The lines PA, PB, PC, ... are produced to A', B', C', ... respectively so that $AA' = PA$, $BB' = PB$, $CC' = PC$, ... Prove that the points A', B', C', ... lie on a straight line parallel to ABC ...

Three equal circles meet in a point P, and their other points of intersection are R, Q, S. Show that the four points P, R, Q, S are such that each of them is the orthocentre of the triangle formed by joining the other three.

62. Two circles in the same plane have radii a, b, and the distance apart of their centres is c. Calculate the lengths of the common tangents, according as c is greater than or less than $(a + b)$.

If $c > (a + b)$ and the difference between the lengths of the two kinds of common tangents is $2l$, prove that $c^2 = a^2 + b^2 + l^2 + a^2b^2/l^2$.

63. Prove that the rectangles contained by the segments of two chords of a circle which meet outside the circle are equal.

$ABCD$ is a cyclic quadrilateral; AB, DC produced meet in E; BC, AD

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produced meet in F; P is a point in FE such that the angles FAP, FEC are equal.

Prove that the rectangle $FP \cdot FE$ is equal to the square on the tangent to the circle from F, and that $EP \cdot EF$ is equal to the square of the tangent from E.

64. Prove that the internal bisector of an angle of a triangle divides the side opposite to that angle in the ratio of the sides which contain the angle.

The internal bisector, AD, of the angle BAC of a triangle ABC meets BC in D; the circle through A, D, C meets AB again in Q, and the circle through A, D, B meets AC again in R. Prove that BQ and CR are equal and that their common value is $BC^2 \div (AB + AC)$.

65. OA is a fixed radius of a circle with centre O, bisecting an arc PQ of the circle. C is any point on OA, inside or outside the circle. QC meets the circle again in R, and PR meets OA (produced if necessary) in D. Prove that the triangles ORC and ODR are similar.

Hence, or otherwise, show that, if C be fixed and P and Q vary, the point D is a fixed point.

66. A point P moves so that its distances from two fixed points are in a constant ratio; prove that P lies on a fixed circle.

Each of two given circles lies wholly outside the other; if their direct common tangents intersect at B, and P be any point on the circle drawn on AB as diameter, prove, by considering the ratio of the distances of P from the centres of the circles, that the two circles subtend equal angles at P.

67. If a circle be drawn to cut two given circles at points A, B and C, D respectively, show that the lines AB and CD meet on the radical axis of the given circles.

Show how to describe a circle to cut two given circles orthogonally and to pass through a given point, not on the line of centres. Prove the validity of your construction.

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68. If $ABCD$ be a quadrilateral inscribed in a circle, prove that

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

If P be a point on the minor arc BC of the circumcircle of an equilateral triangle ABC of side 3 inches, prove that the maximum value of $PB + PC$ is $2\sqrt{3}$ inches.

69. Prove that, if two triangles are equiangular to one another, their corresponding sides are proportional.

Any point C is taken on a circle of centre O, and a diameter AOB of the circle is drawn. If the tangent at C meets AB at T, and meets the tangents at A and B at points P and Q respectively, show that the triangles TAP, TCO are similar, and deduce that the product $AP \cdot BQ$ has the same value for all positions of C on the circle.

70. Prove that, if two polygons be similar, they can be placed so that the lines joining corresponding vertices are concurrent.

Show how to describe a square inside a quadrant of a circle of radius r, so that two of the vertices of the square lie on the arc, and the other two vertices lie on the radii bounding the quadrant.

Prove that the area of this square is $2r^2/5$.

71. A variable point P lies in a fixed plane, and is such that the ratio $PA : PB$ is equal to a constant k, where A, B are two fixed points in the plane.

Prove that P lies on a fixed circle, and find the radius of this circle in terms of AB and k if $k > 1$.

72. If, from a point O outside a circle, OT be drawn to touch the circle at T, and AB be any chord of the circle passing through O, prove that $OT^2 = OA \cdot OB$.

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Find the locus of a point such that the lengths of the tangents drawn from it to two unequal and non-intersecting circles are equal.

73. Three points A, B, C, lying on a straight line, are joined to a point O outside the line, and the lines so drawn are produced to points A', B', C' respectively so that $AO \cdot OA' = BO \cdot BO' = CO \cdot CO'$. Prove that the circle through A', B', C' passes through O.

74. If the internal bisector CX of the angle ACB of a triangle ABC meets AB in X, prove that $AX : XB = AC : CB$.

The triangle ABC is right-angled at C and O is the middle point of AB. If the internal and external bisectors of the angle ACB meet AB and AB produced in X and Y, prove that OC is a tangent to the circle XCY.

75. Prove that the difference of the squares on the sides AB, AC of a triangle ABC is equal to twice the rectangle contained by the base BC and the distance between the middle point of the base and the foot of the perpendicular from A on the base.

Two circles whose centres are A and B touch at C, and from any point P on one of them a tangent PT is drawn to the other. Prove that $PT^2 = 2AB \cdot CN$, where N is the foot of the perpendicular from P on AB.

76. If A, B, C, D are four concyclic points, state and prove the relation known as Ptolemy's theorem, between the lengths of the six lines joining these four points.

ABC is an equilateral triangle, and P is any point on the circumcircle. Prove that one of the three lengths PA, PB, PC is equal to the sum of the other two.

77. Prove that the external and internal bisectors of the angle A of a triangle ABC divide BC externally and internally in the ratio $AB : AC$.

Show that the locus of a point P which moves so that $BP : CP = BA : CA$ is a circle, and find a point R such that www.dbraulibrary.org.in

$$AR : BR : CR = AB \cdot AC : BC \cdot BA : CA \cdot CB.$$

78. If AP, BQ are parallel radii of two circles which are external to one another, show that the line PQ passes through one or other of two fixed points on the line AB, where A, B are the centres of the circles, and deduce that each of the common tangents of the circles passes through one of these points.

Show also that, if another circle touches these two circles, the line joining the points of contact passes through one of these points.

79. AB is the common chord of two circles that cut at right angles. A straight line through A meets the first circle again in P and the second in Q.

Prove that: (i) $\angle PBQ$ is a right angle, (ii) the diameter of the first circle through P is perpendicular to the diameter of the second circle through Q.

80. P, Q are the points of contact of the two tangents from a point T to a circle, and TAB is any straight line through T meeting the circle in A, B. C is the mid-point of the chord AB. Prove that: (i) $TPCQ$ is a cyclic quadrilateral. (ii) $PA \cdot PB = PQ \cdot PC$.

81. Two circles intersect at A and B. P is a variable point on one of the circles and PA and PB, produced if necessary, intersect the other circle in X and Y respectively. Prove that XY and the circumradius of the triangle PXY are of constant lengths.

82. l_1, l_2, l_3 are three non-intersecting straight lines in space, no two of which are parallel. Prove that, through any point on l_1, just one straight line can be drawn intersecting both l_2 and l_3.

If l_1, l_2, l_3 are all parallel to a given plane, and $P_1Q_1R_1$, $P_2Q_2R_2$ are two straight lines meeting l_1 in P_1 and P_2, l_2 in Q_1 and Q_2, l_3 in R_1 and R_2, prove that $P_1Q_1 : P_2Q_2 = Q_1R_1 : Q_2R_2$.

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83. OA, OB, OC are three concurrent straight lines each of which is perpendicular to the other two. The foot of the perpendicular from O to the plane ABC is H. Prove that H is the orthocentre of triangle ABC.

84. l and m are two fixed non-coplanar lines and A and B are two fixed points. Find the locus of a point P which moves so that PA intersects l and PB intersects m. Show that one position of P is on l and one on m.

85. A tetrahedron $ABCD$ has its opposite edges equal in pairs; it is cut by a variable plane which is parallel to both AB and CD. Prove that (i) the section is a parallelogram, (ii) the perimeter of the parallelogram is constant, (iii) the area of the section is a maximum when it is a rhombus.

86. $ABCD$, $EFGH$ are two opposite faces of a rectangular solid, the vertices being opposite in the order named. P, Q, S, T are the middle points of AB, AD, GH, FG respectively. Prove that these points are coplanar.

If the solid be a cube and the plane $PQST$ cut BF in U and DH in R, show that $PQRSTU$ is a regular hexagon.

87. Prove that the shortest distance between two non-intersecting straight lines AB, CD is perpendicular to both.

Prove also that, if P be any point on AB and Q be any point on CD, the locus of X, the mid-point of PQ, is a plane.

88. A triangle ABC is right-angled at A. A point P not lying on the plane ABC is equidistant from A, B, and C. Prove that the line PN, joining P to the middle point N of BC, is perpendicular to the plane ABC.

If $BC = 8$, $AB = AC$, and $PN = 3$, calculate the shortest distance between the skew lines AP and BC.

89. $ABCD$ is a tetrahedron in which the angles CAB and ABD are right angles and $AC = BD$; E and F are the mid-points of AB and CD respectively. Prove that EF is perpendicular to CD and that $AF = BF$.

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90. A and B are fixed points, and P is a variable point such that PA/PB is constant. Show that the locus of P is a sphere. Find the radius of the sphere when $AB = 5$ cm. and $PA/PB = 3/2$.

91. Prove that the locus of points which are equidistant from two given points is the plane which bisects at right angles the line joining the two points.

A plane through the edge AB of a regular tetrahedron $ABCD$ bisects the side CD; find the angles between this plane and the planes CAB, DAB.

92. (*a*) Show, with proof, how to draw a perpendicular to a plane from a point outside it.

(*b*) Show how to draw the common perpendicular to two non-intersecting straight lines, and prove that this perpendicular is the shortest distance between the two lines.

93. Given a point and a plane, such that the point does not lie in the plane, show how to construct the straight line which passes through the point and is perpendicular to the plane. Justify your construction.

$ABCD$ is a face of a cube, E is the vertex diametrically opposite to A, and AF is an edge parallel to CE. Prove that AE is perpendicular to the plane BDF.

94. If two parallel planes are cut by a third plane, prove that the lines of intersection are parallel.

Four planes meet in a point. Show how to draw, through a given point, one of the planes which cuts the four given planes in lines which form the sides of a parallelogram.

95. Prove that two lines, not in the same plane, have a common perpendicular, and that it is the shortest distance between them.

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Find the magnitude of the shortest distance between a pair of opposite edges of a regular tetrahedron of side $2a$.

96. If a straight line be parallel to a line lying in a given plane, prove that it is parallel to the plane.

Two planes have a line of intersection AB. The intersections LM and PQ of these planes with a third plane are parallel. Prove that LM and PQ are each parallel to AB.

97. Prove the following properties of two lines AB and CD, which do not lie in the same plane.

(i) There is a straight line perpendicular to AB and CD.

(ii) The common perpendicular is the shortest distance between AB and CD.

(iii) There is only one line which can be drawn through a point P, which does not lie in AB or CD, to cut AB and CD.

(iv) The locus of points equidistant from AB and CD is a plane.

98. If two straight lines are parallel, and if one of them is perpendicular to a plane, prove that the other is also perpendicular to the same plane.

The point B is the foot of the perpendicular from a point A to a plane XY. If BC and CD are two perpendicular lines in the plane XY, prove that AC is perpendicular to CD.

99. AA', BB', CC' are three straight lines, no two of which are coplanar. Prove that, if P, Q be any two points on CC', one straight line, and only one, can be drawn through each of these points so as to intersect both AA' and BB'.

Prove also that the lines so drawn are not coplanar.

100. Prove that three parallel planes intercept on any two straight lines segments which are in the same ratio.

$ABCD$ is a regular tetrahedron, and X, Y are the mid-points of AB, AC respectively. Through XY a plane is drawn at right angles to the plane ABC. Find the ratios in which this plane divides the edges AD, BD, CD, produced if necessary.

101. Prove that a straight line, which is perpendicular to two given straight lines in a plane, is perpendicular to any other straight line in that plane.

The triangles ABC, ABD are in perpendicular planes, and are right-angled at A and B respectively. P is the mid-point of AB, and Q the mid-point of CD. Show that PQ is perpendicular to AB.

102. Prove that, if a straight line AD is at right-angles to the straight lines AB and AC, it is at right angles to any straight line AF in plane ABC.

If $AB = AC = AD$, and the point F lies on BC and the angle BAC is a right angle, prove that $2 \cot \angle FAB = \sqrt{3} \cot \angle FDB - 1$.

103. Define the angle between two non-intersecting straight lines.

$ABCD$ is a regular tetrahedron and E is the mid-point of CD. Prove that the edges AB and CD are perpendicular and that the angle between AE and BC is $\cos^{-1} (\sqrt{3}/6)$.

104. OA, OB, OC are three straight lines such that OA and OB are each perpendicular to OC. Prove that every straight line in the plane containing OA and OB is perpendicular to OC.

$ABCD$ is a regular tetrahedron, and E, F, G, H are the mid-points of AB, CD, BC, AD respectively. Prove that EF and GH are perpendicular.

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ANSWERS TO EXAMPLES

CHAPTER I (page 22)

8. (*a*) Real, (*b*) real and opposite in sign; $a^4x^2 - [b^4 - 4ab^2c + 2a^2c^2]x + c^4 = 0$.

9. $a = 2b \pm \sqrt{(4b^2 + c^2)}$.

10. (i) $\lambda \leqslant 2, \lambda \geqslant 5$, (ii) $(25ac - 4b^2)x^2 - 3abx + a^2 = 0$.

11. $a = 0$ and $b = c$; $b = -2,\ c = -48$.

12. $8/3$.

13. (i) $4x^2 - 13x + 1 = 0$, (ii) $-4 \leqslant k \leqslant 1$.

14. $y = 6$ or -1.

15. $p^2 - 2(q - q_1) - pp_1$.

16. $(a - 2b + c)x^2 - (6a - 10b + 4c)x + (9a - 12b + 4c) = 0$.

18. $p^3 + q^3 + r^3 = 3pqr$.

21. $16x^2 - 24x - 11 = 0$.

22. $a = \frac{1}{2}[2 + d \pm \sqrt{(d^2 - 4n + 4)}],\ b = \frac{1}{2}[(2 - d) \pm \sqrt{(d^2 - 4n + 4)}]$.

23. Maximum and minimum value is $(ac - b^2)/a$.

25. $a^2p^2x^2 - abpqx + (acq^2 + b^2pr - 4acpr) = 0$.

27. (i) 45, (ii) $-5/4$.

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28. $x = (a - c)/b$ or $(a - b)/c$;
$(a - b)(a - c)x^2 + (b^2 + c^2 - ab - ac)x + bc = 0$.

31. $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

32. (i) complex, (ii) real, opposite signs, greater negative, (iii) real, opposite signs, greater positive, (iv) real, both negative.

33. $a(a^3 + b^3 + c^3 - 3abc) = 0$.

34. (i) $ax^2 + bx + c = 0$, (ii) $3 > y \geqslant 2$.

35. (ii) $ax^2 - 4bx + c = 0$.

36. (i) Minimum value $47/8$, (ii) $4x^2 + 19x + 49 = 0$.

37. $x^2 - x \geqslant -\frac{1}{4}$; if $x > 2$, or if $-1 < x < 0$.

38. $k = 3$.

39. $k = 1$ or -3; $x = -1$ (twice), $x = 1$.

40. (i) $x = 25$, (ii) $x = -3,\ y = 2$; $x = -27/5,\ y = 2/5$.

41. (i) $x = -\frac{1}{2},\ y = -6$; $x = 2,\ y = -1$, (ii) $x = -6$.

42. $x = 2$.

43. (i) $x = \frac{2}{3},\ y = -\frac{1}{3},\ z = 0$, (ii) $x = 5,\ y = 1$; $x = -19/5,\ y = -17/5$.

44. (i) $x = 3 \pm \sqrt{6}$, (ii) $x = 5,\ y = 1$; $x = 1,\ y = 5$.

45. $a = -6,\ b = 4,\ c = 2,\ d = 5/2$; $y = -15/16$.

46. $u^4 - 4u^2v + 2v^2$; $x = 5,\ y = 2$ and $x = 2,\ y = 5$.

47. (*a*) $x = -2$, (*b*) $x = \pm 1/\sqrt{2},\ y = 0$ and $x = \pm 2/\sqrt{6},\ y = \mp 1/\sqrt{6}$.

48. $x = 1,\ y = 0,\ z = 6$; $x = 3,\ y = 4,\ z = -2$.

49. (i) $x = -6$, (ii) $x = 14/5,\ y = 15/7$ and $x = 1,\ y = 3$.

50. (i) $x = 5$, (ii) $x = \pm 4/3,\ y = \pm\frac{3}{4},\ z = \pm 12$, $x = \pm 5\frac{17}{24},\ y = \pm 6\frac{7}{24}$, $z = \mp 4\frac{23}{24}$.

51. (i) $4x/(x^2 - 2x - 2)$, (ii) $x = 25/7$ or $-1/7$.

52. (*a*) $x = 57/16$, (*b*) $x = 1,\ y = 2$ and $x = 2,\ y = 3$.

53. $x = 4,\ y = 3,\ z = 5$; $x = 8,\ y = -3,\ z = 7$.

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54. (i) $x = \pm 1$, $y = \mp 2$ and $x = \pm 3\sqrt{\frac{5}{23}}$, $y = \pm 5\sqrt{\frac{5}{23}}$,

(ii) $x = 1$, ± 2, or -3.

55. $x = \pm 3$, $y = \mp 5$; $x = \pm 2\sqrt{2}$, $y = \mp 3\sqrt{2}$.

56. (i) $x = -1/17$, (ii) $x = 5/2$, $y = -3/2$ and $x = -3/2$, $y = 5/2$.

57. (i) $x = 2$, $y = 6$ and $x = 6$, $y = 2$, (ii) $x = 5{\cdot}129$, $y = 2$.

58. (i) $x = \pm 3\sqrt{10}/8$.

59. (i) $x = 2$, $\frac{1}{2}$, -3, or $-\frac{1}{3}$, (ii) $z^2 + 2z + 1 = 0$; $x = 3$, $y = 4$ and $x = -4$, $y = -3$.

60. (ii) $x = 9/2$, $y = 6$, $z = 4/3$; $x = -9/2$, $y = -6$, $z = -4/3$.

61. (i) $x = \frac{1}{3}(a - 4b)$.

62. (i) $x = 2{\cdot}322$, (ii) $x = \pm\sqrt{(pr/q)}$, $y = c \pm \sqrt{(pq/r)}$, $z = -c \pm \sqrt{(qr/p)}$.

63. $x = \dfrac{c(c - b)}{a(a - b)}$, $y = \dfrac{c(a - c)}{b(a - b)}$.

64. (i) $x = 7$ or 3, (ii) $x = \pm 2$, $y = \mp 3$ and $x = \pm\dfrac{\sqrt{5}}{5}$, $y = \pm\dfrac{8\sqrt{5}}{5}$.

65. (i) $x = \pm 3$, $y = \mp 1$ and $x = \pm 1$, $y = \mp 3$, (ii) $x = 9$ or -4.

66. $x = \frac{1}{2}$, $y = 10$; $x = -2$, $y = 15$.

67. (i) $x = 3$, $y = -5/3$ and $x = 16/9$, $y = -\frac{3}{4}$, (ii) $6p^2 > q^2$.

CHAPTER II (page 37)

1. (i) x^2y^3/z^9.

2. $\sqrt{12} - \sqrt{6}$.

3. $(x^{2/3} + y^{-4/5})(x^{1/3} + y^{-2/5})$.

4. (i) $2\sqrt{3} - \sqrt{7}$, (ii) $11x^3 + 2x^2 - x + 1$.

5. (i) $9/4 + \frac{1}{4}(5\sqrt{5})$, (ii) $\frac{1}{8}(105 + 51\sqrt{5})$.

6. $(p + \sqrt{q} - \sqrt{r})(p - \sqrt{q} + \sqrt{r})(-p + \sqrt{q} + \sqrt{r})$; $(2 + \sqrt{2} - \sqrt{6})/4$.

7. (ii) $y = 2$.

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8. (i) $0{\cdot}435$, (ii) $\log_{10} 5 = 0{\cdot}6989700043$, $\log_{10} 0{\cdot}125 = \bar{1}{\cdot}0969100129$.

9. (i) 22, (ii) $x = 4$, $y = 2$.

10. (i) $x = 3{\cdot}2$.

11. (ii) (*a*) $x = 2$ or -4, (*b*) $x = -0{\cdot}6027$.

12. $-0{\cdot}03476$; $0{\cdot}2368$; $1{\cdot}73159$.

13. $2{\cdot}373$.

14. $\log_6 4 = 0{\cdot}774$, $\log_{1/6} 4 = -0{\cdot}774$, $\log_4 6 = 1{\cdot}29$.

15. $\log 28 = 2{\cdot}17074$; $a = 4{\cdot}642$.

16. (i) $x = -3{\cdot}513$, (ii) $x = -2{\cdot}517$, $y = 1{\cdot}839$.

17. (i) $\log_8 5 = 0{\cdot}7740$, (ii) $x = 2\log_{10} 2 - 1 = -0{\cdot}39794$.

18. $\log 5 = 16a - 4b + 7c$, $\log 3 = 11a - 3b + 5c$.

19. (ii) $x = 0$ or $0{\cdot}8611$.

20. (i) $9{\cdot}452$, (ii) $-0{\cdot}5413$.

21. $n = 163$, 164, 165, 166; $x = 1{\cdot}25$.

23. $0{\cdot}2779$.

25. $y = 1{\cdot}35x^{1\cdot 5}$; $x = 2{\cdot}995$.

26. $E = 861{\cdot}9$, $\log_7 E = 3{\cdot}474$.

27. $a = 64$, $b = 2\sqrt{2}$, $c = \sqrt{2}$.

28. $x = 3{\cdot}166$, $y = 2{\cdot}485$.

29. 15; $x = \pm 0{\cdot}6309$.

31. $\log_{10} 2 = 0{\cdot}301$.

32. (ii) $x = 0{\cdot}262$ approximately.

33. (i) $x = 0{\cdot}322$.

34. $\log_{10} 108 = 2p + 3q$, $\log_{10} 10{\cdot}8 = 2p + 3q - 1$, $\log_{10}\sqrt{(3/5)} = \frac{1}{2}(p + q - 1)$

(ii) $x = 0$, or $\dfrac{\log 2}{\log 3} = 0{\cdot}631$.

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35. (i) $x = 2$, $y = 3$, (ii) $n = 31$.
36. (i) 1·585, (ii) $x = 2{\cdot}585$, $y = 2$.
37. (i) $x = 2{\cdot}66$.
38. (i) $x = \dfrac{\log 3}{\log 5}$ or $\dfrac{\log 2}{\log 5}$, (ii) $\log_{10} 96 = 1{\cdot}982271$, $\log_{10} 0{\cdot}0375 = \bar{2}{\cdot}574031$.

CHAPTER III (page 58)

2. $A = 0$, $B = \frac{1}{6}$, $C = \frac{1}{2}$, $D = \frac{1}{3}$.
4. H.C.F. $= (x + 2)$, L.C.M. $= (x - 1)(x + 1)(x + 2)^2(x + 3)(x - 3)$.
5. (i) z, (ii) $(x - 2y + 3z)(2x + y - z)$.
6. $-14x/(9x^2 - 1)$; $\frac{7}{3}$.
7. $(x^2 + y^2 + z^2 - yz - zx - xy)$; $2(c^3 - 3d^2) = a(3b^2 - a^2)$.
8. $2x^3 + 5x^2 - x - 6$.
9. $(x^2 + y^2 + z^2 - yz - zx - xy)$; $xyz = -4$.
10. $p = -3$, $q = 2$ or $p = 2$, $q = -3$; $A = -1$, $B = 1$, $C = 2$, $D = 1$ (cancelling by 25).
11. H.C.F. $= a + 3b$, L.C.M. $= (a + 3b)^2(a - b)^2(a - 2b)(a - 3b)$.
12. (i) $A = 3$, $B = -12$, $C = 7$, (ii) $\alpha = 2$, $\beta = -3$ or $\alpha = -5$, $\beta = 4$.
13. 1 : 2 : 2.
15. $\lambda = 7$, factors $(x + 2y - z)(x - y + 3z)$; $\lambda = -5$, factors $(x + 2y + 3z)(x - y - z)$.
16. $3abc(b + c)(c + a)(a + b)$.
17. $(b - c)/a$: $(c - a)/b$: $(a - b)/c$.
18. $(bx - ay)^2 + (ax + by - 1)^2$; $x = a/(a^2 + b^2)$, $y = b/(a^2 + b^2)$.
19. $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$; $(1 - \sqrt{2} - \sqrt{3})$, $(1 - \sqrt{2} + \sqrt{3})$, $(1 + \sqrt{2} - \sqrt{3})$.
20. $x + 2k$.
21. $(x^2 + 4x + 2)$, $(x^2 - 4x + 2)$; $x = \pm 2 \pm \sqrt{2}$.
22. $A = 1/7$, $B = -12/49$, $C = 24/49$.
23. (b) $-2(b - c)(c - a)(a - b)(a + b + c)$.
25. $a = \frac{1}{3}$, $b = -4/3$, $c = 1$; $x = 1$ or $\frac{1}{2}(-1 \pm \sqrt{13})$.
26. $p = 3/2$, $4x^2 - \frac{1}{2}x - 2$; $p = -1$, $4x^2 - 3x + 3$.
28. $x = 2$, $y = 3/2$, $z = 1/3$, or $x = 23/8$, $y = 48/11$, $z = 44/69$.
29. $x = 0{\cdot}45$ or $3{\cdot}52$.
30. $x = 1$ or $3{\cdot}23$.
31. $k = 3{\cdot}6$, $x = 0{\cdot}358$ or $-3{\cdot}358$.
32. $x = 0{\cdot}11$.
33. $n = 0{\cdot}44$, $k = 14{\cdot}5$.
34. $x = 1{\cdot}62$ or $-0{\cdot}62$.
35. $[-2 \pm \sqrt{(6/m)}, \pm\sqrt{(6m)}]$.
36. $y = 2x - 5$; $(5/2, 0)$.
37. $k < 1 - \dfrac{\sqrt{3}}{2}$ and $k > 1 + \dfrac{\sqrt{3}}{2}$; $3 < x < 4$.
38. $x = 2{\cdot}32$.
39. Two roots; $x = 0{\cdot}54$.
40. $x = 3{\cdot}11$, $0{\cdot}1001$, or $-3{\cdot}21$.
41. $x = 2{\cdot}3$.
42. $x = 1{\cdot}1$ or $-1{\cdot}4$.
42. $m = \frac{1}{2}$.
44. $x = 1{\cdot}32$.

CHAPTER IV (page 78)

1. Sum $= (3 + x^2)/(x^2 - 2x + 3)$.
2. (i) 3 and 75, (ii) First term $= 2kl/(k^2 + l)$, common ratio $(k^2 - l)/(k^2 + l)$.

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4. (i) 11,570, (ii) Sum $= 2n \cdot 3^{n+1} - 15/2(3^n - 1)$.
6. 22.
7. (i) $6[1 - (\frac{1}{2})^n]$, (ii) $10 - \frac{1}{2^n}(4n + 10)$.
8. Sum $= 3[1 - 3^{-n}]$, number of terms = 8.
9. $a = 3 - 6n$, $b = 3n^2 - 3n + 1$; sum $= r(r + 1)(2r + 1)/6$.
10. (ii) $125/2$, $-75/2$; $-27/2$, $45/2$; sum to infinity $= 625/16$.
11. Sum $= \frac{a^n - b^n}{(a - b)a^{n-2}}$; number of oscillations = 24.
14. $(1 - x^n)/(1 - x)$; $-1 < x < 1$; sum to infinity $= 1/(1 - x)(1 - x^2)$.
15. 17th.
17. £3,751 10s.
19. (i) 2, 4, 6, . . .
21. £1,308 11s.
22. £2,716.
24. $a = 845{\cdot}3$, $r = 0{\cdot}6762$; sum to infinity = 2,611.
25. £1,405.
27. £702 approximately; 41 years.
28. -7, $-4\frac{1}{2}$, -2, $\frac{1}{2}$, 3, $5\frac{1}{2}$, 8, $10\frac{1}{2}$, 13.
29. (i) 8, (ii) Sum $= \frac{2 + x - (3n + 2)x^n + (3n - 1)x^{n+1}}{(1 - x)^2}$.
30. $a = 2b$, sum of first 6 $= 364b/27$; $a = -3b$, sum $= -63b$.
31. 3,654.
32. £78 10s.
33. (ii) 9 years.
34. (i) 9 or 16, (ii) $r = 1/(1 + p)$.
36. $n = 11$.
37. (ii) $A = 3$, $B = 2$; sum = 2,211.
38. 22 years.
39. (i) First term $= p + q - 1$, common difference $= -1$.
40. (i) 332,667, (ii) eleventh term = 1, sum to infinity = 271.
41. (i) $\frac{1}{6}n(n + 1)(3n^2 + 5n + 1)$, (ii) $-n^2(4n + 3)$. (iii) $\frac{1}{4}n(n + 1)(n + 2)(n + 3)$.

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CHAPTER V (page 103)

1. (i) 2,860, (ii) 120.
2. 182.
4. (i) $n(n - 1)(n - 2)$, (ii) 126; 56.
5. 256; sum = 711,040.
6. 3,360; 1,200.
7. 155; 25; 30.
8. $\frac{1}{2}m(m - 1) + n(n - 1) + 2mn$.
9. 625; 505; 306.
10. 73.
11. $15!/(5!)^3$.
12. $\frac{13!}{(4!)(2!)}$.
14. (ii) 72.
15. $n(n - 1)(n - 2)/6$.
16. 126,720; greatest term is ninth.
18. 1,001; 2,002; 3,003.
19. (i) $-704,000$, (ii) $r = 5$.
22. $n = 8$.
23. (i) $2^7 \cdot 3^7 \cdot 7$, $2^5 \cdot 3^7 \cdot 5 \cdot 7$, (ii) $n = 7$, coefficient of $x^4 = 56$.

24. $-51 \cdot 2^{19}$; $a = 16$, $b = 90\frac{2}{3}$.
25. (i) Fifth term.
26. (i) ${}^{19}C_r \cdot 4^{19-r} \cdot 3^r \cdot x^r$; ${}^{19}C_8 \cdot 4^{11} \cdot 3^8$.
28. (ii) $x^8 - 8x^7 + 36x^6 - 104x^5 + 214x^4 - 312x^3 + 324x^2 - 216x + 81$.
29. (ii) $1 - \frac{15}{2}x + \frac{35}{2}x^2 - \frac{15}{4}x^3 - \frac{515}{16}x^4$.
30. $-{}^{21}C_{10} \cdot 3^{10} \cdot 2^{11} = -2^{13} \cdot 3^{11} \cdot 7 \cdot 13 \cdot 17 \cdot 19$; 18th term $= 2^{17} \cdot 3^{19} \cdot 5 \cdot 7 \cdot 19$.
31. $-1{,}760$; $2{\cdot}00900420$.
33. $n = 2$, expansion $= 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6$;
$n = 3$, expansion $= 1 + 3x + 5x^2 + 7x^3 + 7x^4 + 5x^5 + 3x^6 + x^7$.
34. $1 + 2nx + (2n^2 - m)x^2$; $p = 2a + 1$.
35. Coefficient of x is -200, coefficient of x^5 is $-1{,}052$.
39. $a = 2(1 - n)$; coefficient of x^3 is $-4n(n-1)(2n-1)/3$, coefficient of x^4 is $-2n(n-1)(2n-1)(n-3)/3$.
40. 7, 21, 35.
42. 9,602.
43. Coefficient of x^9 is $-1{,}280a^2$; $-1{,}001 \times 2^5 \times 3^{11}$.
45. (ii) $a^8 + 2a^7b - 2a^6b^2 - 6a^5b^3 + 6a^3b^5 + 2a^2b^6 - 2ab^7 - b^8$.
48. (i) -72, (ii) $n = 9$, $c = 2$.
49. (i) $-231/8$, (ii) $r = 9$, $c_r = 2{,}002$, $c_{r+1} = 1{,}001$; sixth term and seventh term.
50. (i) $n = 14$, $r = 4$, (ii) 175 ways.
51. (i) 4,095, (ii) 2,520, (iii) (*a*) $\frac{3}{4}$, (*b*) $\frac{1}{4}$.
52. 246 ways.
53. 450 ways; $3/11$.
54. (i) 120; 24; 72.
55. $n = (k + 3)/(k - 1)$, $r = 2/(k - 1)$.
56. $\frac{1}{r}$; $\frac{a\mu}{r^2}$; $\frac{a^2}{2r^3}(3\mu^2 - 1)$; $\frac{a^3}{2r^4}\mu(5\mu^2 - 3)$.
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57. $1 + nx + \frac{n(n+1)}{2}x^2 + \frac{n(n+1)(n+2)}{6}x^3$.
58. $1{\cdot}10462$; (*a*) $10{\cdot}462$, (*b*) $-0{\cdot}052$.
59. (ii) Coefficient of $x^5 = n(n - 1)$, coefficient of $x^7 = \frac{1}{3}n(n-1)(n-2)$.
60. (i) $1{,}120/81$.
61. (i) $16\frac{1}{2}$, (ii) $10{\cdot}049876$.
62. 970.
63. (ii) 5,151 ways.
64. (i) 3,780, (ii) $\frac{1}{2}(1 - x + \frac{3}{8}x^2 - \frac{7}{8}x^3)$, $|x| < 1$.
65. (i) $1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$; $0{\cdot}986084$.
(ii) 2,640.
66. (i) 25,740; 3,960, (ii) $-1{,}050$.
67. (i) 105; 70, (ii) ${}^nC_r a^{n-r} b^r x^r$, $a = \sqrt{3}/3$.

CHAPTER VI (page 150)

1. $37° 5'$, $30°$.
2. 6·608 ml., S. 68° E.
6. $\tan^{-1} 4 = 75° 58'$; $\tan^{-1} 2\sqrt{2} = 70° 32'$
8. $\frac{1}{3}ab^2 \tan \theta$ cubic units.
9. 23·46 feet, 24° 31′.
10. 465 feet; error approximately 5 feet.
11. 23·5 feet.
12. $p\sqrt{(p^2 + q^2)}/(p \cos \theta + q \sin \theta)$.
14. (i) 79° 6′, (ii) 59° 6′.
15. 1·465 inches, 20° 53′.

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16. $12/\sqrt{29}$ cm.
18. 34·4 miles; 13′, 17′.
19. 18·9 feet.
20. 29° 20′, 38° 48′, 111° 52′.
21. 43° 41′.
22. 40·10 yards.
23. 4° 20′ N. of E.; 137·9 feet.
24. $a\sqrt{2}/\sqrt{(\cot^2\alpha + \cot^2\gamma - 2\cot^2\beta)}$; about 90 feet.
25. 25° 55′; 25° 40′.
28. Tower 28·5 yards, BC = 65·6 yards.
29. θ = 30° 36′.
30. (i) $13\cos(x + \alpha)$; $\alpha = \tan^{-1} 2\cdot4$ = 67° 23′; 292° . 37 .
31. (0°, 30°), (60°, 90°), (0°, 150°).
32. (i) A = 37° 55′, B = 25° 37′; A = 154° 23′, B = 142° 5′,
(ii) x = 14° 2′, 123° 41′, 194° 2′, or 303° 41′.
34. (iii) $-\frac{1}{2}$; $2/\sqrt{3}$.
35. (*a*) $n\,.\,180° + (-1)^n\,.\,18°$, $m\,.\,180° - (-1)^m\,.\,54°$. (*b*) $n\,.\,180° + 90°$, and $\dfrac{1}{p+1}(m\,.\,360° \pm 60°)$.
36. $\theta = n\pi/4$, n any integer.
37. θ = 0·26, 1·57 or 2·98.
38. x = 18°, 90°, 162°, 234°, 306°; $x = 18°(4m + 1)$.
40. $r = \sqrt{(a^2 + b^2)}$, $\sin\alpha = \dfrac{b}{\sqrt{(a^2+b^2)}}$, $\cos\alpha = \dfrac{a}{\sqrt{(a^2+b^2)}}$; θ = 68° 48′.
41. θ = 4° 34½′ or 273° 13½′.
42. (ii) $\theta = (4n + 1)\,.\,9°$ or $(2m - 1)\,.\,45°$.
43. $\theta = n\,.\,360° \pm 134° 26′ + 36° 52′$; 171° 18′, 262° 26′.
45. $\theta = n\,.\,360° \pm 60°$ or $m\,.\,360° \pm 36° 52′$.
46. $\theta = n\,.\,360° + 240°$.
47. (i) $\pm(1 + t)/(3 + t)$.
48. $\sin(\alpha - \beta) = -60/901$, $\cos(\alpha + \beta) = 451/901$.
49. Maximum 7, minimum 2.
52. $\theta = \pi/6$, $\varphi = -\pi/6$.
54. Least value $\cos^2\frac{1}{2}k$.
56. $A = \frac{1}{2}\sqrt{[(a - c)^2 + b^2]}$, $\tan B = b/(c - a)$, $C = \frac{1}{2}(a + c)$; greatest value 9·5, least value −5·5; θ = 71° 5′.
57. $x = (2n + 1)\,.\,180°$ or $m\,.\,360° + 53° 8′$; −306° 52′, −180°, 53° 8′, 180°.
59. $\cos\dfrac{\pi}{10} = \left(\dfrac{5 + \sqrt{5}}{8}\right)^{\frac{1}{2}}$, $\cos\dfrac{3\pi}{10} = \left(\dfrac{5 - \sqrt{5}}{8}\right)^{\frac{1}{2}}$,
$\cos\dfrac{7\pi}{10} = -\left(\dfrac{5 - \sqrt{5}^{\frac{1}{2}}}{8}\right)$, $\cos\dfrac{9\pi}{10} = -\left(\dfrac{5 + \sqrt{5}}{8}\right)^{\frac{1}{2}}$.
60. $x = n\,.\,360° \pm 63° 26′ - 26° 34′$; (i) x = 37°, 270°, −90°, −323°, (ii) x = 0·76 approximately.
61. $\frac{1}{4}\sqrt{(10 - 2\sqrt{5})}$ = 0·5878.
62. $\alpha = k\,.\,180° \pm 50° 42′$, $\beta = k\,.\,180° \pm 9° 17′$;
$\alpha = k\,.\,180° \pm 9° 17′$, $\beta = k\,.\,180° \pm 50° 42′$.
64. $BC = \dfrac{a(h^2 + a^2)}{h^2 - a^2}$, $CD = \dfrac{a(h^2 + a^2)^2}{(h^2 - a^2)(h^2 - 3a^2)}$.
65. 13° 17′, 103° 17′, 193° 17′, 283° 17′.
68. (ii) 26° 34′, 63° 26′, 206° 34′, 243° 26′.
69. (i) $\theta = n\,.\,180° + 30° 58′ + (-1)^n\,.\,20° 4′$, (ii) $\theta = n\,.\,180° + 45°$ or $m\,.\,180° + 18° 20′$.

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71. $\sin 18° = 0{\cdot}30902$, $\cos 18° = 0{\cdot}95106$.
72. 60°, 120°, or −96° approximately.
73. $x = 2{\cdot}33$ or $3{\cdot}30$ approximately; $-0{\cdot}84 < k < 0{\cdot}35$.
74. 206,266; $x = 130°$ approximately.
75. $\alpha = 0{\cdot}49$ approximately.
76. $x = 3\pi/20$, $3\pi/4$, or $97\pi/60$ approximately.
77. $x = 1{\cdot}03$ approximately.
78. Shorter is 11·15 feet.
80. $CD = 108{\cdot}1$ feet; width $= 64{\cdot}8$ feet.
82. $\tan\theta = \frac{1}{2}(u - 1/u)$, $\tan\frac{1}{2}\theta = (u-1)/(u+1)$; 45°, 135°.
86. $\tan 15° = 2 - \sqrt{3}$.
87. $\tan\frac{1}{2}\theta = \frac{1}{4}$ or $\frac{1}{3}$; $\theta = 28°\,4'$ or $36°\,52'$.

CHAPTER VII (page 189)

1. $c = 14{\cdot}69$ or $4{\cdot}49$.
2. $\varphi = 62°\,27'$, $a = 16{\cdot}65$ cm.
3. $74°\,7'$, $49°\,29'$.
4. $57°\,24'$.
6. $A = 66°\,12'$, $c = 20{\cdot}17$ inches.
7. $B = 22°\,3'$, $C = 126°\,15'$, $c = 10{\cdot}74$.
8. 2·275 miles.
9. 4·711 feet.
10. $AX = \frac{1}{2}\sqrt{673}$ feet, $BY = 2\sqrt{37}$ feet, $CZ = \frac{1}{2}\sqrt{505}$ feet; area $\triangle ABC = 84$ square feet, area second $\triangle = 63$ square feet, ratio 4 : 3.
12. $(\sqrt{2} - 1)$ inches; $[1 - \pi(1 - \sqrt{2}/2)]$ square inches.
14. 16·35 square inches.
16. $100°\,48'$, $32°\,58'$, $46°\,14'$; area $= 67{\cdot}16$ square inches.
17. 13·19 m.p.h.; $8°\,21'$ N. of W.
18. $4\sqrt{5}$ inches.
19. $B = 85°\,36\frac{1}{2}'$, $C = 43°\,23\frac{1}{2}'$; $BD = 3{\cdot}365$ inches.
20. (i) $AD = 3a$, (ii) Volume $= \frac{2}{3}a^3$ cubic units.
22. $28\sqrt{5} \div 7$, 9, 12, i.e. 8·94, 6·96, 5·22 feet.
24. $29°\,20'$, $38°\,48'$, $111°\,52'$.
26. $A = 72°\,49'$.
31. N., $60°\,10'$ E.
32. $C = 54°\,38'$, $B = 85°\,22'$, $b = 10{\cdot}389$ inches or $C = 125°\,22'$, $B = 14°\,38'$, $b = 2{\cdot}6333$ inches.
33. External bisector $= (2bc\sin\frac{1}{2}A)/(b-c)$, $b > c$.
37. $B = 30°\,47'$, $C = 51°\,35'$, $a = 19{\cdot}91$ inches.
39. 7 inches and 8 inches; radius $16/\sqrt{15}$ inches.
40. $B = 44°\,48'$, $C = 100°\,12'$, $c = 24{\cdot}04$ or $B = 135°\,12'$, $C = 9°\,48'$, $c = 4{\cdot}15$.
41. Area $= 16\sqrt{15}/15$ square inches; angle $ABC = 75°\,31'$.
42. 8·30, 6, 3·70.
43. $\frac{1}{2}(B + C)$, $\frac{1}{2}(C + A)$, $\frac{1}{2}(A + B)$.
45. $B = 65°\,47'$, $C = 48°\,55'$, $a = 430{\cdot}3$ feet, area $= 70{\cdot}06$ square feet.
47. $49°\,6'$, $49°\,6'$, $81°\,48'$.
48. $l^2 = a^2 + 4x(a + x)\sin^2(\pi/n.)$
51. $A = 22°\,56'$, $B = 34°\,12'$, $C = 122°\,52'$; area $= 34{,}040$ square feet.
52. $PQ = 6{\cdot}481$, $AP = 13{\cdot}36$.
54. $B = 37°\,13'$, $C = 91°\,39'$, $a = 27{\cdot}42$.
56. $\sin\frac{1}{2}A = \frac{1}{2}(\sqrt{5} - 1)$ therefore $A = 76°\,24'$; $B = C = 51°\,48'$.
57. $q = (2bc\sin\frac{1}{2}A)/(c - b)$.
59. Angle $BCD = 89°\,25'$.

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60. $R = 5{\cdot}77$ inches, $OH = 2{\cdot}99$ inches.
61. $A = 45° 34'$, $r_1 = 7{\cdot}14$ cm.
62. $a = 15{\cdot}2$, $B = 44° 33'$, $C = 28° 1'$.
65. $A = 31° 34'$, $B = 43° 18'$, $C = 105° 8'$.
66. 76·4.
68. 64° 28′, 25° 32′.
70. $A = 45° 14'$, $a = 7{\cdot}455$ inches, $r = 1{\cdot}894$ inches.

CHAPTER VIII (page 221)

1. $2x + y = 10$, $2y = x + 5$; area $= 78/5$ square units.
3. (4, 2), (0, 4).
4. $3x - y - 3 = 0$, $x + 3y - 11 = 0$; area = 5 square units.
6. $P \equiv (28/5, -1/5)$.
7. $B \equiv (4, 4)$.
8. $y - k = (b/a)(x - h)$; $C \equiv (25, 16)$, $M \equiv (9, 4)$.
9. $x_1 = (1/5)(3x_2 - 4y_2 + 2)$, $y_1 = (1/5)(3y_2 - 4x_2 + 4)$; $(\frac{1}{3}, \frac{1}{3})$.
10. $x + 4y = 11$; $\tan^{-1} 5/3$ and $\tan^{-1} 9/2$.
12. $Q \equiv \left(\dfrac{x_1 + 4y_1 - 6}{3}, \dfrac{2x_1 - y_1 + 6}{3}\right)$;
Equation of curve $4xy + 17y^2 + 12x - 50y + 63 = 0$.
13. $C \equiv (2, 14)$, $B \equiv (6, 17)$; $P \equiv (-4, 19/2)$.
14. $3x + 4y = 2$, $3x + 4y = 50$, $3x - 4y = 10$, $3x - 4y + 14 = 0$; (10, 5), (6, 8), (−2, 2).
15. $3x + y = 12$, $B \equiv (0, 12)$; (24/7, 12/7).
16. (13/17, 23/17).
17. $13x + 21y + 1 = 0$, $17x + 15y + 5 = 0$; area = 4/9 square units.
18. (*a*) (1/5, 7/5), (*b*) (17/5, 19/5), (−3, −1).
20. AP is $y = m(x - 5)$, BQ is $y = m(x + 5)$; $m = 4/7$ or $-4/13$.
21. 1 square unit.
22. $2x + y = 8 + \sqrt{5}$, $x - 2y = -1 + \sqrt{5}$, $3x - y = 2 - \sqrt{10}$.
23. $D \equiv (3, 6)$, $E \equiv (2, 1)$, $F \equiv (10, -1)$; $\triangle ABC = 63$ square units, $\triangle DEF = 21$ square units.
24. $p_2 = \dfrac{(b^2 - a^2)p_1 - 2abq_1 - 2ac}{a^2 + b^2}$, $q_2 = \dfrac{(a^2 - b^2)q_1 - 2abp_1 - 2bc}{a^2 + b^2}$; (3/5, −19/5).
25. 1 : 2; $14x + 5y = 0$.
26. From A $2x + y - 6 = 0$, from B $5x - 3y - 2 = 0$, from C $x + 6y - 16 = 0$; lines meet at (20/11, 26/11).
27. $x + 3y - 2 = 0$, $3x - y + 4 = 0$; length $= 6\sqrt{5}/5$.
28. (20/9, 40/9).
29. Area = 63/65 square units, $LM = 2$ (very nearly), equation of LM is $43x - 6y - 42 = 0$.
30. $D \equiv (3, 5)$, $E \equiv (-3, 3)$, $F \equiv (-1, -1)$; areas 7/2, 14.
31. AD is $x - y - 6 = 0$, DC is $2x + y = 0$; $D \equiv (2, -4)$, mid-point (2, 0).
32. $A \equiv (3, 3)$, $E \equiv (4, -1)$, $F \equiv (-5/3, 5/3)$.
33. $2xx_0 + 2yy_0 = x_0^2 + y_0^2$.
34. $a_1a_2 + b_1b_2 = 0$; $12x + 5y = 17$, $P \equiv (16, -35)$.
35. AB is $3x - 4y - 2 = 0$, BC is $4x - 3y - 12 = 0$; required lines are $2x - 5y + 8 = 0$, $10x - 11y - 16 = 0$.
36. (i) $A \equiv (3, 0)$, $C \equiv (1, 2)$, (ii) AC is $x + y - 3 = 0$, (iii) $P \equiv (7, 8)$.
37. (9, 0); area 20 square units.
38. $AB \cos \theta$; projections on OX 3, 1, −3, −1, projections on OY 1, 1, 4, −6.
39. $(x - 4)^2 + (y - 3)^2 = 9$, $(x - 7)^2 + (y - 6)^2 = 9$; $PQ = \frac{3}{2}\pi$.

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40. $y = 0$, $(3, 0)$; $4x = 3y$, $(9/5, 12/5)$.
41. $4\frac{4}{5}$.
42. $x^2 + y^2 - 10x - 5y + 25 = 0$; $(3, 4)$.
43. $x^2 + y^2 + 5x - 5y = 0$.
44. $x + 1 = 0$; $x^2 + y^2 + 2x - 4y = 0$, radius $\sqrt{5}$, centre $(-1, 2)$.
45. $x^2 + y^2 - 2x - 4y - 20 = 0$; $3x - 4y + 20 = 0$, $3x - 4y = 10$.
46. $3y = 4x$.
47. $x^2 + y^2 - 12x + 24 = 0$.
48. $4x^2 + 4y^2 - 5x - 20y + 25 = 0$; $4x^2 + 4y^2 - 13x - 4y + 1 = 0$; $\sqrt{5}$.
49. Centres $(2, 1)$, $(4, 2)$; radii 1, 2; $2x + y = 6$.
50. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$; $2x + y = 0$, $x + 2y = 0$.
51. $\pm \dfrac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$; $k = -39$ or -91; $x^2 + y^2 - 10x - 2y + 18 = 0$.
52. Centre $(-g, -f)$, radius $\sqrt{(g^2 + f^2 - c)}$.
53. $x^2 + y^2 - 2x - 4y + 4 = 0$; $4y = 3x$.
54. $2x + y = 5$; 4 units, $73° 44'$.
56. (i) Centre $(3, 2)$, radius 5 units; (ii) centre $(3, 0)$, radius 4 units.
57. $x^2 + y^2 - 10x + 14y = 0$; $c = \pm 5\sqrt{15} - 13$.
58. (i) $(10, 2)$, (ii) 2 units.
59. 4 units; $x^2 + y^2 - 2x - 4y + 1 = 0$.
60. $x^2 + y^2 + 5x - 5y = 0$.
61. $(-1, \pm\sqrt{15})$.
62. $5x^2 + 5y^2 - 28x - 44y + 91 = 0$; $2{\cdot}154$.
63. $x^2 + y^2 - 9x - 11y + 18 = 0$; $5{\cdot}70$ radius.
64. $(62/13, 50/13)$.
65. $x^2 + y^2 - 8x - 7y + 22 = 0$, $45° \cdot 14'$.
66. $x^2 + y^2 - 10x \pm 2y\sqrt{21} + 21 = 0$; centres $(5, \pm\sqrt{21})$, radius 5.
67. $y + 2 = 0$, $12x + 5y - 26 = 0$; $\tan^{-1} 12/5 = 67° 23'$.
68. $x^2 + y^2 - 2ax - 2ay + a^2 = 0$.
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69. $(3, 0)$, $(12/5, 9/5)$; tangents $y = 0$, $3x - 4y = 0$.
70. Circle $x^2 + y^2 - 22x - 4y + 100 = 0$; tangents $3x = 4y$, $7x + 24y = 0$.
71. $x^2 + y^2 - 7x + 37y = 68$, centre $(7/2, -37/2)$, radius $\frac{1}{2}(13\sqrt{10})$; tangents intersect at $(7/2, 173/82)$.
72. Centre $(-g, -f)$, radius $\sqrt{(g^2 + f^2 - c)}$; circle $x^2 + y^2 - x + y - 8 = 0$.
73. $y = mx \pm a\sqrt{(1 + m^2)}$; $2y = x \pm 5$ and $2x + y = \pm 5$.
74. $x(a - p) + y(b - q) + c - r = 0$; $x^2 + y^2 - 5x - 15y + 20 = 0$.
75. $x^2 + y^2 - 30x - 20y + 225 = 0$; $2x^2 + 2y^2 - 85y = 0$.
76. $x^2 + y^2 - 4x - 2y + 4 = 0$; $x^2 + y^2 - 12x - 6y + 36 = 0$, radius 3 units.

CHAPTER IX (page 264)

2. $G \equiv (x_1 + 4, 0)$.
3. $k = a/m^2$.
6. $[apq, a(p + q)]$.
9. $m^2(x^2 + y^2) - ax(1 + m^4) + 2amy(m^2 - 1) - 3a^2m^2 = 0$.
10. $(t_1 + t_2)y = 2(x + at_1t_2)$; $ty = x + at^2$.
11. $25x + 10y + 4a = 0$.
12. $4x + 2y + a = 0$; $(\frac{1}{4}a, -a)$.
15. $(1, 2)$ twice, $(\frac{1}{4}, -1)$, $(9/4, -3)$; common tangent $x - y + 1 = 0$.
18. Tangent $3y = x + 18$, normal $3x + y = 66$.
19. $4y = x + 5$, $20y = 7x - 29$.
20. Tangent $x + y\sqrt{3} = 2a$, normal $y\sqrt{3} = 3x - 2a$.
21. $Q \equiv \left(\dfrac{2a^2 - b^2}{2a}\cos\theta,\ \dfrac{b}{2}\sin\theta\right)$.
24. $x/a \cos\theta + y/b \sin\theta = 1$.

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26. $bx \sin\theta + ay(1 - \cos\theta) = ab\sin\theta,\ ay(1 + \cos\theta) = b\sin\theta(x + a)$.
27. $y = mx \pm \sqrt{(a^2m^2 + b^2)}$.
28. $xy_1 + yx_1 = 2c^2$.

CHAPTER X (page 306)

1. $3x^2$; $9x + y + 5 = 0$.
3. (i) $1/2\sqrt{x}$.
4. $\frac{3a}{8}\sqrt{13}$.
5. (i) $\cot x - x\,\mathrm{cosec}^2 x$, (ii) $\frac{2(1 + 2x)}{(1 + x)^3}$, (iii) $-\frac{1}{(1 + \sqrt{x})^2\sqrt{x}}$.
6. $6(2x - 1)(x^2 - x)^5$; $3\sin(2 - 3x)$; $2x\sin 2x + 2x^2\cos 2x$.
8. (i) $(1 - 1/x^2)\cos x - (x + 1/x)\sin x$, (ii) $(a^2 + b^2)/(a\cos x - b\sin x)^2$.
9. $m = 3,\ n = 4$.
10. $-1/(x + a)^2$; tangent at $(0, 1)$ is $y = 1$, tangent at $(\frac{2}{3}, 0)$ is $27x + 4y = 18$.
11. (i) $4x^3 - 2x$, (ii) (a) $-30x(1 - 3x^2)^4$, (b) $3\cos(3x + \pi/7)$, (iii) $x = 5$ or -1.
12. $a\cos ax$.
13. $x^{m-1} + ax^{m-2} + a^2x^{m-3} + \ldots + a^{m-1}$; $A = \sqrt{3}/6$, $B = 0$, $C = 1 - \sqrt{3}/6$.
14. (i) $-2\sin 2x$.
15. $a = \frac{3}{4},\ b = -9$.
16. $\cos x$; $2\pi/3$.
17. $\frac{15}{4}x$; $\frac{9 - x^2}{(x^2 + 9)^2}$; $x\sin x$; (i) $x^2 - \frac{9}{x^4}$, (ii) $-2(1 + x)^{-3}$, (iii) $-2\sin 2x - \sin\frac{1}{2}x$.
18. 1; $-\sin 2x$; $(2x^3 + 3x^2 + 1)/(x + 1)^2$.
19. $\sqrt{3}$.

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20. $(\frac{1}{2}, -\frac{5}{4})$, $(-\frac{1}{2}, \frac{5}{4})$; $x = \pm\frac{6}{11}$.
21. $(\sqrt{2}, 2 - \sqrt{2})$, $(-\sqrt{2}, 2 + \sqrt{2})$.
22. x 0, 1, 2, 3. y 0, 2, 4, 0. $\frac{dy}{dx}$ 0, 3, 0, -9. $\frac{d^2y}{dx^2}$ 6, 0, -6, -12.
23. $2(1 - x^2)/(1 + x^2)^2$.
24. (i) $\frac{3}{2}x^{\frac{1}{2}}$, (ii) $a\cos ax$; $a = \pm 1$.
25. (i) $\frac{1}{3}x^{-2/3}$, (ii) $-\frac{2 - 4x - 5x^2}{(x - x^2)^2}$, (iii) $\sin x(2\cos x\cos\frac{1}{2}x - \frac{1}{2}\sin x\sin\frac{1}{2}x)$.
28. $-1/x^2$; $1/\sqrt{2}$; $-\sqrt{2}$.
29. (i) $\frac{2x^2 + 2x - 7}{(2x + 1)^2}$, (ii) $\frac{6x\cos 3x - \sin 3x}{2x^{3/2}}$.
30. $(x\cos x - \sin x)/x^2$.
31. $a = -3/16,\ b = 0,\ c = 9/4,\ d = -1$; $(-4, 2)$.
32. (i) $\cos x - x\sin x$.
33. $5\pi/16$ square feet.
34. $y = 3t^2x - 2t^3$; $y = 28x^5$.
35. $1 < x < 5$.
36. $-3(1 + x)^{-4}$; $3\cos 3x$.
37. $a = 3,\ b = -2,\ c = \frac{1}{3}$; $x = 1$ or 3.
38. $y = x(1 - 3x_1^2) + 2x_1^3$; $y = x - 5x^3$.
39. $3\left(\frac{1 + x_2^2}{1 - 3x_1^2}\right)^2$; gradient $= 1$, when $x^1 = \pm\sqrt{\frac{3 + 2\sqrt{3}}{3}}$.
40. Tangent $2y = t_1(3x - 4t_1^2)$.
41. $y - y_1 = (4x_1^3 - 4a^2x_1 + b^3)(x - x_1)$.
43. $(x\cos x - \sin x)/x^2$

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44. (i) (*a*) $12x^3 + \frac{4}{3}x^{-5/3}$, (*b*) $2x \cos 5x - 5x^2 \sin 5x$.

45. $\frac{2(3x - 2)}{(x + 2)^3}$; $4 \sin (8x - 6)$.

48. $2ab$.

50. 8 miles.

51. $a = 1$, $b = -6$, $c = 9$, $d = -4$; (1, 0) maximum, (3, −4) minimum, (2, −2) point of inflexion.

52. $-1/(2 + x)^2$; minimum value 4, maximum value −8.

53. $x = 3a/4$.

54. $\theta = 60°$.

55. Maximum +4, minimum 0; point of inflexion.

56. (ii) $v^2/(v - f)$.

58. Maximum point (1, 11), minimum point (−2, −16); slope is −24.

59. £8(1,125/v + v^2/5); v = 14·12 m.p.h.

61. Maximum a^3, minimum $a^3/9$.

62. 88·57 square feet.

63. Sum $= \pi r^2 + (8 - \pi r)^2/(3 + 2\sqrt{2})$; $r = 8/(\pi + 3 + 2\sqrt{2})$.

64. $-2x^{-3}$; maxima 35, −19; minima 33, −21.

65. Maximum $32a^3$, minimum 0.

66. $-2 \sin 2x$; $x = \pi/2$, function minimum value $\frac{1}{2}$, $x = 7\pi/6$, function maximum value 11/4, $x = 3\pi/2$, function minimum value 5/2, $x = 11\pi/6$, function maximum value 11/4.

67. (ii) $4 \operatorname{cosec} 4x$, (iii) $(2x^2 - 3x + 1)e^{x^2 - 3x}$.

68. (*b*) $\frac{3x - 4}{x(x - 2)}$.

69. (*a*) (i) $-\frac{105}{2}x^2(1 - x)^{\frac{1}{2}}$, (ii) $-4(x^2 + 4x + 3)e^{-2x}$, (*b*) acceleration −8 maximum speed −2.

70. (*b*) $-e^x(/1 + e^x)^3$.

72. $18x^3 \log (x + 1)$. www.dbraulibrary.org.in

73. (*a*) (i) $\frac{-4}{(x^2 + 4x)^{3/2}}$, (ii) $\frac{-x^2 e^{-x}}{(x + 2)^2}$, (*b*) $x = \pm c$.

74. (*b*) $x = e^{-\frac{1}{2}}$.

75. (*a*) $\frac{2x^2(x + 3)}{(2 + x)^2}$; $4 \cos 2x(1 + \sin 2x)$, (*b*) $A = -6$, $B = -1$.

CHAPTER XI (page 347)

1. (i) 1, (ii) $\frac{1}{6}$, (iii) $\frac{2}{3}$.

2. (ii) $y = \frac{2}{3}x^3 - 3x^2 + 6x + 1$.

3. (*a*) $-6 \sin x$; $23\frac{33}{40}$; $\frac{1}{8}$.

4. $\frac{dy}{dx} = -\frac{1}{6}A(l - x)^3 - \frac{1}{2}B(l - x)^2 + C$, $y = \frac{1}{24}A(l - x)^4 + \frac{1}{6}B(l - x)^3 + Cx + D$.

When $x = l$, $\frac{dy}{dx} = \frac{1}{6}l^2(2Al + 3B)$, $y = \frac{Al^4}{4} + \frac{Bl^3}{3}$.

5. $10\frac{2}{3}$ square inches.

6. $y = x^3 - 2x^2 - 4x + 8$; $(\frac{1}{3}, 6\frac{13}{27})$.

7. (*a*) 17/4, (*b*) $(\pi - 2)/8$, (*c*) $\frac{5}{8}$, (*d*) $\frac{7}{12}$.

8. $20\pi/3$ cubic units.

9. $\left(\sqrt{3} + \frac{2\pi}{3}\right)\pi$ cubic units.

10. $\frac{1}{3}a$.

11. (4, 0); $6\frac{3}{4}$ quare units.

13. (0, 0), (a, a); $7a^2/9$ square units; $\pi a^3/17$ cubic units.

14. $10\frac{2}{3}$ square units.

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15. $y = \frac{1}{4}(x^2 - 6x + 5)$; $-8/3$ square units.
16. $1/12$ square units; $(3/4, 9/4)$; $2 : 3$.
17. $x + y + 4 = 0$, $x + y - 28 = 0$; 48 square units.
18. 45·2 square units.
19. $5\pi R^3/24$ cubic units.
20. $1{,}056\pi$ cubic units; $8/11$.
21. (i) $3/121$, $\frac{9}{8}$, (ii) $14\frac{23}{40}\pi$ cubic units.
22. $2/3\pi$ in./sec.
23. 8 square units; total area $8\pi(1 + \sqrt{2})$ square units, volume $\frac{40}{3}\pi$ cubic units.
24. Area = 36 square units; centroid $(1, 18/5)$.
25. 22·2 cubic inches.
27. $a(6 + 2\sqrt{6})/3$.
28. Area = 62·8 square cm.; volume = 54·5 cc.
29. $25 : 2$.
30. $(48 - 9\pi/2)$ cubic inches = 33·86 cubic inches; 3·56 inches.
32. $405\pi/2$ cubic inches.
33. $(12 - 4\sqrt{3})$ inches = 5·072 inches.
34. 96,830 square miles.
35. $325\pi/3$ cubic inches.
36. $5\pi R^3/24$ cubic units.
37. $3\pi a^3/2$ cubic units.
38. $\sqrt{3{,}960 \cdot 25}$ miles = 62·93 miles, 3960π square miles approx.
39. Volume = $(1{,}920 + \frac{1550}{3} \tan 35°)$ cubic feet = 2,281·8 cubic feet.
Area = 240 sec 35° square feet = 2,930·0 square feet.
40. $\frac{7}{24}\pi a^2 - (\frac{1}{8}\sqrt{3})a^2$.
41. 194·5 cubic feet; 0·7256, limiting value $27 : 37$.
44. $2\pi r(r - d)$.
45. 1 inch.
46. Area = 26π square inches, volume = $\frac{352}{3}\pi$ cubic inches.
47. $4\pi a^3(3 + 2\sqrt{2})$ cubic units.
48. $3 : 5$.
49. $\frac{1}{3}\pi r^2 h$ cubic units; $\frac{1}{3}\pi a^2(2h - 3a)$ cubic units.
52. 97,800 square miles.
53. 70° 32′.
55. Volume = $16\pi\sqrt{2}/3$ cubic inches; length = $6\sqrt{3}$ inches.
57. 50 square cm., 25 square cm., $25\sqrt{5}$ square cm., $25\sqrt{2}$ square cm. inclinations 45° and $\tan^{-1} 2 = 63° 26′$.
62. 611·3 square feet.
63. $256\pi/3$ cubic inches.
66. (i) Radius $5\sqrt{3}/8$ inches, (ii) $15/8$ inches, (iii) $125\pi/384$ cubic inches.
70. $5/27$, $8/63$.
71. 7·68 units.
72. (i) $\tan^{-1} 1/9$, (ii) $\frac{5\pi a^2}{9}\sqrt{82}$ square units, (iii) $\frac{8}{9}\pi a^3$.
73. $25 : 2$.
74. $3a^3/2$ cubic units; 126° 51′.
75. $[2a^2 \sin^{-1} x/a + 2x\sqrt{(a^2 - x^2)}]$ square units.
76. 87·08 square inches.
79. $\frac{2\sqrt{2}}{3}a^3$ cubic units.
81. (i) $C - (10/9)e^{-0 \cdot 3v}$, (ii) $C - (1 + 3e^{2\theta})/3e^{3\theta}$, (iii) $\frac{3}{7}e^{7x/3} + C$, (iv) $C - \frac{2}{3}\log_e(1 - 3y)$, (v) $\frac{1}{3}\log_e(2 + 3y) + C$, (vi) $\frac{1}{2}\log_e 3 = 0 \cdot 5493$.
82. (i) 0·2987, (ii) 0·2983.
83. 3·611.

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CHAPTER XII (page 356)

2. (i) $3x + 4y = 1$, (ii) $x^2 + y^2 = 3x + 4y$; $(\frac{1}{3}, \frac{1}{4})$, $(0, 0)$, $(3, 0)$, $(0, 4)$; $5/2$, $(3/2, 2)$.
4. $65° 22'$, $114° 38'$.
5. $\theta = 51° 49\frac{1}{2}'$, or $308° 10\frac{1}{2}'$, $r = \frac{1}{2}(1 + \sqrt{5})$.
6. $PQ = 2$.
7. $r^2(a^2 \cos^2\theta + b^2 \sin^2\theta) + 2r(g\cos\theta + f\sin\theta) + c = 0$.
9. $\sin\theta = \theta - \theta^3/3! + \theta^5/5!$, $\cos\theta = 1 - \theta^2/2! + \theta^4/4!$, $\theta = 0{\cdot}2$.
10. $\log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$,
$\log_e(1 - x) = -x - x^2/2 - x^3/3 - x^4/4 - \ldots$,
$\log_e(1 + x)/(1 - x) = 2(x + x^3/3 + x^5/5 + \ldots)$,
$$\log_e\frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right\},$$
$\log_e 8 = 2{\cdot}07944$.
11. (*b*) $a = \frac{1}{2}$, $b = 0$, $c = -1/24$, $d = 0$.
12. (*a*) $x < -2/3$ or $x > 4$.
(*b*) $\log_e(1 + x) = x - x^2/2 + x^3/3 - x^4/4$, $e^x = 1 + x + x^2/2! + x^3/3!$
13. $h - h^2/2 + h^3/4 - h^4/8 + \ldots$
14. (*a*) $z = 3 + 2x - x^2 - \frac{10}{3}x^3 - \frac{7}{2}x^4$.
16. $(1, 0)$, $(3, 0)$; (i) $x^2 + y^2 - 4x - 4y + 3 = 0$,
(ii) $x^2 + y^2 - 4x \pm 2y\sqrt{3} + 3 = 0$,
(iii) $x^2 + y^2 - 4x \pm 4y + 3 = 0$, $x^2 + y^2 - 4x - 2y + 3 = 0$.
18. $x^2 + y^2 = 9$; $x^2 + y^2 \pm 8x - 9 = 0$; $x^2 + y^2 \pm 8y - 9 = 0$.
22. (*a*) (i) -59, (ii) $5x^2 + 17x - 8$, (iii) $6 + 4\sin\theta - 3\cos\theta$.
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(*b*) (i) $x = 4/5$, (ii) $x = 0$ or -3.
23. (*a*) (i) -12, (ii) 887; (*b*) -349, -379; (*c*) -452.
24. (*a*) $$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$
(*b*) 3040.
26. (*a*) $x = -5$, (*b*) $(a - b)(b - c)(c - a)(a + b + c)$.
27. (*a*) $x = 0$ or ± 2,
(*b*) $\frac{d}{dx}(vy) = v\frac{dy}{dx} + y\frac{dv}{dx}$, $\frac{d^2vy}{dx^2} = v\frac{d^2y}{dx^2} + 2\frac{dv}{dx}\frac{dy}{dx} + y\frac{d^2v}{dx^2}$.
28. (*a*) When $\theta = 0$ determinant $= 4$, and when $\theta = \pi$ it equals -4.
30. (i) $(b - c)(c - a)(a - b)(a + b + c)$,
(ii) $(b - c)(c - a)(a - b)(a + b + c)(a^2 + b^2 + c^2)$.
31. $(b - c)(c - a)(a - b)(bc + ca + ab)$; $c = a, b, -ab/(a + b)$;
$x{:}y{:}z = 1{:}0{:}-1$, $0{:}1{:}-1$, $b^2(2a + b){:}-a^2(a + 2b){:}(b^2 - a^2)(a - b)$.

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3. (*a*) $\frac{dy}{dx} = 1 - \frac{x}{\sqrt{(1 - x^2)}}\sin^{-1}x$; y, (*b*) $-2a(1 + t^2)^{3/2}$; $-\sqrt{(2)}AB$.
4. (*a*) (i) $2\tan^{-1}x$, (ii) $\frac{1}{2\sqrt{[(x - 1)(2 - x)]}}$, $1 \leqslant x \leqslant 2$.
(*b*) minimum, $x(2 + \log_e x)^{3/2}$.

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5. (*a*) $\dfrac{1}{(1+x)\sqrt{x}}$; $\dfrac{1}{(1+x)\sqrt{x}}$, (*b*) $\dfrac{1}{ab}[a^2\sin^2 t + b^2\cos^2 t]^{3/2}$; $b^3:a^3$.
6. (*a*) 32/15, (*b*) $\pi/12$, $1 - 2e^{-1}$.
7. (*a*) (i) $18x^2\log_e(x+1)$, (ii) $9x^2\sin^{-1}x$.
8. (i) (*a*) $6x^2\tan^{-1}x$, (*b*) $-4/(x^2+4x)^{3/2}$.
9. (*a*) (i) $\dfrac{-8}{5x^2+18}$, (ii) $\dfrac{-x^2 e^{-x}}{(x-2)^2}$, (*b*) $x = \pm c$.
10. (*a*) (i) $-(2x^2-2x+1)e^{-x}$, (ii) $4(x^2+8)/(x^4+64)$.
11. (i) $x^x(1+\log_e x)$.
12. (*a*) 2, (*b*) (i) $1+\pi/2$, (ii) $\log_e 2$.
13. (*a*) (i) $\log_e 4/3$, (ii) $\frac{1}{4}$, (*b*) $\pi/6$ cu. units.
14. (*a*) $\log_e \dfrac{(x+1)^2}{(x+2)} + C$, (*b*) 1/40.
15. (*a*) $\log_e 8/3$, (*b*) $-\frac{1}{4}\log_e 3$.
16. $\frac{1}{2}\log_e 5$; $\dfrac{1}{5-x} - \dfrac{x}{1+x^2}$, $\dfrac{1}{2}\log_e\dfrac{125}{9}$.
17. (i) 0, (ii) 3, (iii) $\frac{1}{8}$.
18. (i) $\dfrac{1}{6}\log_e\dfrac{x^2-3}{x^2} + C$.
19. $2\tan^{-1}\sqrt{x} - C$, $\frac{1}{3}\cos^3 x - \cos x + C$.
20. (*a*) $\dfrac{1}{2}\left\{\dfrac{1}{x-1} - \dfrac{1}{x+1}\right\}$, $\frac{1}{2}\log_e 1{\cdot}8$, (*b*) $\dfrac{1}{2}\left\{\dfrac{\pi}{12} - \dfrac{\sqrt{3}}{4} + \dfrac{1}{2}\right\}$.
21. (ii) (*a*) $\pi/4$, (*b*) $\pi/6$.
22. (ii) $\dfrac{dy}{dx} = \dfrac{b}{a}\tan\dfrac{50}{2}$.

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23. (i) (*a*) $\dfrac{1}{\sqrt{(4-x^2)}}$, (*b*) $\sin^{-1}x + \dfrac{x}{\sqrt{(1-x^2)}}$, (ii) $\dfrac{\pi}{4}$, $\dfrac{1}{\sqrt{3}}\log_e(2+\sqrt{3})$,
$\sin x - \frac{1}{3}\sin^3 + C$.
24. (i) $\cos^{-1}\sqrt{\dfrac{b-x}{b-a}}$.
25. (*a*) $\pi/2$, (*c*) $5/3\pi$.
26. (i) $2a/\pi$, (ii) $\pi^2/4$.
27. 3·988 sq. units, 0·997.
28. (*b*) $(3a/5, 2a/35)$.
29. (i) $\frac{2}{3}$, (ii) $2/\pi\log_e 2$, (iii) $\frac{1}{4}(9\log_e 3 - 4)$.

CHAPTER XIV (page 415)

1. (i) $1 - 2e^{-1}$, (ii) $2\log_e 2 - \frac{3}{4}$, (iii) $\pi/2 - 1$.
2. (i) π, (ii) $C - (\cos x)\log_e(\sin x) + \log_e(\tan x/2) + \cos x$,
(iii) $C - x^2\cos x + 2x\sin x + 2\cos x$.
3. (i) $\frac{1}{2}[x^2\tan^{-1}x - x + \tan^{-1}x] - C$,
(ii) $x\sec x - \log_e(\sec x + \tan x) + C$.
4. (*a*) $C - \dfrac{e^{-x}}{2}(\sin x + \cos x)$, (*b*) $x\tan x + \log_e(\cos x) - C$,
(*c*) $\pi/4 - \frac{1}{2}\log_e 2 + \pi^2/32$.
5. $P = \dfrac{e^{ax}(a\sin bx - bc\cos bx)}{a^2+b^2}$, $Q = \dfrac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2}$.

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6. $v = \frac{1}{k}[g - e^{c-kt}]$, distance $= \frac{g}{k^2}\,[kT + e^{-kT} - 1]$

7. (i) $y = 1 - e^{-x}$, (ii) $ky\frac{dx}{dt} = x\frac{dy}{dt}$ or $x\frac{dy}{dx} = ky$; $y = Cx^k$.

8. (i) $y = (x + 1)/(x - 1)$, (ii) $\sin 2y - 2y = 2x^2 + 6x + C$.

9. (i) $1/(y + 2) + \frac{1}{2}e^{x^2} + C = 0$, (ii) $45x^3y = 2(1 + x^3)^{5/2} + C$.

10. $y = (\frac{1}{2}x^2 + C)e^{-x}$.

11. (i) $x^2y = C + \sec^2 x$, (ii) $y \sin x = C - \frac{1}{4}\cos 2x$.

12. (i) $y\,e^{2x} = x + C$, (ii) $dz/dx = \frac{1}{4} - z^2$, $\log_e (x + 2y)/(x - 2y) = x - C$.

13. $y = x(k \sin x + 2)$, $\left\{\left(\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}\right)\sin x\right\}^2 = \left\{\left(\frac{y}{x} - 2\right)\cos x\right\}^2$

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4. On QR construct square $QABR$ outside $\triangle PQR$. Join PA, PB meeting QR in C and D respectively. Draw perpendiculars to QR at C and D meeting PQ at F and PR at E. $CDEF$ is required square.

17. $AX = \frac{[bc(a + b + c)(b + c - a)]^{\frac{1}{2}}}{b + c}$, $AY = \frac{[bc(a + b - c)(a - b + c)]^{\frac{1}{2}}}{b - c}$.

22. $a = \frac{2}{3}\sqrt{46}$, $b = \frac{2}{3}\sqrt{31}$, $c = \frac{2}{3}\sqrt{10}$; $\cos^{-1} -5/2\sqrt{310}$.

24. 12 : 25.

25. $PQ = 2QR = \frac{2abc}{(b + c)^2}$.

30. 10 cm., $BX = 58/3$ cm., $CX = 50/3$ cm.

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32. 160 feet.

49. Circle through O.

60. With any point B on AX draw a circle touching AY. Join AP meeting the circle in L and M. Draw PS and PR parallel to BL and BM respectively. Then circles, centres R and S, radii PR and PS, are the required circles.

71. Radius $= (k \cdot AB)/(k^2 - 1)$.

72. Straight line (radical axis).

88. 12/5.

90. 6 cm.

91. $\tan^{-1} 1/\sqrt{2}$.

95. $a\sqrt{2}$.

100. Ratios all 1 : 3, interior for AD and exterior for BD and CD.

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